A fault-tolerant one-way quantum computer

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BIRS @Banff Center, July 31, 2006

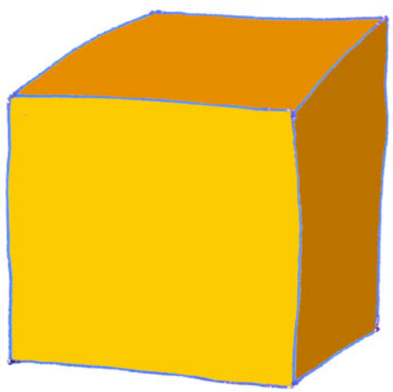
Talk outline

Part I: One-way quantum computer ($QC_{\mathcal{C}}$) and cluster states

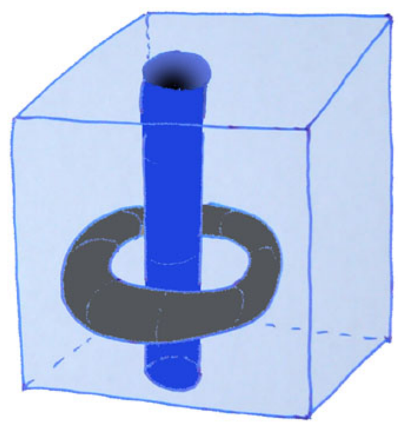
What is the one-way quantum computer?

Part II: Fault-tolerance

Which methods are used? What is the threshold?



3D cluster state = fault-tolerant substrate

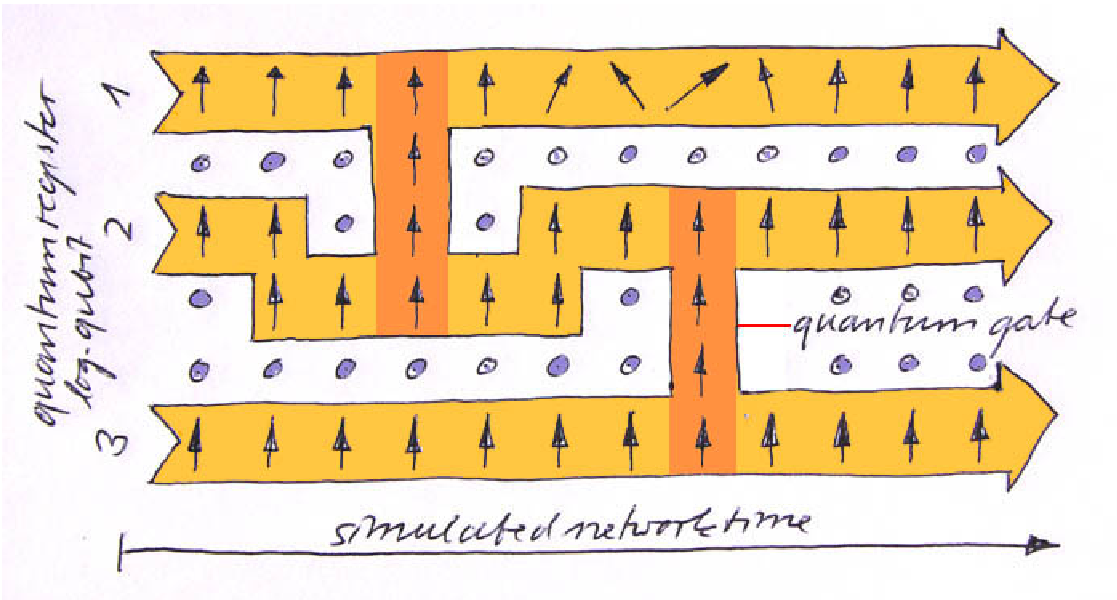


Logic from non-trivial boundary conditions

Part I:

The one-way quantum computer and cluster states

The one-way quantum computer



measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

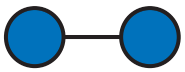
- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

Cluster states - creation

1. Prepare product state $\bigotimes_{a \in \mathcal{C}} \frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}$ on d-dimensional qubit lattice $\mathcal{C}$.

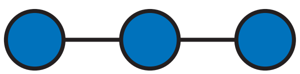
2. Apply the Ising interaction for a fixed time T (conditional phase of π accumulated).

Cluster states - simple examples



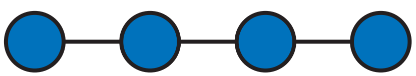
$$|\psi\rangle_2 = |0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2$$

Bell state



$$|\psi\rangle_3 = |+\rangle_1|0\rangle_2|+\rangle_3 + |-\rangle_1|1\rangle_2|-\rangle_3$$

GHZ-state



$$|\psi\rangle_4 = |0\rangle_1|+\rangle_2|0\rangle_3|+\rangle_4 + |0\rangle_1|-\rangle_2|1\rangle_3|-\rangle_4 + \\ + |1\rangle_1|-\rangle_2|0\rangle_3|+\rangle_4 + |1\rangle_1|+\rangle_2|1\rangle_3|-\rangle_4$$

Number of terms exponential in number of qubits!

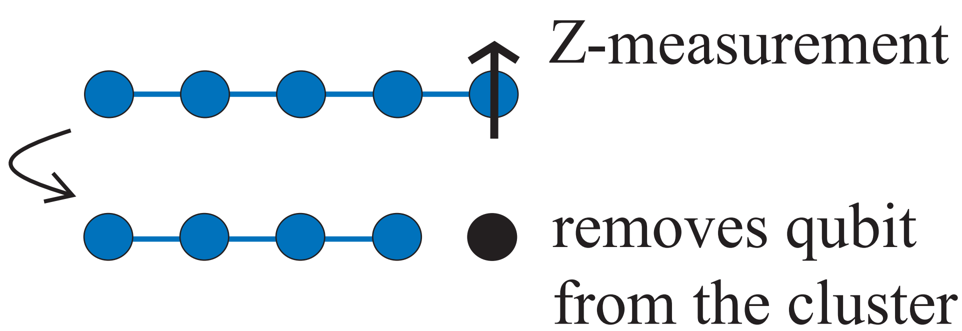
Cluster states - definition

A cluster state $|\phi\rangle_{\mathcal{C}}$ on a cluster $\mathcal{C}$ is the single common eigenstate of the stabilizer operators

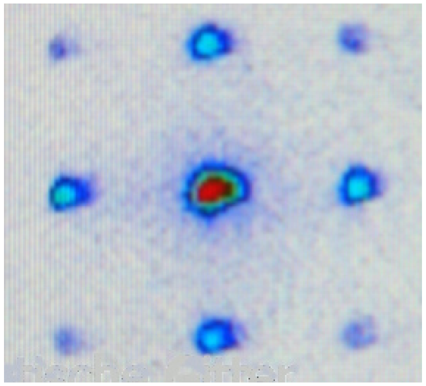
$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in \mathcal{C}, \tag{1}$$

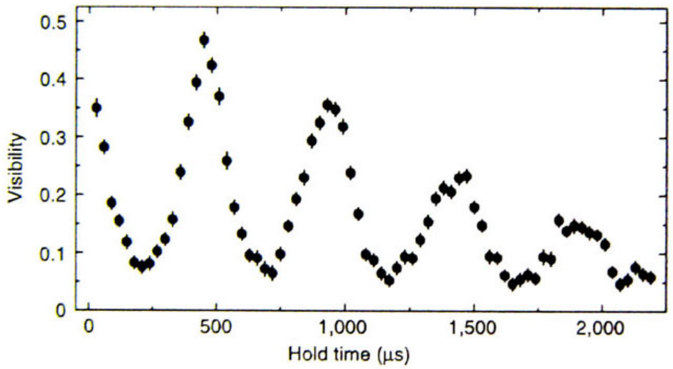
where $b \in N(a)$ if a,b are spatial next neighbors in $\mathcal{C}$.

Z-**Rule:**



Cluster states - experiment





Cold atoms in optical lattices

Coherent interaction among neighboring atoms

Greiner, Mandel, Esslinger, Hänsch, and Bloch, *Nature* 415, 39-44 (2002),
Greiner, Mandel, Hänsch and Bloch, *Nature*, 419, 51-54 (2002).

Part II:

Fault-tolerance

The threshold theorem

Theorem*: *Assume a suitable noise model for a universal quantum computer. If the noise per elementary operation is below a constant non-zero threshold ϵ then arbitrarily long quantum computations can be performed with arbitrary accuracy at small operational overhead.*

What is a suitable noise model?	What is the value of ϵ?	What is a small overhead?
Standard: Indepedent probabilistic errors	$\epsilon = 10^{-10}..10^{-2}$	Polylogarithmic. $S \longrightarrow S \log^{\gamma} S$
Generalized	Improve threshold!	Reduce overhead!

*: Aharonov & Ben-Or (1996), Kitaev (1997), Knill & Laflamme & Zurek (1998), Aliferis & Gottesman & Preskill (2005)

Known threshold values

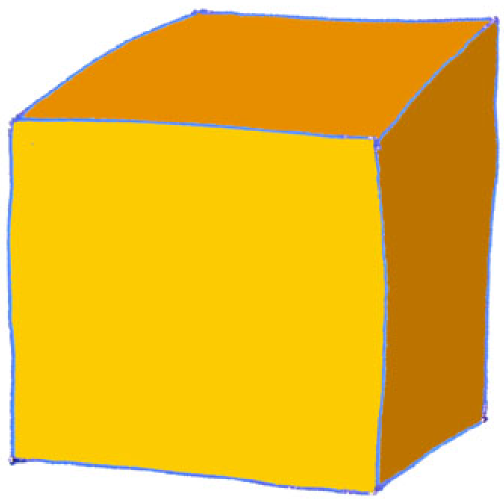
no constraint	geometric constraint: 3D	geometric constraint: 2D	geometric constraint: 1D
[1]! — 0.03, est.			
[2] — 10^{-3}, est.	[5]* — 10^{-3}, est.		
[3] — 10^{-4}, est.			
[4] — 10^{-5}, bd.		[6] — 10^{-5}, est.	
			[7] — 10^{-8}, est.

!: 10^{14} bare gates for 1000 encoded gates [Knill, (2004)]

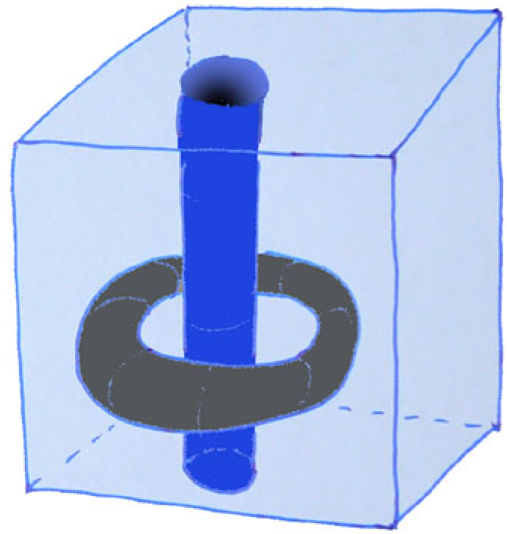
[1] Knill, (2005); [2] Zalka (1999); [3] Dawson & Nielsen (2005); [4] Aliferis & Gottesman & Preskill (2005), [5] Raussendorf & Harrington & Goyal, quant-ph/0510135; [6] Cross (unpublished), [7] Aharonov & Ben-Or (1999)

Fault-tolerant one-way QC

Main idea: *Replace 2D cluster state by 3D cluster state!*

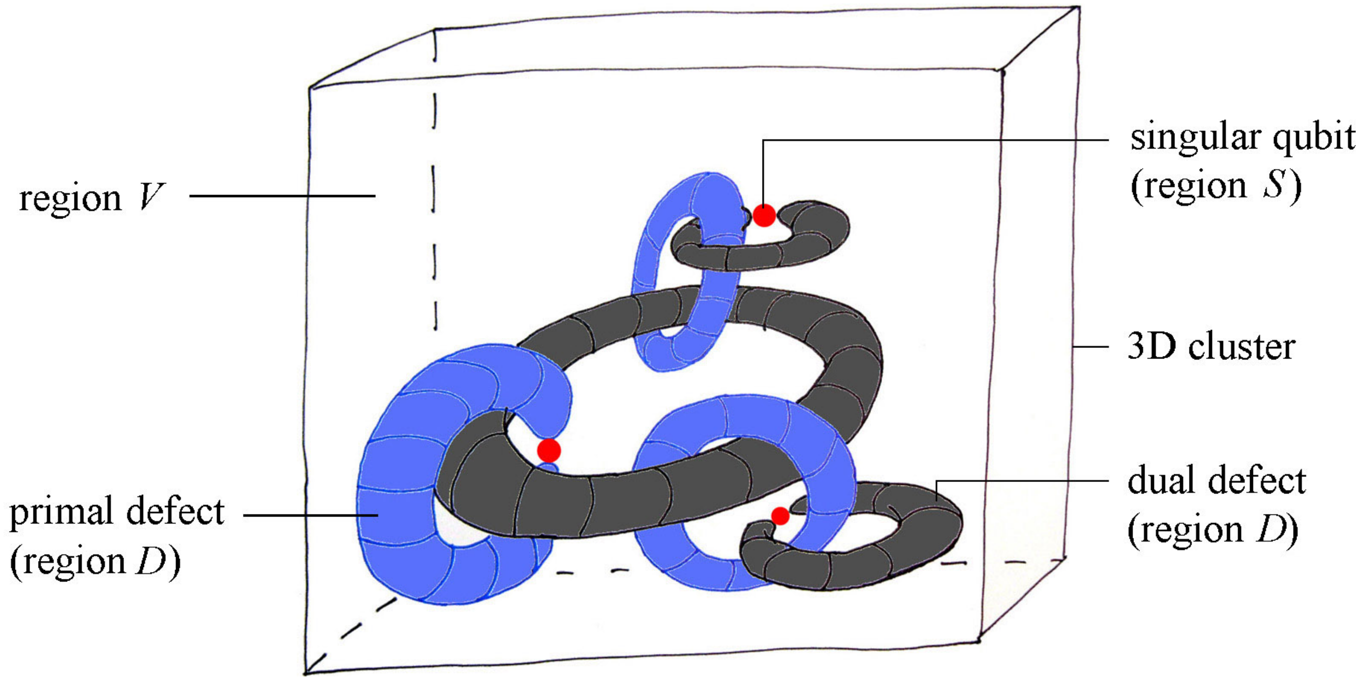


3D cluster state = fault-tolerant substrate



Logic from non-trivial boundary conditions

Macroscopic view

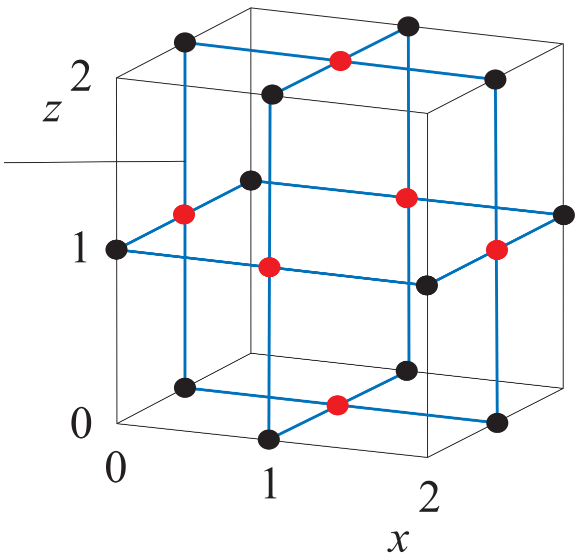


Three cluster regions:

V (Vacuum), D (Defect) and S (Singular qubits).

Qubits $q \in V$:	local X-measurements,
Qubits $q \in D$:	local Z-measurements,
Qubits $q \in S$:	local measurements of $\frac{X \pm Y}{\sqrt{2}}$.

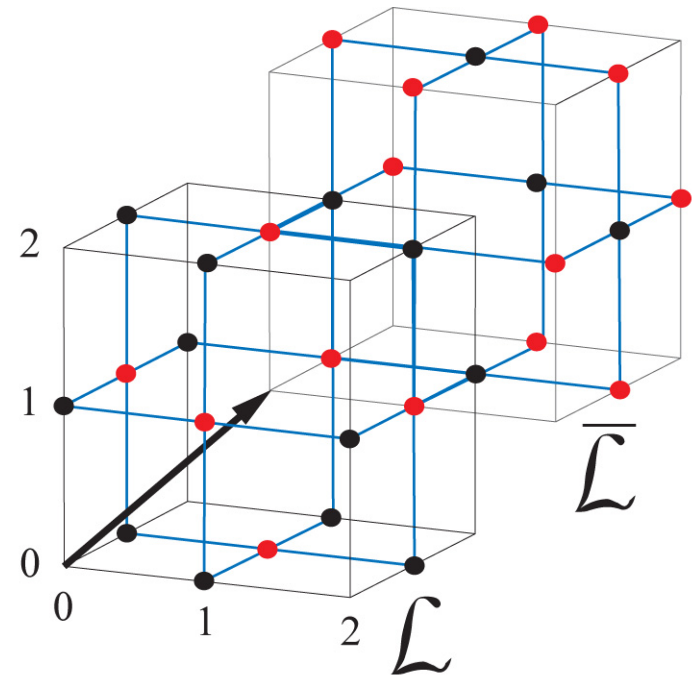
Microscopic view



elementary cell of $\mathcal{L}$

qubit location:	(even, odd, odd)	-	face of $\mathcal{L}$,
qubit location:	(odd, odd, even)	-	edge of $\mathcal{L}$,
syndrome location:	(odd, odd, odd)	-	cube of $\mathcal{L}$,
syndrome location:	(even, even, even)	-	site of $\mathcal{L}$.

Lattice duality $\mathcal{L} \longleftrightarrow \overline{\mathcal{L}}$



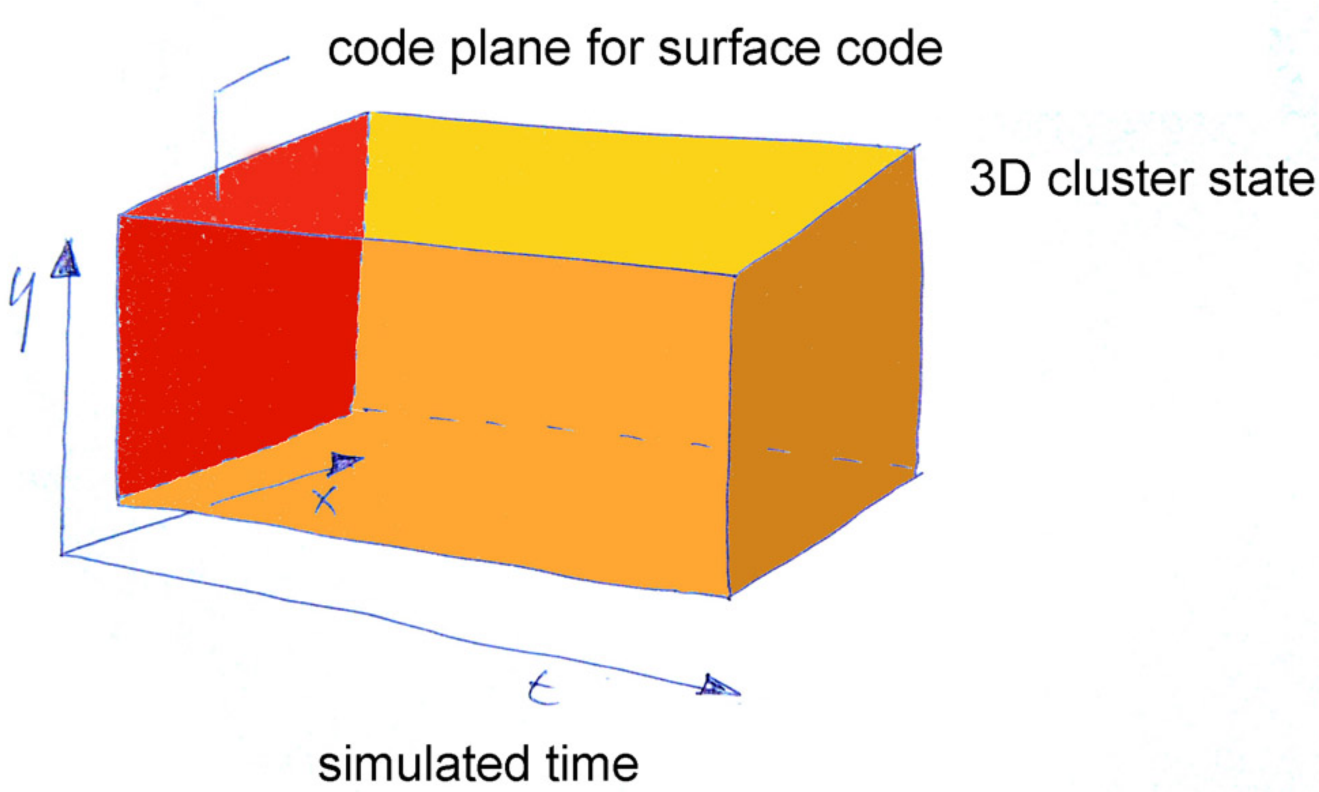
Translation by vector $(1,1,1)^T$:

- Cluster $\mathcal{C}$ invariant,
- $\mathcal{L}$ (primal) $\longrightarrow$ $\overline{\mathcal{L}}$ (dual).

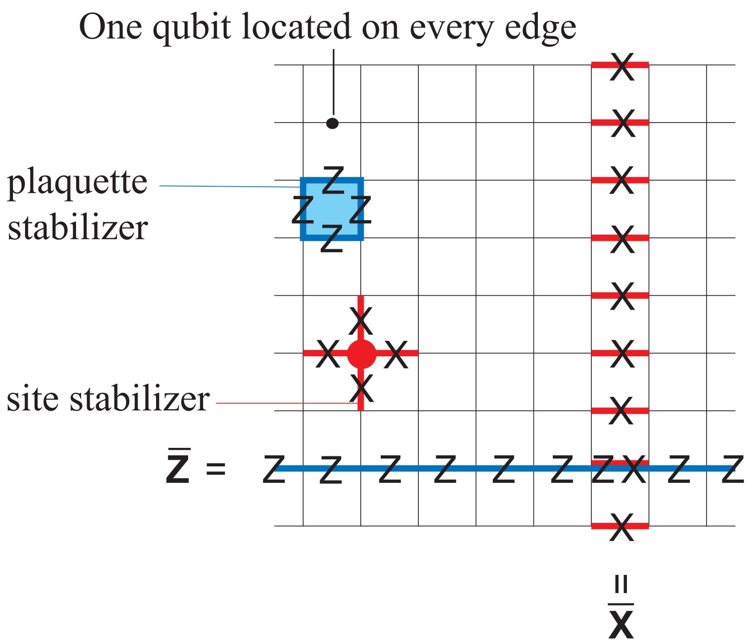
$$\begin{aligned} \text{face of } \mathcal{L} \;&-\; \text{edge of } \overline{\mathcal{L}}, \\ \text{edge of } \mathcal{L} \;&-\; \text{face of } \overline{\mathcal{L}}, \\ \text{site of } \mathcal{L} \;&-\; \text{cube of } \overline{\mathcal{L}}, \\ \text{cube of } \mathcal{L} \;&-\; \text{site of } \overline{\mathcal{L}}, \end{aligned} \tag{2}$$

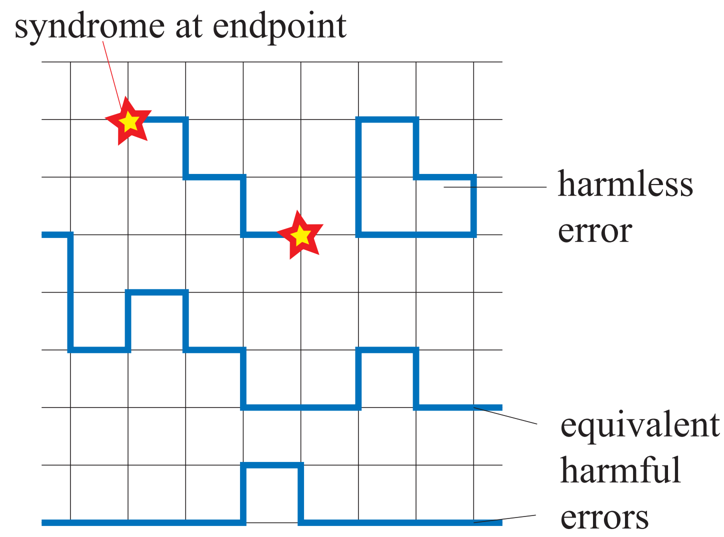
- Many objects in this scheme exist as 'primal' and 'dual'.

Key to scheme



Surface codes

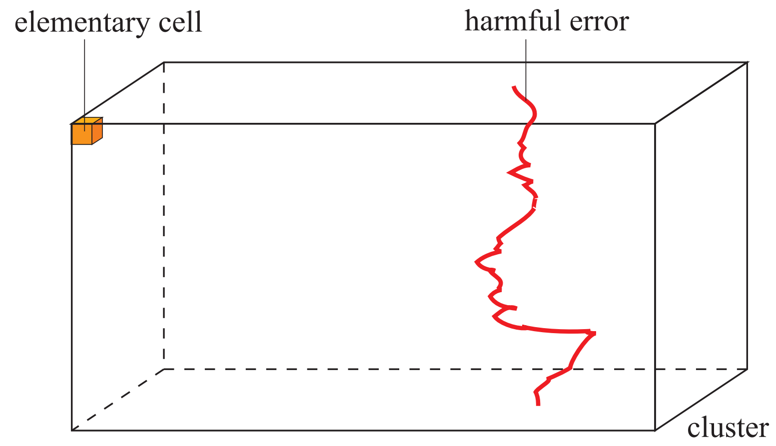




- Errors are represented by chains.
- Homologically equivalent chains correspond to physically equivalent errors.
- Harmfull errors stretch across the entire lattice (rare events).

A. Kitaev,quant-ph/9707021 (1997).

$QC_\mathcal{C}$: topological error correction in V



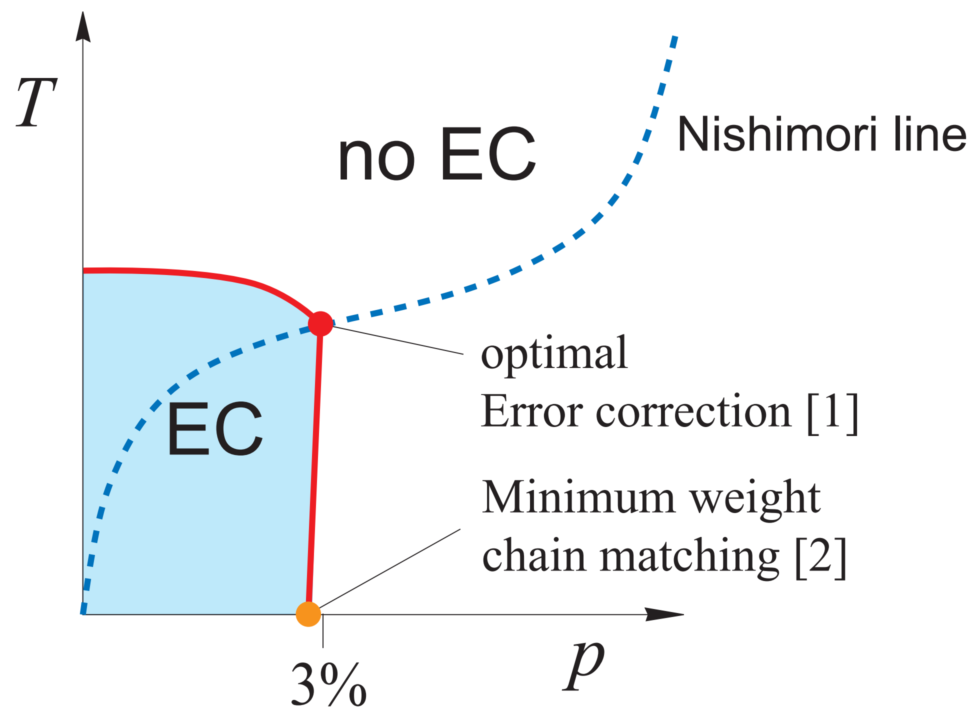
- Errors are represented by chains.
- Homologically equivalent chains correspond to physically equivalent errors.
- Harmfull errors stretch across the entire lattice.

-> Leads to *Random plaquette Z_2-gauge model* (RPGM) [1].

[1] Dennis et *al.*, quant-ph/0110143 (2001).

RPGM: schematic phase diagram

Map error correction to statistical mechanics:

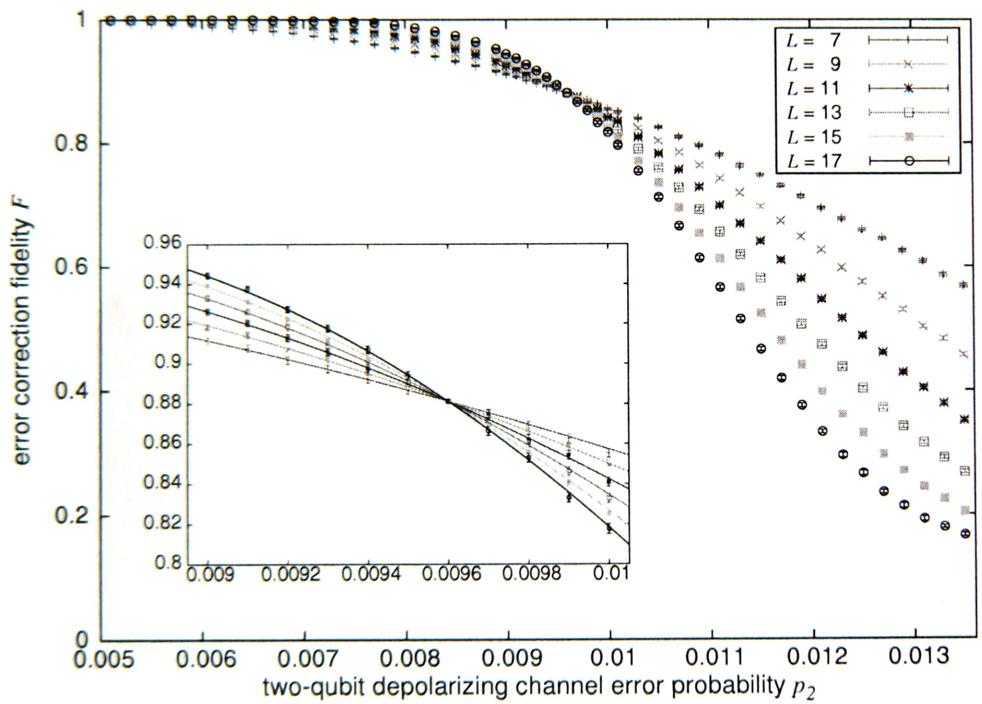


[1] T. Ohno et *al.*, quant-ph/0401101 (2004). [2] E. Dennis et *al.*, quant-ph/0110143 (2001); J. Edmonds, Canadian J. Math. **17**, 449 (1965).

Error model:

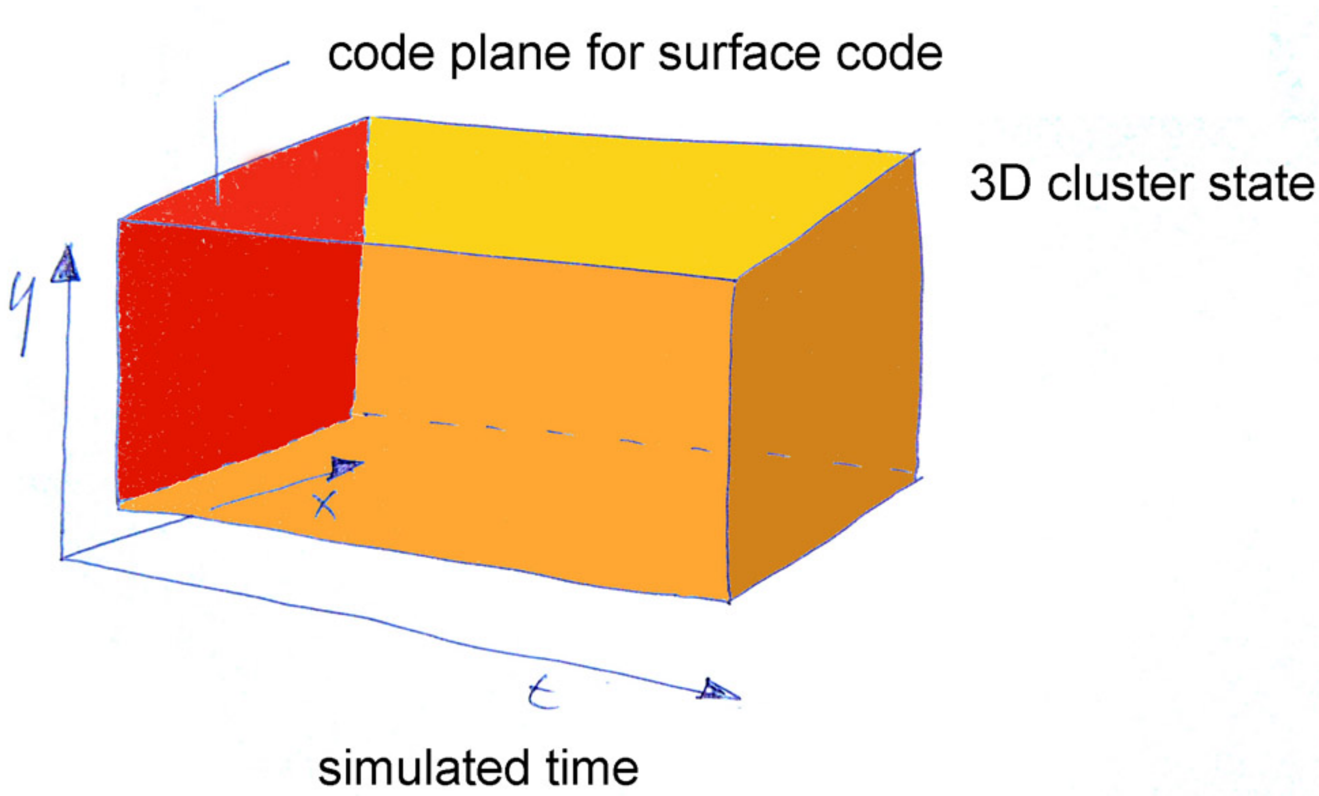
- Cluster state created in a 4-step sequence of $\Lambda(Z)$-gates from product state $\bigotimes_{a\in\mathcal{C}} |+\rangle_a$.
- Error sources:
  - $|+\rangle$-preparation: Perfect preparation followed by 1-qubit partially depolarizing noise with probability p_P.
  - $\Lambda(Z)$-gates: Perfect gates followed by 2-qubit partially depolarizing noise with probability p_2.
  - Memory: 1-qubit partially depolarizing noise with probability p_S per time step.
  - Measurement: Perfect measurement preceeded by 1-qubit partially depolarizing noise with probability p_M .
- 3D cluster state created in slices of fixed thickness.
- Instant classical processing.

Fault-tolerance threshold in V

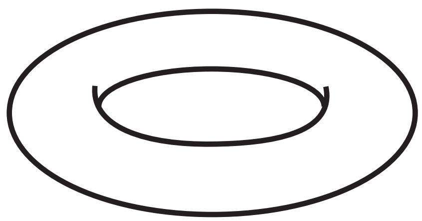


$$\begin{aligned} p_{2,c} &= 9.6 \times 10^{-3}, && \text{for } p_P = p_S = p_M = 0, \\ p_c &= 5.8 \times 10^{-3}, && \text{for } p_P = p_S = p_M = p_2 =: p. \end{aligned} \qquad (3)$$

Fault-tolerant quantum logic

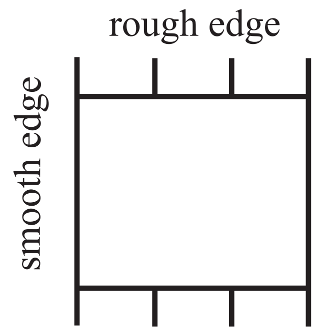


Surface codes



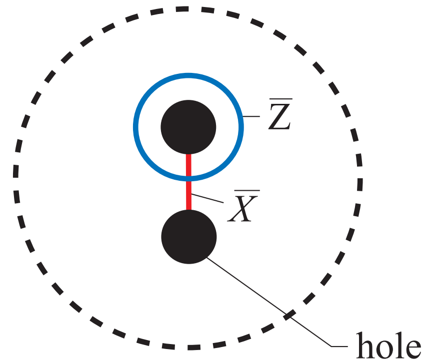
Torus

2 Qubits



Plane segment

1 Qubit

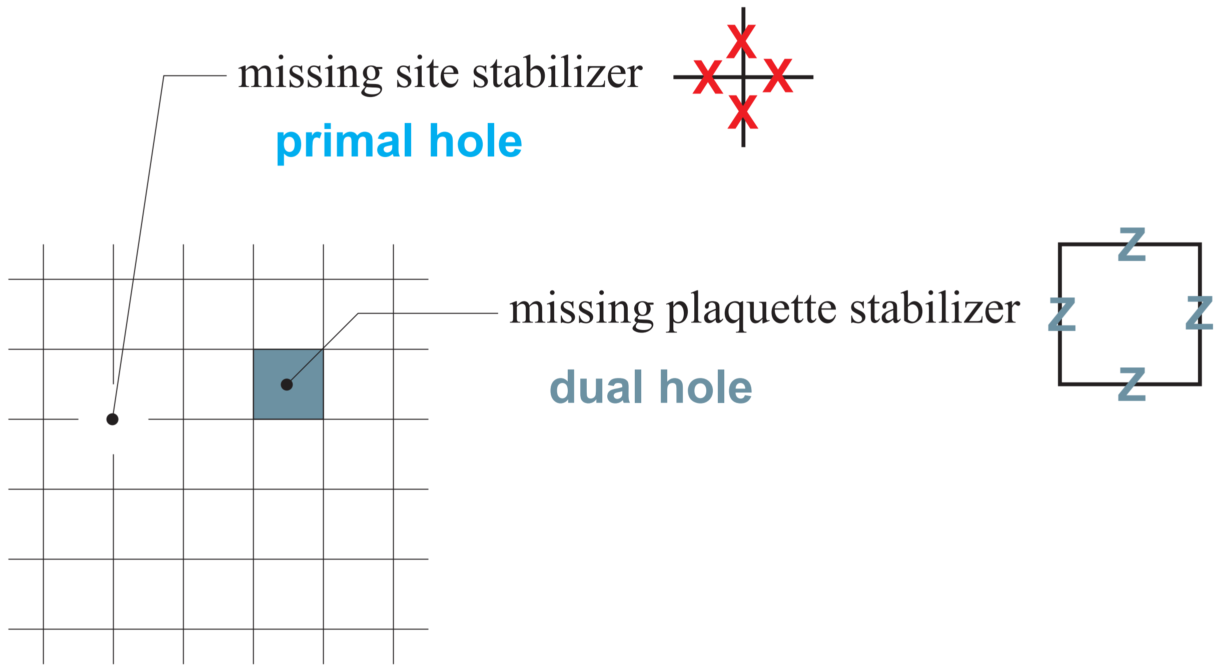


Plane with 2 holes

1 Qubit

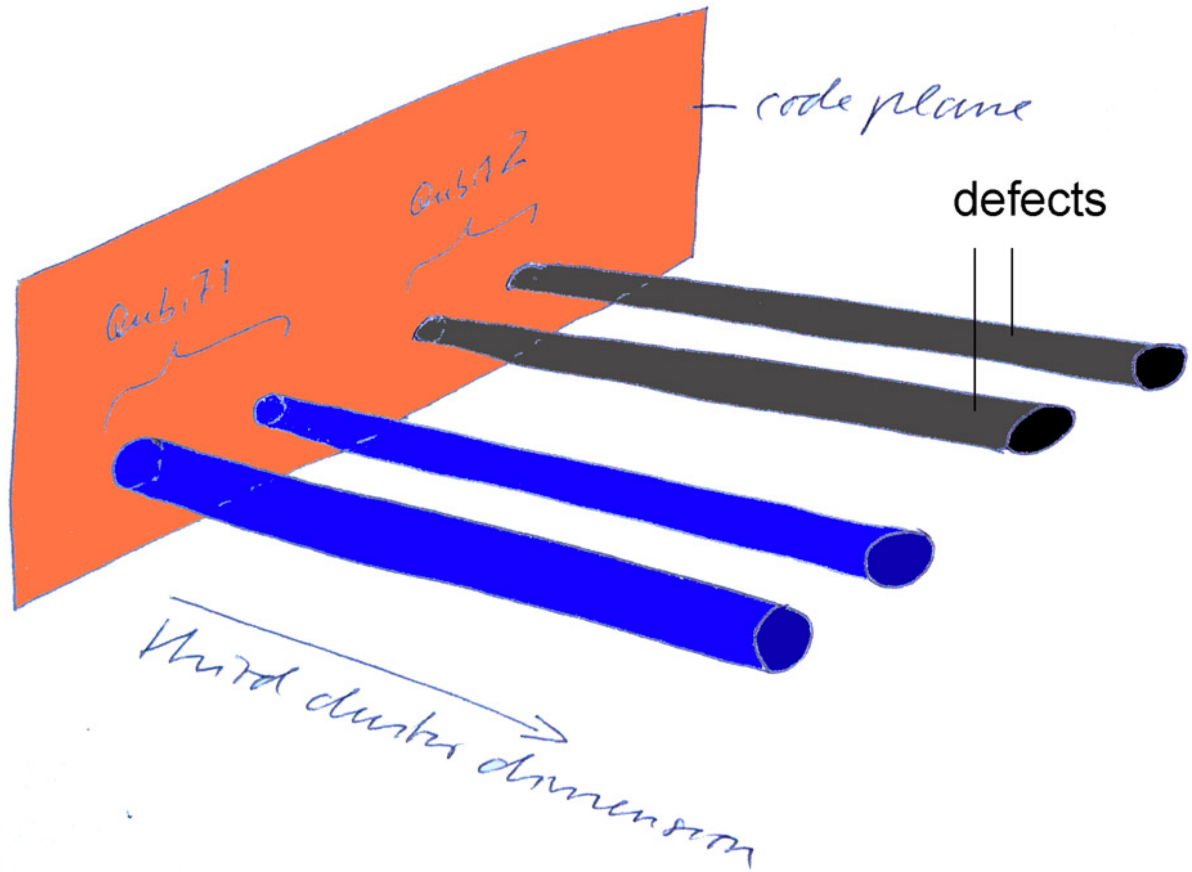
- Storage capacity of the code depends upon the topology of the code surface.

Surface codes



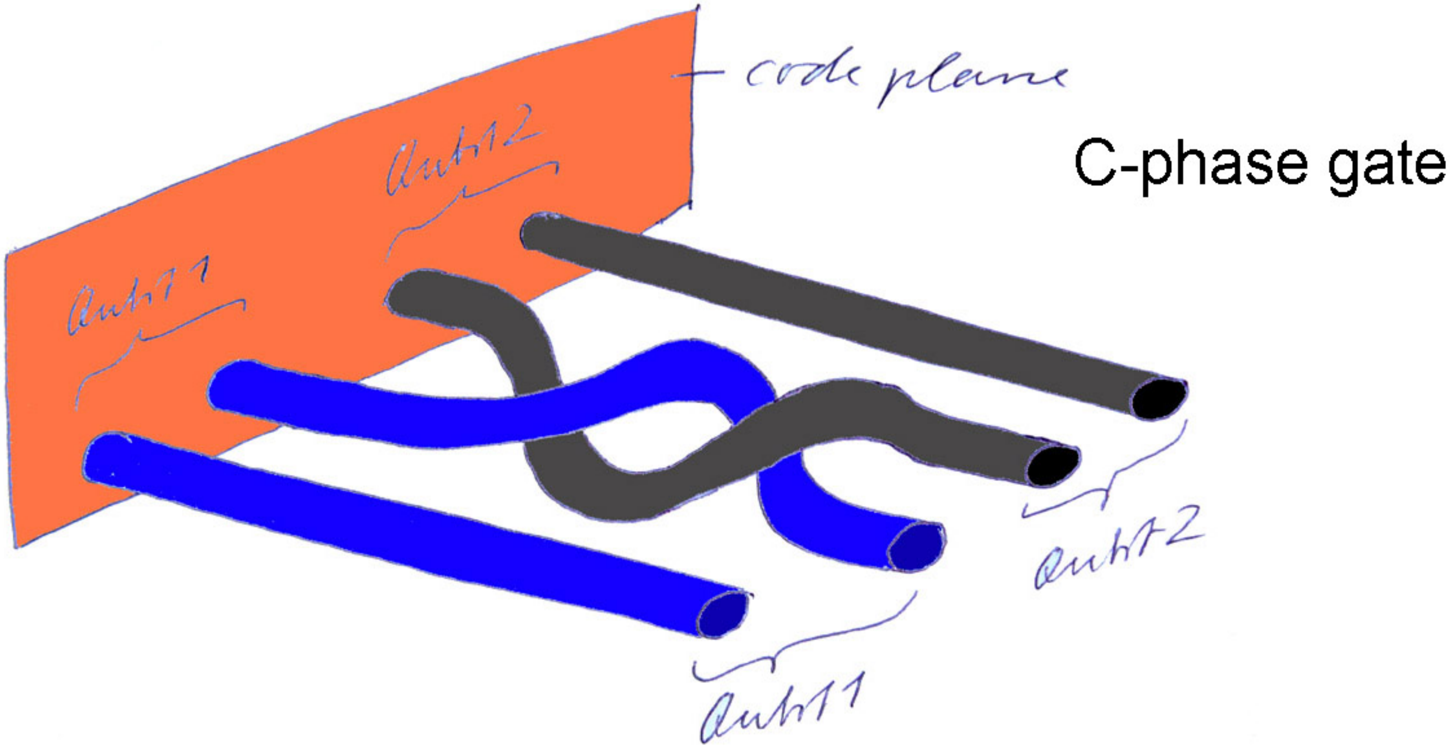
- There are two types of holes: primal and dual.
- A pair of same-type holes constitutes a qubit.

Defects for quantum logic



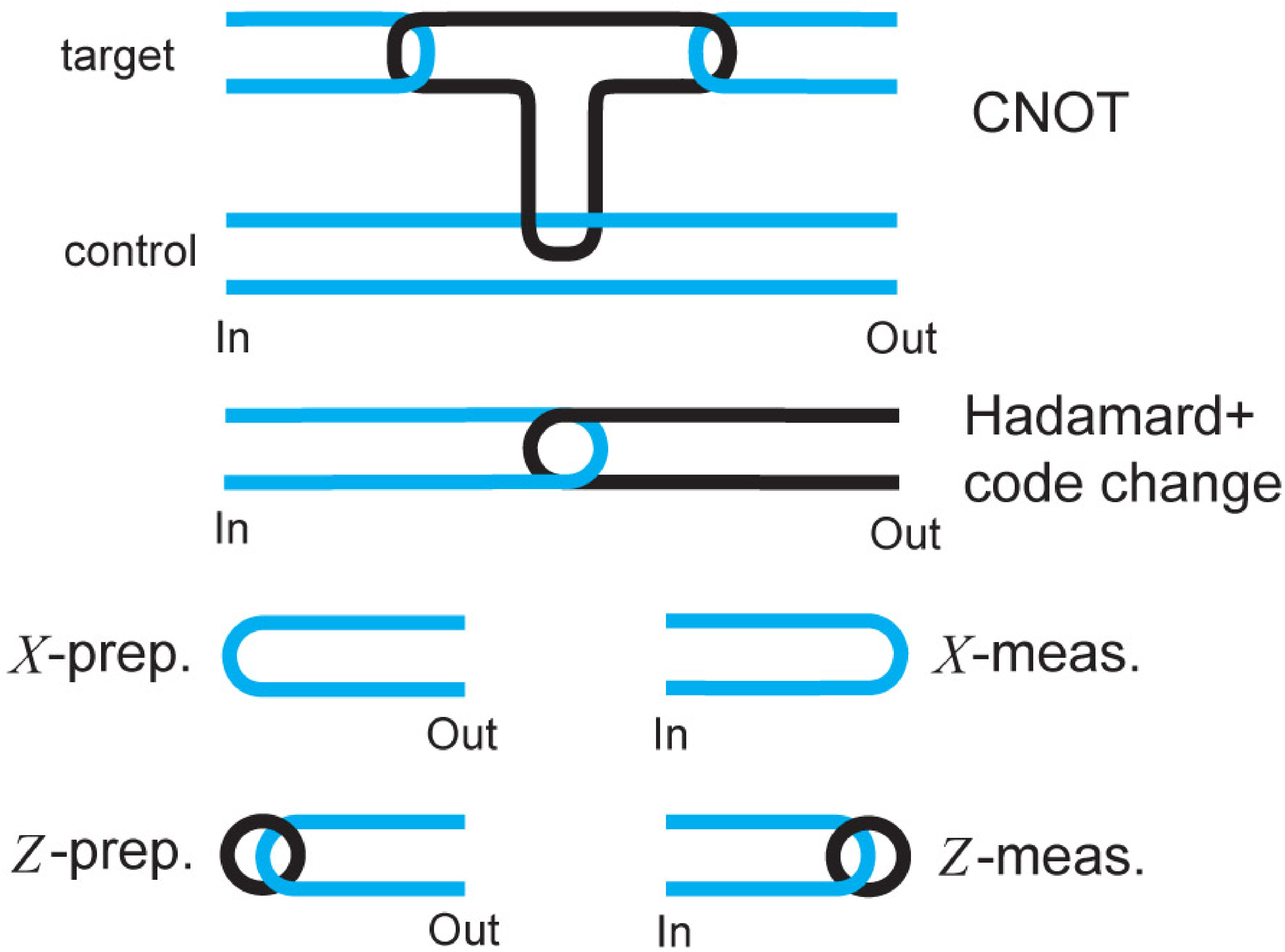
Defects are the extension of holes in the code plane to the third dimension.

Defects for quantum logic



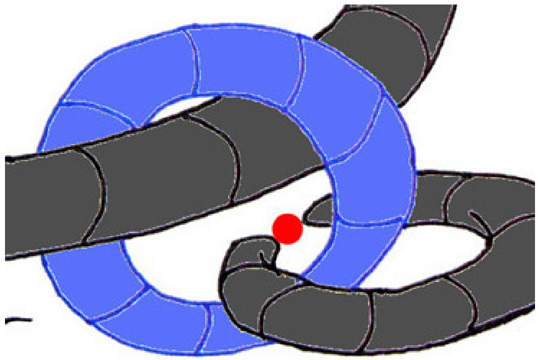
A quantum circuit is encoded in the way primal and dual defects are wound around another.

Quantum gates, Part I



- Displayed fault-tolerant gates are not universal.
- Need one non-Clifford element:

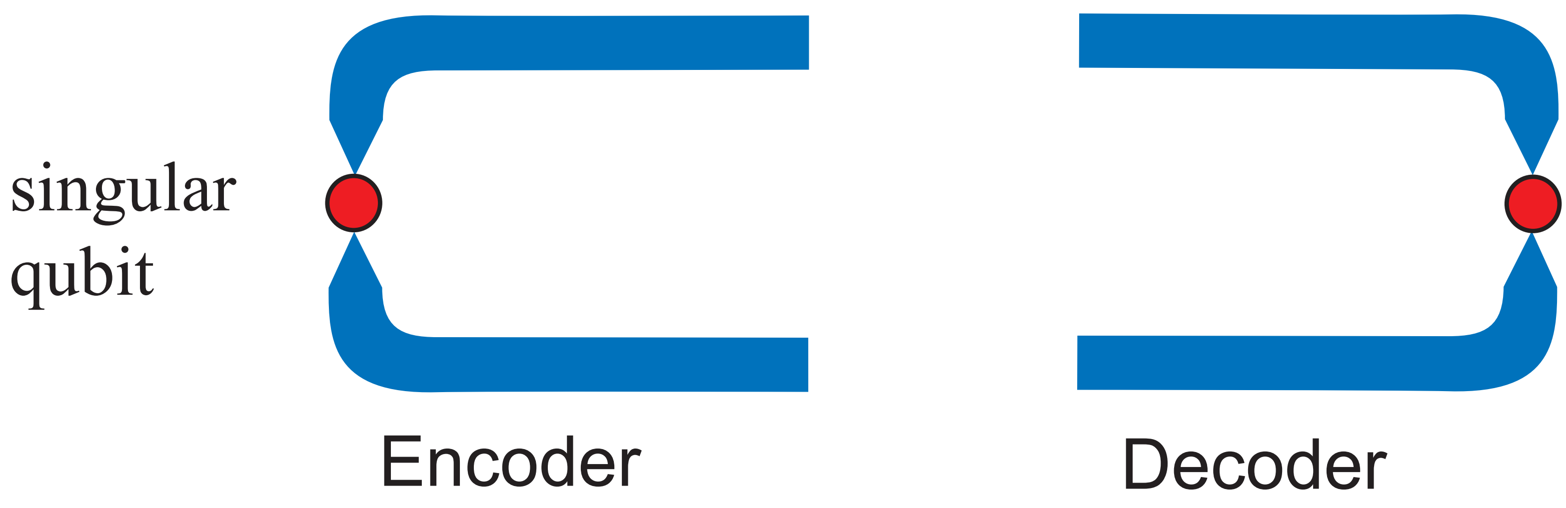
  fault-tolerant measurement of $\frac{X \pm Y}{\sqrt{2}}$.



Singular Qubits

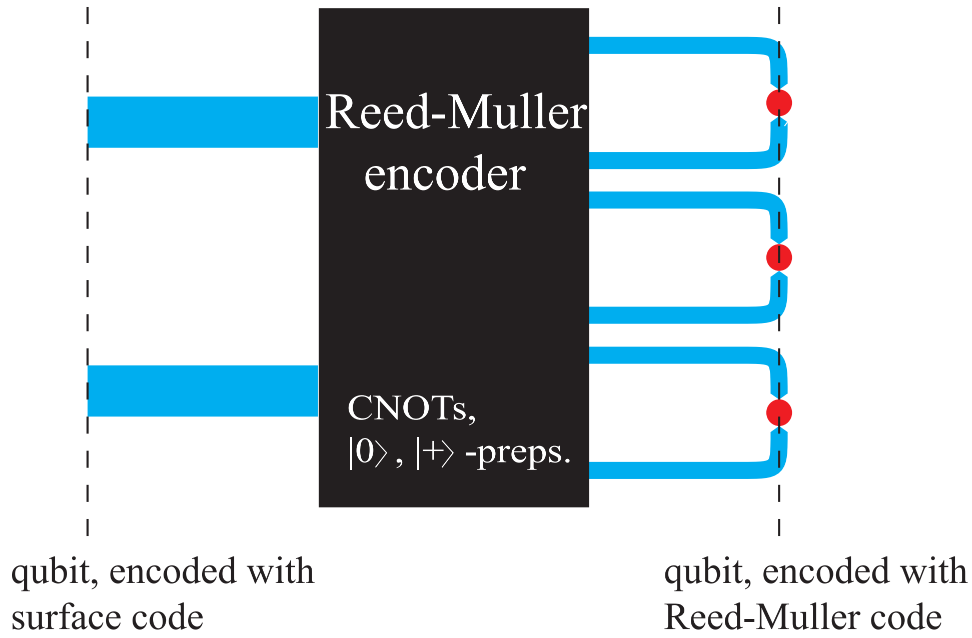
Quantum gates, Part II

Encoder and decoder for surface code:



Quantum gates, Part II

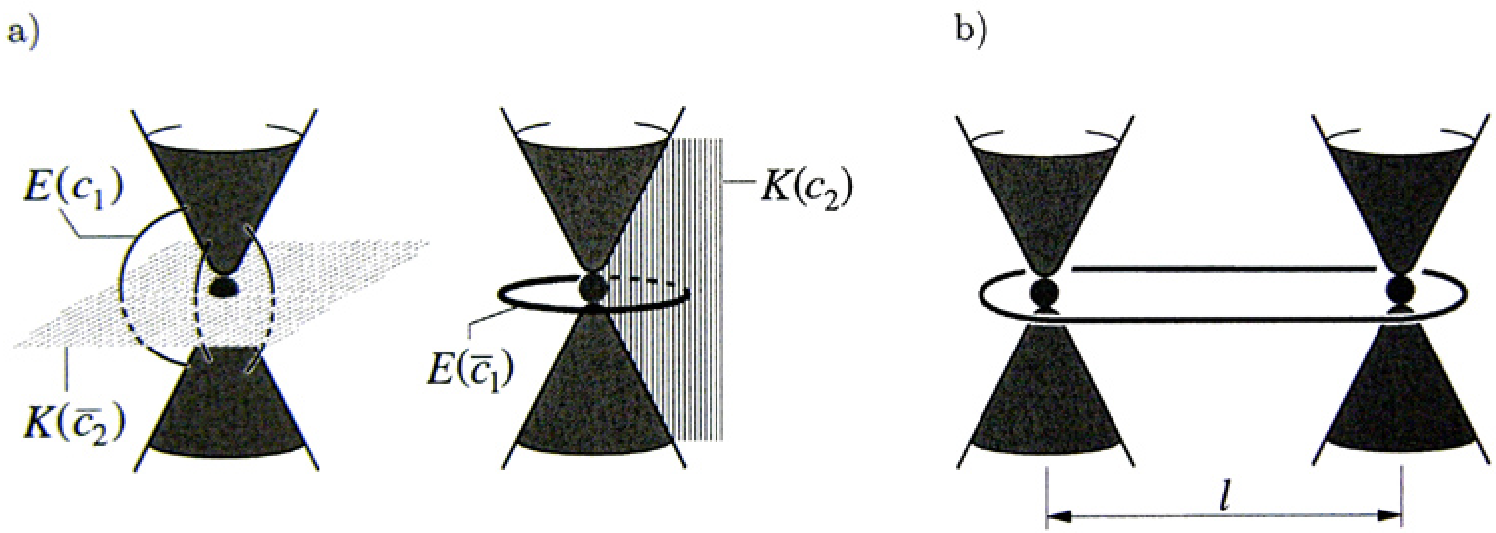
A circuit for code-conversion:



- Reed-Muller code: Fault-tolerant measurement of $\frac{\overline{X}\pm\overline{Y}}{\sqrt{2}}$ via *local* measurements of $\frac{X_a\pm Y_a}{\sqrt{2}}$ and classical post-processing.

-> Fault-tolerant universal gate set complete.

Fault-tolerance threshold in S



- Topological error correction breaks down near the S-qubits.
- Leads to an effective error on S-qubits.
- This effective error is *local*.

Fault-tolerance threshold in S

Error budget from Reed-Muller concatenation threshold:

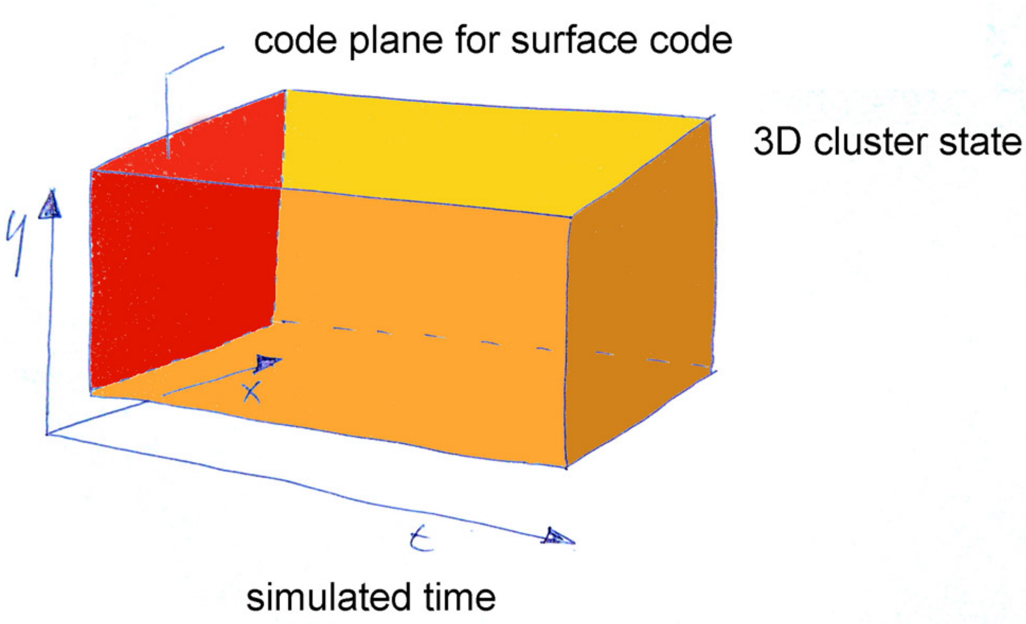
$$\frac{76}{15}p_2 + \frac{2}{3}p_P + \frac{4}{3}p_M + \frac{4}{3}p_S < \frac{1}{105}. \tag{4}$$

Specific parameter choices:

$$\begin{aligned} p_{2,c} &= 2.9 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = 0, \\ p_c &= 1.1 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = p_2 =: p. \end{aligned} \tag{5}$$

The Reed-Muller code sets the overall threshold.

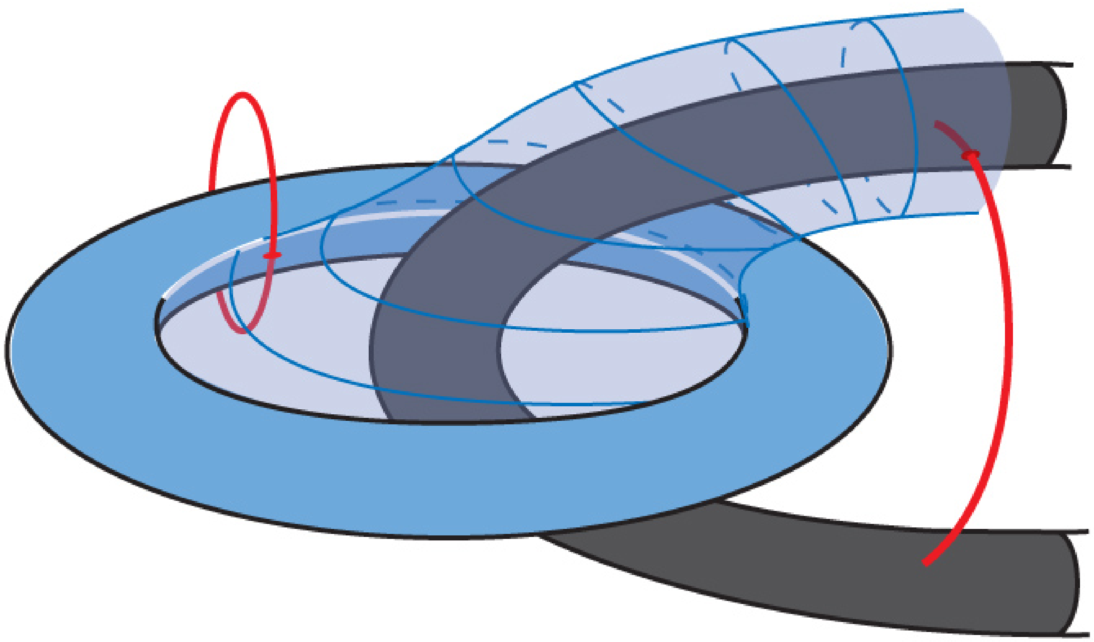
Remark 1 - mapping to 2D



- Make "simulated time" real time. Entangle slice-wise.

$\longrightarrow$ 2D qubit lattice suffices.

Remark 2 - Homology



Undetectable errors $\cong$ *1-cycles,*
measured correlations $\cong$ *2-cycles.*

Summary

[quant-ph/0510135]

Numbers:

- Fault-tolerance threshold of 1.1×10^{-3} in 3D local architecture.

Methods:

- Cluster states in three spatial dimensions provide intrinsic topological error correction.