A fault-tolerant one-way quantum computer

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Talk outline

Part I: One-way quantum computer ($QC_{C}$) and cluster states

What is the one-way quantum computer?

Part II: Fault-tolerance

Which methods are used? What is the threshold?

3D cluster state = fault-tolerant substrate

Logic from non-trivial boundary conditions
Part I:
The one-way quantum computer and cluster states
The one-way quantum computer

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

measurement of $Z (\bigcirc)$, $X (\uparrow)$, $\cos \alpha X + \sin \alpha Y (\nearrow)$
Cluster states - creation

1. Prepare product state \( \bigotimes_{a \in \mathcal{C}} \frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}} \) on \( d \)-dimensional qubit lattice \( \mathcal{C} \).

2. Apply the Ising interaction for a fixed time \( T \) (conditional phase of \( \pi \) accumulated).
Cluster states - simple examples

\[ |\psi \rangle_2 = |0 \rangle_1 |+ \rangle_2 + |1 \rangle_1 |- \rangle_2 \]
Bell state

\[ |\psi \rangle_3 = |+ \rangle_1 |0 \rangle_2 |+ \rangle_3 + |- \rangle_1 |1 \rangle_2 |\rangle_3 \]
GHZ-state

\[ |\psi \rangle_4 = |0 \rangle_1 |+ \rangle_2 |0 \rangle_3 |+ \rangle_4 + |0 \rangle_1 |- \rangle_2 |1 \rangle_3 |- \rangle_4 + |+ \rangle_1 |- \rangle_2 |0 \rangle_3 |+ \rangle_4 + |1 \rangle_1 |+ \rangle_2 |1 \rangle_3 |- \rangle_4 \]

Number of terms exponential in number of qubits!
Cluster states - definition

A cluster state $|\phi\rangle_C$ on a cluster $C$ is the single common eigenstate of the stabilizer operators

$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in C,$$  \hspace{1cm} (1)

where $b \in N(a)$ if $a, b$ are spatial next neighbors in $C$.

**Z-Rule:**

A $Z$-measurement removes qubit from the cluster
Cluster states - experiment

Part II:
Fault-tolerance
The threshold theorem

Theorem*: Assume a suitable noise model for a universal quantum computer. If the noise per elementary operation is below a constant non-zero threshold $\epsilon$ then arbitrarily long quantum computations can be performed with arbitrary accuracy at small operational overhead.

What is a suitable noise model? What is the value of $\epsilon$? What is a small overhead?

Standard: Independent probabilistic errors

$\epsilon = 10^{-10}..10^{-2}$ Polylogarithmic. $S' \rightarrow S' \log \gamma S$

Generalized Improve threshold! Reduce overhead!

# Known threshold values

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<td>0.03, est.</td>
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*: 10^{14} bare gates for 1000 encoded gates [Knill, (2004)]

Fault-tolerant one-way QC

Main idea: *Replace 2D cluster state by 3D cluster state!*

3D cluster state = fault-tolerant substrate  
Logic from non-trivial boundary conditions
Three cluster regions:
- **V** (Vacuum), **D** (Defect) and **S** (Singular qubits).

- **Qubits** $q \in V$: local $X$-measurements,
- **Qubits** $q \in D$: local $Z$-measurements,
- **Qubits** $q \in S$: local measurements of $\frac{X \pm Y}{\sqrt{2}}$. 
Microscopic view

- **qubit** location: \((\text{even, odd, odd})\) - face of \(\mathcal{L}\),
- **qubit** location: \((\text{odd, odd, even})\) - edge of \(\mathcal{L}\),
- **syndrome** location: \((\text{odd, odd, odd})\) - cube of \(\mathcal{L}\),
- **syndrome** location: \((\text{even, even, even})\) - site of \(\mathcal{L}\).
Lattice duality $\mathcal{L} \leftrightarrow \overline{\mathcal{L}}$

Translation by vector $(1,1,1)^T$:
- Cluster $\mathcal{C}$ invariant,
- $\mathcal{L}$ (primal) $\rightarrow \overline{\mathcal{L}}$ (dual).

\begin{align*}
\text{face of } \mathcal{L} & \quad \text{edge of } \overline{\mathcal{L}}, \\
\text{edge of } \mathcal{L} & \quad \text{face of } \overline{\mathcal{L}}, \\
\text{site of } \mathcal{L} & \quad \text{cube of } \overline{\mathcal{L}}, \\
\text{cube of } \mathcal{L} & \quad \text{site of } \overline{\mathcal{L}},
\end{align*}

(2)

- Many objects in this scheme exist as ‘primal’ and ‘dual’.
Key to scheme

- Code plane for surface code
- 3D cluster state
- Simulated time
Surface codes

- Errors are represented by chains.
- Homologically equivalent chains correspond to physically equivalent errors.
- Harmful errors stretch across the entire lattice (rare events).

$QCC$: topological error correction in $V$

- Errors are represented by chains.
- Homologically equivalent chains correspond to physically equivalent errors.
- Harmful errors stretch across the entire lattice.

$\Rightarrow$ Leads to *Random plaquette $Z_2$-gauge model* (RPGM) [1].

RPGM: schematic phase diagram

Map error correction to statistical mechanics:

Error model:

- Cluster state created in a 4-step sequence of $\Lambda(Z)$-gates from product state $\otimes_{a \in C} |+\rangle_a$.

- Error sources:
  - $|+\rangle$-preparation: Perfect preparation followed by 1-qubit partially depolarizing noise with probability $p_P$.
  - $\Lambda(Z)$-gates: Perfect gates followed by 2-qubit partially depolarizing noise with probability $p_2$.
  - Memory: 1-qubit partially depolarizing noise with probability $p_S$ per time step.
  - Measurement: Perfect measurement preceeded by 1-qubit partially depolarizing noise with probability $p_M$.

- 3D cluster state created in slices of fixed thickness.

- Instant classical processing.
Fault-tolerance threshold in $V$

\[ p_{2,c} = 9.6 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = 0, \]
\[ p_c = 5.8 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = p_2 =: p. \]
Fault-tolerant quantum logic
Surface codes

- Storage capacity of the code depends upon the topology of the code surface.
Surface codes

- There are two types of holes: primal and dual.
- A pair of same-type holes constitutes a qubit.
Defects for quantum logic

Defects are the extension of holes in the code plane to the third dimension.
Defects for quantum logic

A quantum circuit is encoded in the way primal and dual defects are wound around another.
Quantum gates, Part I

- **CNOT**
  - Control input (In) and target input (In)
  - Control output (Out) and target output (Out)

- **Hadamard+ code change**
  - Input (In) and output (Out)

- **X-prep.**
  - Input (In) and output (Out)

- **X-meas.**
  - Input (In) and output (Out)

- **Z-prep.**
  - Input (In) and output (Out)

- **Z-meas.**
  - Input (In) and output (Out)
• Displayed fault-tolerant gates are not universal.

• Need one non-Clifford element: 
  fault-tolerant measurement of $\frac{X \pm Y}{\sqrt{2}}$. 

Singular Qubits
Quantum gates, Part II

Encoder and decoder for surface code:

Encoder

Decoder
Quantum gates, Part II

A circuit for code-conversion:

- Reed-Muller code: Fault-tolerant measurement of $\frac{X\pm Y}{\sqrt{2}}$ via local measurements of $\frac{X_a\pm Y_a}{\sqrt{2}}$ and classical post-processing.

$\Rightarrow$ Fault-tolerant universal gate set complete.
Fault-tolerance threshold in $S$

- Topological error correction breaks down near the $S$-qubits.
- Leads to an effective error on $S$-qubits.
- This effective error is local.
Fault-tolerance threshold in $S$

Error budget from Reed-Muller concatenation threshold:

$$\frac{76}{15}p_2 + \frac{2}{3}p_P + \frac{4}{3}p_M + \frac{4}{3}p_S < \frac{1}{105}.$$  \hfill (4)

Specific parameter choices:

\begin{align*}
  p_{2,c} &= 2.9 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = 0, \\
  p_c &= 1.1 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = p_2 =: p. \hfill (5)
\end{align*}

The Reed-Muller code sets the overall threshold.
Remark 1 - mapping to 2D

- Make “simulated time” real time. Entangle slice-wise.

→ 2D qubit lattice suffices.
Remark 2 - Homology

Undetectable errors $\simeq$ 1-cycles, measured correlations $\simeq$ 2-cycles.
Summary

Numbers:
- Fault-tolerance threshold of $1.1 \times 10^{-3}$ in 3D local architecture.

Methods:
- Cluster states in three spatial dimensions provide intrinsic topological error correction.