

Topological Entanglement & Quantum Computation

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BIRS Spin, Charge, and Topology in low dimensions workshop

July 30, 2006

Outline:

Motivation of Topological QC

Two models of Topological QC

An **entanglement** theoretic study of Topological QC

Some related subjects



a major problem in QC: quantum information is more error-prone than classical information

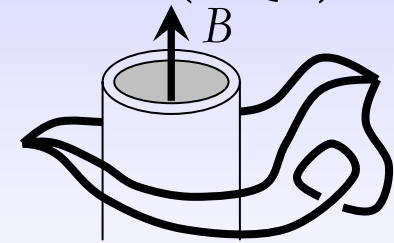
controlling errors \longrightarrow **decoherence**

Remedy: **error-correction** methods & other **fault-tolerant** schemes

Fault-tolerant QC: encoding quantum information in degrees of freedom that are **error-resilient**

\longrightarrow **Topological Quantum Computation** (TQC)

Aharonov-Bohm effect



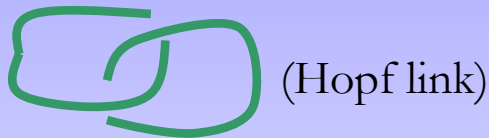
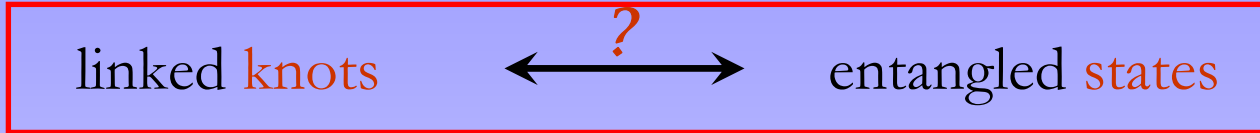
Abelian & non-Abelian **anyons** (FQHE)

another motivation: new powerful quantum **algorithms**

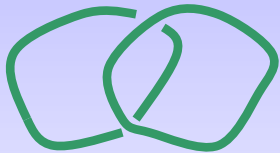


topological entanglement \longleftrightarrow quantum entanglement

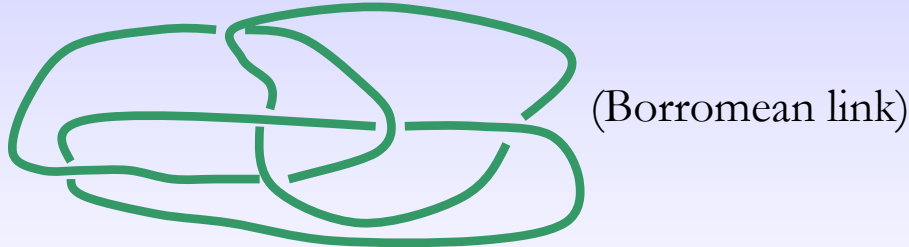
Aravind's hypothesis:



$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$



$$|\Phi\rangle \otimes |\Psi\rangle$$



$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$



$$|\Lambda\rangle = \frac{1}{2}(|000\rangle + |110\rangle + |101\rangle + |011\rangle)$$



Issues: Inconsistencies

- ① particle $\overset{?}{\longleftrightarrow}$ link component
topological self-entanglement



- ② local invariance of entanglement

$$|\Lambda\rangle = H \otimes H \otimes H |\text{GHZ}\rangle$$

- ③ measurement $\overset{?}{\longleftrightarrow}$ cutting a link
post-measurement state depends on measurement basis & result

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \xrightarrow{\{|0\rangle, |1\rangle\}} \begin{array}{l} |0\rangle: \text{ entangled} \\ |1\rangle: \text{ separable} \end{array}$$

...



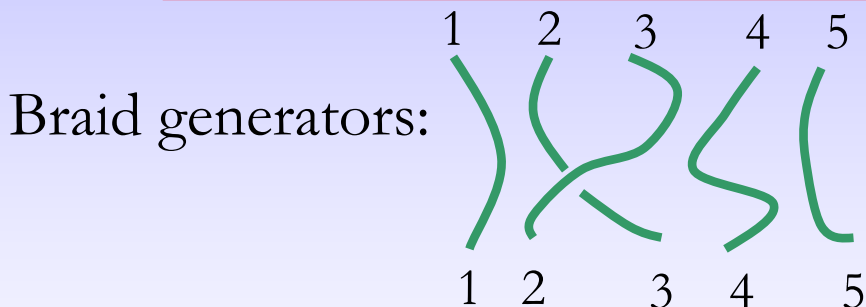
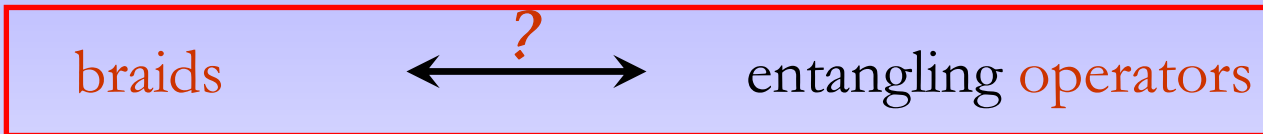


knots \longleftrightarrow states

Inconsistent!

second approach: motivated by **Alexander's Theorem**

Theorem: Any knot can be obtained from closure of some **braid**



$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, \dots, n - 2$$

Yang-Baxter equation: $R: V \otimes V \rightarrow V \otimes V$

$$\tilde{R}_{12} \tilde{R}_{13} \tilde{R}_{23} = \tilde{R}_{23} \tilde{R}_{13} \tilde{R}_{12} \quad (R = S \cdot \tilde{R}) \longrightarrow \text{solutions of YBE?}$$



$V = \mathbb{C}^2 \longrightarrow$ a full classification of solutions (Hietarinta, Hlavaty, Dye)

Now, braiding operators (solutions of YBE) can be considered as 2-qubit quantum operators.

Question: Can braiding operators be used for QC?



Are braiding operators **sufficient** for **universal** QC?

Universality Theorem: 1-qubit unitaries + any **entangling** 2-qubit unitary

$$U \in \text{SU}(4) \quad \exists |\alpha\rangle, |\beta\rangle: U(|\alpha\rangle \otimes |\beta\rangle) \neq |\alpha'\rangle \otimes |\beta'\rangle$$

E.g.: CNOT $|x\rangle \otimes |y\rangle \rightarrow |x\rangle \otimes |x \oplus y\rangle$

Local equivalence:

$$U \equiv U' \iff \exists u_1, v_1, u_2, v_2 \in \text{SU}(2); \quad U' = (u_1 \otimes v_1)U(u_2 \otimes v_2)$$



Theorem: Canonical decomposition $U \in \text{SU}(4)$

$$U = (u_1 \otimes v_1) \underbrace{e^{-i(c_1 \sigma_x \otimes \sigma_x + c_2 \sigma_y \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z)}}_{\text{Heisenberg interaction}} (u_2 \otimes v_2)$$

→ classification of local equivalence: $\begin{cases} U \equiv [c_1, c_2, c_3] \\ G_i[U]: \text{local invariants} \end{cases}$

- If we can find an **entangling** gate among YBE solutions, then braiding operators are able to do **universal** QC.

→ selection of **unitary** solutions of YBE: (by a brute-force search)

(up to some symmetries & local equivalence)

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & 1 \\ & 1 & -1 & \\ & & 1 & 1 \\ -1 & & & 1 \end{pmatrix} \equiv \text{CNOT}$$

$$R' = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \quad (|a_i| = 1)$$

⇒ **universality** ✓

$a_i = 1 \Rightarrow \text{SWAP}$
 $a_{i \neq 3} = 1, a_3 = -1 \Rightarrow \text{DCNOT}$



Question: How entangling braid operators are?!

quantification of **entangling power**: $e_p(U) := \overline{E(U | \psi \rangle \otimes | \phi \rangle)}_{|\psi \rangle, |\phi \rangle}$

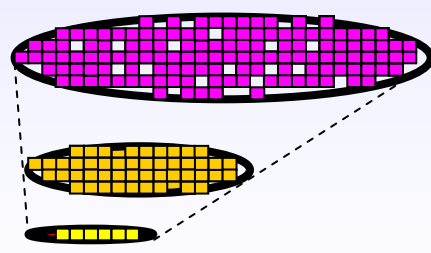
$\left\{ \begin{array}{l} \text{——— } |\psi \rangle, |\phi \rangle: \text{ average on the manifold of separable states with the probability } p \\ E: \text{ an entanglement measure for states, e.g.: linear entropy } E(| \Psi \rangle_{AB}) := 1 - \text{tr}(\rho_A^2) \end{array} \right.$

Entangler: $\exists |\alpha \rangle, |\beta \rangle: U(|\alpha \rangle \otimes |\beta \rangle) \neq |\alpha' \rangle \otimes |\beta' \rangle \longrightarrow$ entangled
 entangling some separable state

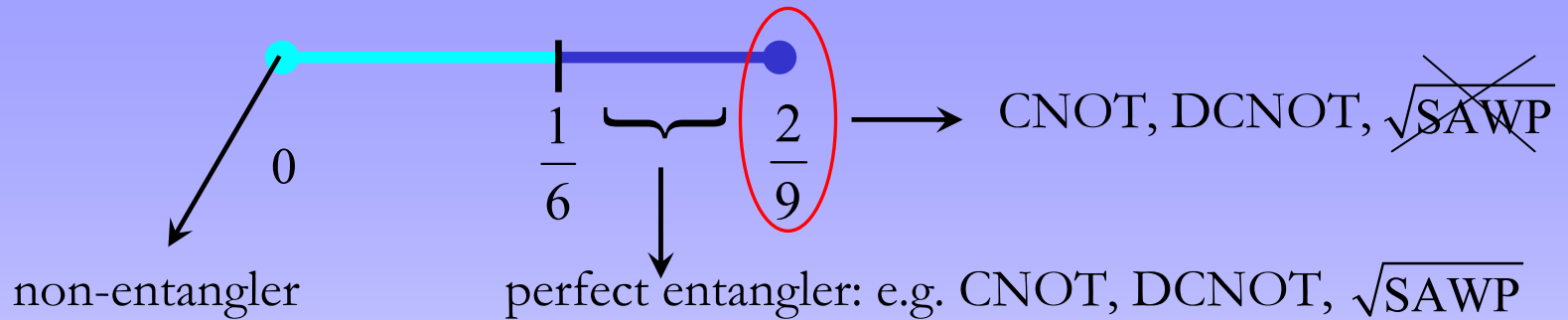
Perfect entangler: $\exists |\alpha \rangle, |\beta \rangle: U(|\alpha \rangle \otimes |\beta \rangle) \neq | \Psi_{\alpha\beta} \rangle \longrightarrow$ maximally entangled
 E.g.: CNOT, DCNOT, $\sqrt{\text{SAWP}}$

Special perfect entangler: **maximally** entangling some full **separable basis** ?
 E.g.: CNOT, DCNOT, ~~$\sqrt{\text{SAWP}}$~~

the only **braiding** operators that are **perfect entanglers**:
 CNOT, DCNOT



$$e_p(U) = \frac{1}{18} [3 - (\cos 4c_1 \cos 4c_2 + \cos 4c_2 \cos 4c_3 + \cos 4c_3 \cos 4c_1)]$$



Question: Is there any very strong entangler other than CNOT & DCNOT?

If **NOT**, all strongest possible entanglers are braid entanglers. 😊

Answer: Theorem:

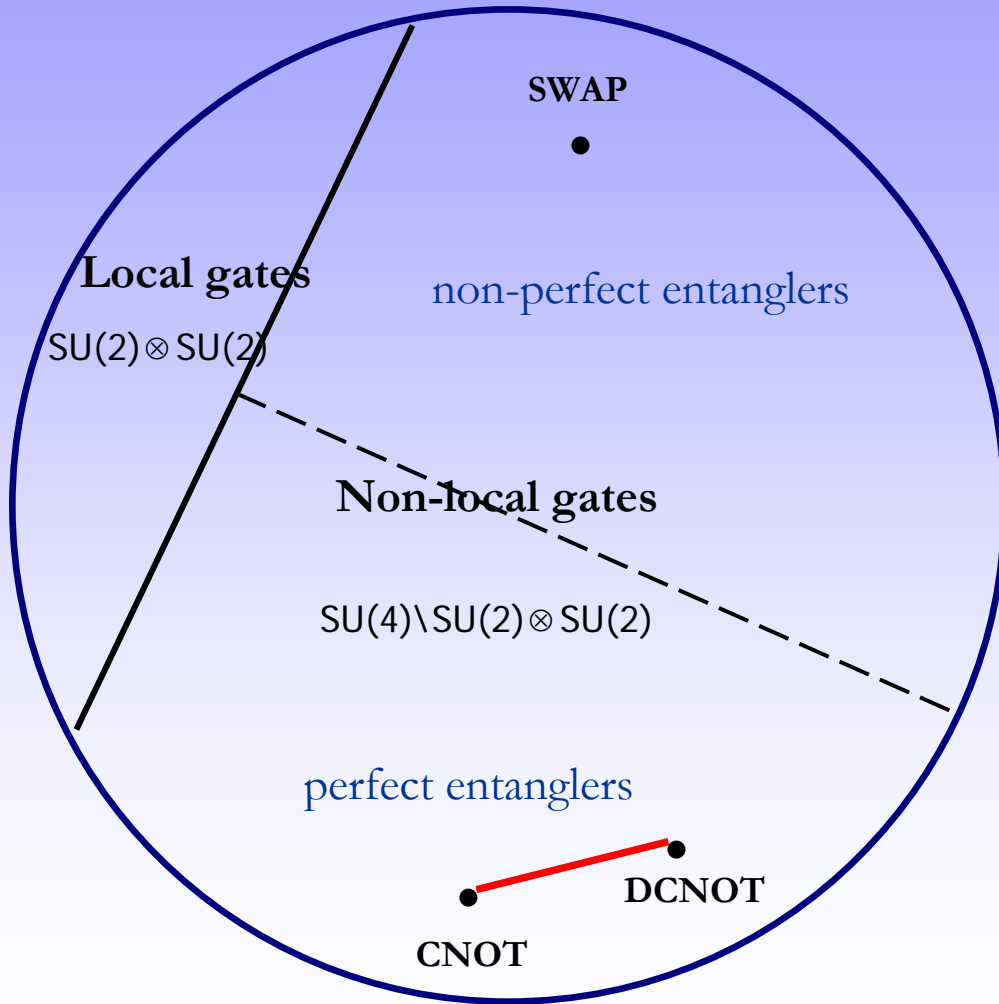
$$e_p(U) = e_{p_{\max}} = \frac{2}{9} \iff U \equiv \left[\frac{\pi}{4}, \varphi, 0 \right] \iff U : \text{special perfect entangler}$$

$$\text{CNOT} \quad 0 \leq \varphi \leq \frac{\pi}{4} \quad \text{DCNOT}$$

$$C[\varphi] := e^{-i\left(\frac{\pi}{4}\sigma_x \otimes \sigma_x + \varphi\sigma_y \otimes \sigma_y\right)}$$

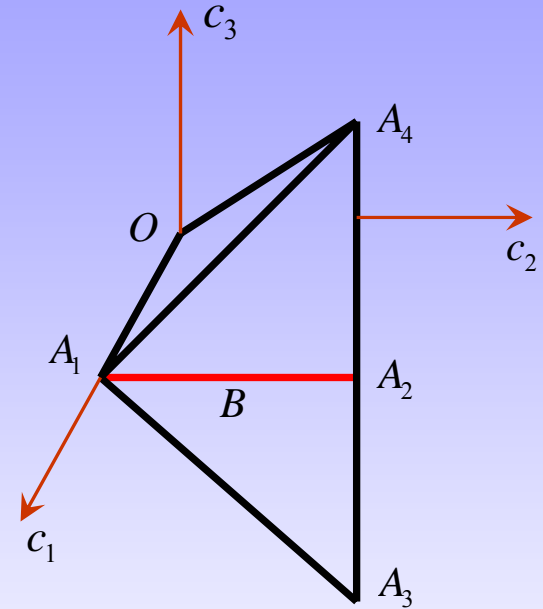


Geometric picture:



$$\frac{\pi}{4} \geq c_1 \geq c_2 \geq |c_3|$$

the Weyl chamber



$$A_1 \left(\left[\frac{\pi}{4}, 0, 0 \right] \right) \equiv \text{CNOT}$$

$$A_2 \left(\left[\frac{\pi}{4}, \frac{\pi}{4}, 0 \right] \right) \equiv \text{DCNOT}$$

$$B \left(\left[\frac{\pi}{4}, \frac{\pi}{8}, 0 \right] \right)$$



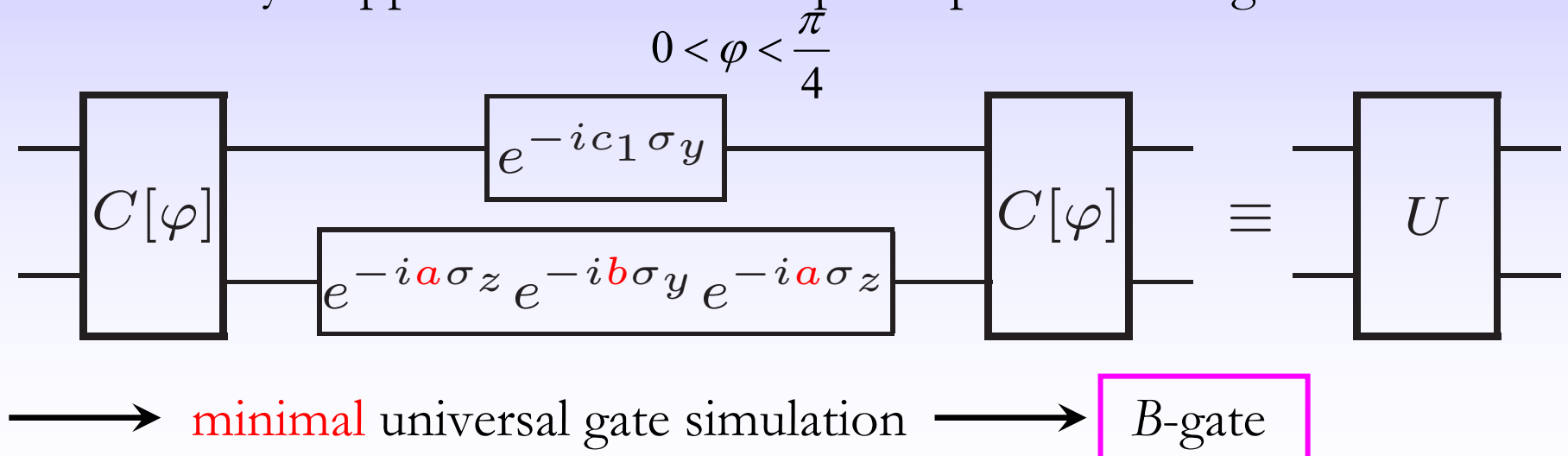
Then, braiding entanglers are not the only possible strongest entanglers!

further investigation: importance of special perfect entanglers ?

their special role in universal gate construction

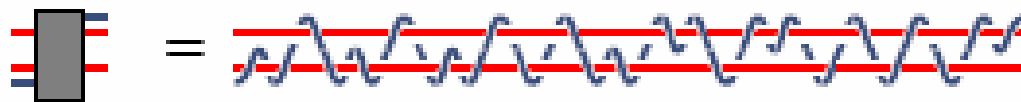
By using **at least 3** CNOTs or DCNOTs any unitary gate can be constructed or simulated,

while: only **2** applications of **other** special perfect entanglers is sufficient!



Some related subjects :

- an improvement: Universal TQC with only one mobile quasiparticle (Simon et al.)



- Fault-tolerant one-way QC with **cluster states** (Raussendorf et al.)
- a quantum algorithm calculation of the **Jones polynomials**:
a BQP-complete problem (Freedman, Kitaev, Aharonov, et al.)
- **Quantum/Topological order**: root of order in **FQH** states (Wen)
topological degeneracy of the ground state of Kitaev's model
relation of topological order and entanglement of ground state (Kitaev & Preskill)
braid group analysis of topological order (Sato et al.)

