Exact Duality and Magic Circles of the Dissipative Hofstadter Model

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I. Introduction

- Classical dissipative system: EQ of motion

$$M\frac{d^2q}{dt^2} + \eta\frac{dq}{dt} + \frac{dV}{dq} = 0$$

η: friction coefficient

- Microscopic action by Calderia-Leggett

$$\begin{aligned} S &= \int dt \left[\frac{M}{2}\left(\frac{dq}{dt}\right)^2 - V(q) \right. \\ &\quad \left. + \frac{1}{2}\sum_\alpha m_\alpha \left(\left(\frac{dx_\alpha}{dt}\right)^2 - \omega_\alpha^2 x_\alpha^2 \right) - q\sum_\alpha C_\alpha x_\alpha \right] \end{aligned}$$

x_α: bath of an infinite number of degrees of freedom or an environment (Calderia-Leggett, 1981, 1983)

- Classical equation of motion

$$
\begin{aligned}
M\frac{d^2q}{dt^2} &= -\frac{dV}{dq} - \sum_\alpha C_\alpha x_\alpha, \\
m_\alpha \frac{d^2x_\alpha}{dt^2} &= -m_\alpha\omega_\alpha^2 x_\alpha - C_\alpha q
\end{aligned}
$$

- Solving for x_α, (Fourier transformed)

$$
\begin{aligned}
-M\omega^2\tilde{q}(\omega) &= -\frac{d\tilde{V}}{dq}(\omega) + \tilde{K}(\omega)\tilde{q}(\omega) \\
\tilde{K}(\omega) &= -\sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha^2}\frac{\omega^2}{\omega^2-\omega_\alpha^2}
\end{aligned}
$$

- Density of states

$$
\tilde{J}(\omega) = \frac{\pi}{2}\sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha), \quad \tilde{J}(\omega) = Im\tilde{K}(\omega).
$$

- Ohmic condition

$$\tilde{J}(\omega) = \eta\omega$$

- Frictional term

$$-M\omega^2\tilde{q}(\omega) = -\frac{d\tilde{V}}{dq}(\omega) + i\eta\omega\tilde{q}(\omega)$$

- Quantum Mechanics: Non-local interaction

$$S_{CL} = -\int dtdt' q(t)\left[\int \frac{d\omega}{2\pi}\tilde{J}(\omega)e^{-\omega|t|}\right] q(t')$$

Imposing the Ohmic condition

$$S_{CL} = \frac{\eta}{4\pi}\int dtdt' \frac{(q(t)-q(t'))^2}{(t-t')^2}$$

II. Dissipation and Boundary State

- String (boundary) theory: $t \to \sigma \in [-\pi, \pi]$

$$
\begin{aligned}
S_\eta[X] &= \int_{-\pi}^{\pi} d\sigma \int_{-\infty}^{\infty} d\sigma' \frac{(X(\sigma) - X(\sigma'))^2}{(\sigma - \sigma')^2} \\
&= \frac{1}{4} \int_{-\pi}^{\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \frac{(X(\sigma) - X(\sigma'))^2}{\sin^2 \frac{(\sigma - \sigma')}{2}} \\
&= 4\pi^2 \alpha' \sum_n x_n \bar{x}_n
\end{aligned}
$$

where

$$
X(\sigma) = x + \sqrt{\frac{\alpha'}{2}} \sum_{n=1} \frac{1}{\sqrt{n}} \left(x_n e^{in\sigma} + \bar{x}_n e^{-in\sigma} \right),
$$

- Boundary State

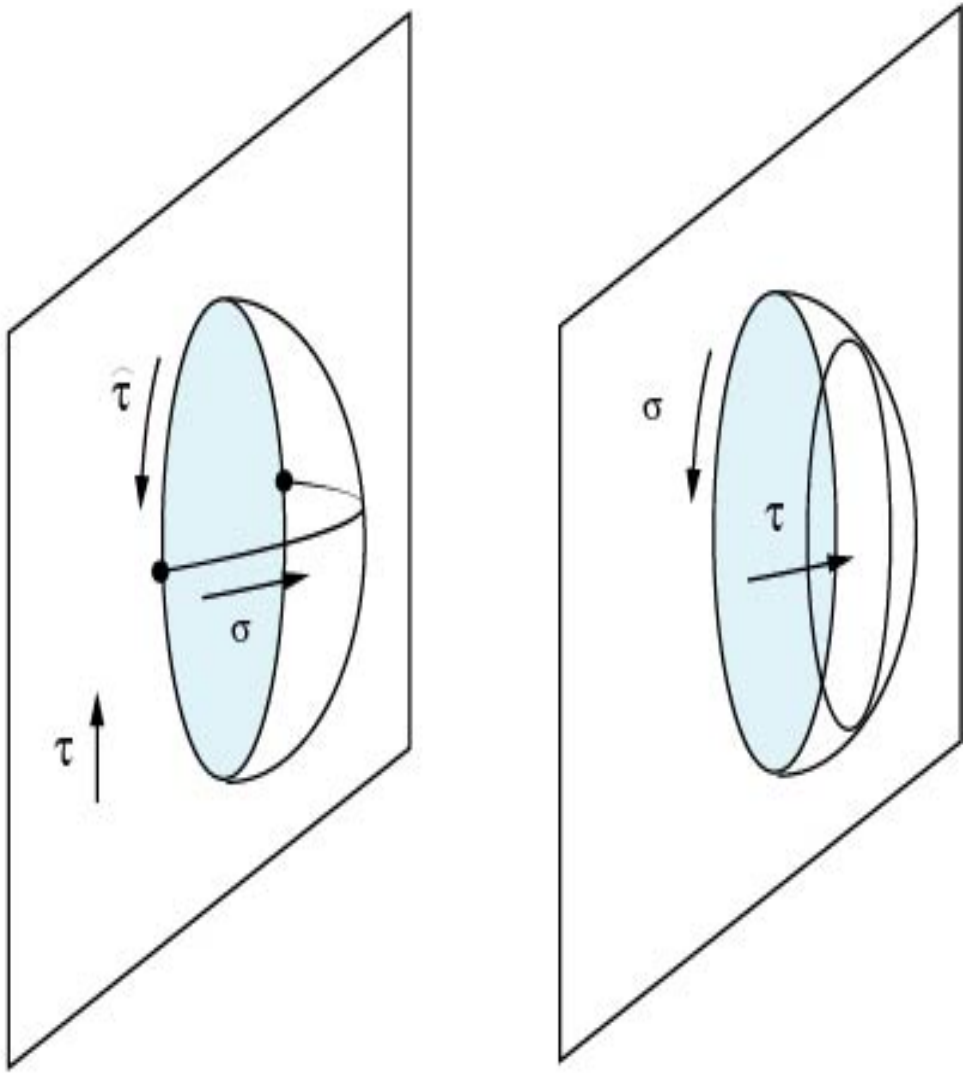
$$
\begin{aligned}
|X\rangle &= \prod_n \exp\left\{-\frac{1}{2}x_n\bar{x}_n - a_n^\dagger\tilde{a}_n^\dagger + a_n^\dagger x_n + \bar{x}_n\tilde{a}_n^\dagger\right\}|0\rangle \\
\langle 0|X\rangle &= e^{S_\eta[X]}
\end{aligned}
$$

- Partition Function

$$
\begin{aligned}
Z &= \int D[X]\exp\left(S_\eta + \int d\sigma V[X]\right) \\
&= Z_{Disk} = \langle 0|B\rangle, \\
|B\rangle &= \int D[X]\exp\left(\int d\sigma V[X]\right)|X\rangle
\end{aligned}
$$

(Callan and Thorlacius, 1990)

Boundary State



III. Schmid Model

- Caldeira-Leggett model with a periodic potential

$$S_{SM} = \frac{\eta}{4\pi\hbar}\int_{-T/2}^{T/2} dtdt' \frac{(X(t)-X(t'))^2}{(t-t')^2} - \frac{V_0}{\hbar}\int_{-T/2}^{T/2} dt \cos\frac{2\pi X}{a}.$$

- String theory where $t = \frac{T}{2\pi}\sigma, \quad X \to \frac{a}{2\pi}X$

$$S_{SM} = \frac{\eta}{4\pi\hbar}\left(\frac{a}{2\pi}\right)^2\int_{-\pi}^{\pi} d\sigma d\sigma' \frac{(X(\sigma)-X(\sigma\prime))^2}{(\sigma-\sigma')^2} - \frac{V_0}{\hbar}\frac{T}{2\pi}\int_{-\pi}^{\pi} d\sigma \frac{1}{2}\left(e^{iX}+e^{-iX}\right).$$

- Comparison:

$$\frac{\eta}{4\pi\hbar}\left(\frac{a}{2\pi}\right)^2 = \frac{1}{8\pi^2\alpha'}, \quad \frac{V_0}{\hbar}\frac{T}{2\pi} = \frac{g}{2}, \quad \frac{1}{\alpha'} = \alpha$$

- Renormalization

$$g_0(\mu) = g_0(\Lambda)\left(\frac{\mu}{\Lambda}\right)^{1/\alpha-1}$$

- Critical Point:

$$\alpha = \alpha' = 1$$

This is also the self-dual point of the duality (Schmid 1983)

$$\alpha \leftrightarrow 1/\alpha$$

- At the critical point S_{SM} coincides with the full brane action of the rolling tachyon

IV. Schmid Model at Critical Point: Fermionization

- Fermion in terms of boson

$$\begin{aligned}\psi_L(0,\sigma) &= :e^{i\sqrt{2}\phi_L}: \\ &= e^{\sum_{n=1}\frac{1}{n}e^{in\sigma}\alpha_{-n}}e^{ix_L}e^{i\sigma(p_L+\frac{1}{2})}e^{-\sum_{n=1}\frac{1}{n}e^{-in\sigma}\alpha_n}\end{aligned}$$

- In order to rewrite the potential using fermion fields, we need to introduce an auxiliary boson

$$\phi_1 = \frac{1}{\sqrt{2}}(X+Y) \quad , \quad \phi_2 = \frac{1}{\sqrt{2}}(X-Y)$$

(Polchinski and Thorlacius (1994) for open string)

- Fermions:

$$\begin{aligned}\psi_{1L}(z) &= \zeta_{1L} : e^{-\sqrt{2}i\phi_{1L}(z)} : \quad \psi_{2L}(z) = \zeta_{2L} : e^{\sqrt{2}i\phi_{2L}(z)} : \\ \psi_{1R}(\bar{z}) &= \zeta_{1R} : e^{\sqrt{2}i\phi_{1R}(\bar{z})} : \quad \psi_{2R}(\bar{z}) = \zeta_{2R} : e^{-\sqrt{2}i\phi_{2R}(\bar{z})} :\end{aligned}$$

where $\zeta_{L/R}$ are co-cycles (Lee and Semenoff (2005): boundary state formulation)

- Boundary condition for bosons:

$$\left[-\frac{1}{2\pi}\partial_\tau X(0,\sigma) + i\frac{g}{2}e^{iX(0,\sigma)} - i\frac{\bar{g}}{2}e^{-iX(0,\sigma)}\right]|B,D\rangle = 0$$
$$Y(0,\sigma)|B,D\rangle = 0$$

Note: $|B,D\rangle = |\text{Boundary state for } X\rangle \otimes |\text{Boundary state for } Y\rangle$.

- Boundary condition for fermions:

$$\Big[:\psi_L^\dagger\sigma^3\psi_L: - :\psi_R^\dagger\sigma^3\psi_R: + \pi g\psi_L^\dagger\left(1+\sigma^3\right)\psi_R$$
$$-\pi\bar{g}\psi_L^\dagger\left(1-\sigma^3\right)\psi_R\Big]|B,D\rangle = 0$$

(Hasselfield, Lee, Semenoff and Stamp, 2006)

V. Exact Boundary State for Schmid Model at Critical Point

- Gluing condition = Boundary condition

$$\begin{aligned}\left(\psi_R(0,\sigma) + i\sigma^1 U \psi_L(0,\sigma)\right)|BD> &= 0 \\ \left(\psi_R^\dagger(0,\sigma) + \psi_L^\dagger(0,\sigma) U^{-1} i\sigma^1\right)|BD> &= 0\end{aligned}$$

If we choose

$$U = \begin{pmatrix} e^{-2\pi i A}\sqrt{1-\pi^2 g\bar{g}} & -i\pi g \\ -i\pi\bar{g} & e^{2\pi i A}\sqrt{1-\pi^2 g\bar{g}} \end{pmatrix}$$

where A is topological parameter. U is unitary when A is real, $\bar{g} = g^*$, $|g| < 1/\pi$

- Exact Boundary State in the NS-sector

$$|BD>_{NS}=\prod_{r=\frac{1}{2}}^{\infty}\exp\left[\psi^{\dagger}_{-r}U^{-1}i\sigma^{1}\tilde{\psi}_{-r}-\tilde{\psi}^{\dagger}_{-r}i\sigma^{1}U\psi_{-r}\right]\ |0>$$

and in the R-sector

$$|BD>_{R}=\prod_{n=1}^{\infty}\exp\left[\psi^{\dagger}_{-n}U^{-1}i\sigma^{1}\tilde{\psi}_{-n}-\tilde{\psi}^{\dagger}_{-n}i\sigma^{1}U\psi_{-n}\right]$$
$$\exp\left[\psi^{\dagger}_{0}U^{-1}i\sigma^{1}\tilde{\psi}_{0}\right]\ |-+-+>$$

VI. Exact Calculation of Mobility

- Mobility: $\langle 0|X'(\sigma)X'(\sigma')|B,D\rangle$ where $X' = J_L^3 + J_R^3$
- We choose $|B,D\rangle = |B,D\rangle_{NS}$ since $|0\rangle$ belongs to the NS-sector
- Some exact calculations:

$$\langle 0|J_L^3(\sigma)J_R^3(\sigma')|B,D\rangle = -\frac{1}{16}\mathrm{tr}(\sigma^3 U^{-1}\sigma^1\sigma^3\sigma^1 U)\sin^{-2}\frac{(\sigma-\sigma')}{2}$$

$$\langle 0|\left[J_L^3(\sigma)J_L^3(\sigma') + J_R^3(\sigma)J_R^3(\sigma')\right]|B,D\rangle = -\frac{1}{4}\sin^{-2}\frac{(\sigma-\sigma')}{2}$$

- Exact Result

$$\langle 0|X'(\sigma)X'(\sigma')|B,D\rangle = -\frac{1}{2}(1-\pi^2 g\bar{g})\sin^{-2}\frac{(\sigma-\sigma')}{2}$$

Remarks

- The mobility does not depend on the topological parameter A
- The result does not agree with the previous works of Callan and Freed (1991) and Callan, Felce and Freed (1992) which employed the bosonic theory.
- The mobility has been also calculated by Callan, Klebanov, Maldacena and Yegulalp (1995). But the renomalization effect due to the ordering of operators has been ignored. If it is properly renormalized, their result reduces to the exact result obtained here

$$e^{-\int(\frac{g}{2}\psi_L^\dagger\frac{1}{2}(1+\sigma^3)\psi_R+\frac{\bar{g}}{2}\psi_L^\dagger\frac{1}{2}(1-\sigma^3)\psi_R)}|ND>$$
$$= e^{\chi(g',\bar{g}')}\ e^{i\int(\frac{g'}{2}\psi_L^\dagger\sigma^+\psi_L+\frac{\bar{g}'}{2}\psi_L^\dagger\sigma^-\psi_L)}|ND>$$
$$\sin^2\pi\sqrt{\bar{g}'g'} = \pi^2\bar{g}g\ ,\quad \frac{g'}{\bar{g}'} = \frac{g}{\bar{g}}$$

VII. Dissipative Hofstadter Model

- The Dissipative Hofstadter Model (Wannier-Azbel-Hofstadter (WAH) Model): Electron moving in two dimensions subjects to a magnetic field, a square lattice potential and dissipative force
- Phase transitions between localized and delocalized long-time behavior of the electron: Phase diagram is fractal
- Applications: Quantum dynamics of SQUID, Josephson junction, Polaronic motion through conductors, Tunneling in QM Hall System, New solutions of open string theory in non-trivial background of tachyons and gauge fields
- String theory (Rolling Tachyon): Decay of unstable D-brane in the presence of NS B-field.

Dissipative Hofstadter Model

- The Action

$$
\begin{aligned}
S_{DHM} &= \frac{\eta}{4\pi\hbar}\int_{-T/2}^{T/2} dtdt' \frac{\left(X^i(t)-X^i(t')\right)^2}{(t-t')^2} \\
&+\frac{ieB_H}{2\hbar c}\int_{-T/2}^{T/2} dt\left(\partial_t X^1 X^2 - \partial_t X^2 X^1\right) \\
&-\frac{V_0}{\hbar}\int_{-T/2}^{T/2} dt\left(\cos\frac{2\pi X^1}{a}+\cos\frac{2\pi X^2}{a}\right).
\end{aligned}
$$

- Choosing

$$
t=\frac{T}{2\pi}\sigma, \quad X^i \to \frac{a}{2\pi}X^i,
$$

and introducing dimensionless parameters

$$
2\pi\alpha=\frac{\eta a^2}{\hbar}, \quad 2\pi\beta=\frac{eB_H}{\hbar c}a^2, \quad g=\frac{V_0 T}{\pi\hbar}
$$

VIII. Boundary State for DHM

- we get

$$
\begin{aligned}
S_{DHM} &= \frac{\alpha}{8\pi^2}\int_{-\pi}^{\pi} d\sigma d\sigma' \frac{\left(X^i(\sigma)-X^i(\sigma')\right)^2}{(\sigma-\sigma')^2} \\
&\quad + i\frac{\beta}{4\pi}\int_{-\pi}^{\pi} d\sigma \left(\partial_\sigma X^1 X^2 - \partial_\sigma X^2 X^1\right) \\
&\quad - \frac{g}{2}\int_{-\pi}^{\pi} d\sigma \left(\cos X^1 + \cos X^2\right).
\end{aligned}
$$

where $\alpha = 1/\alpha', \quad \beta = 2\pi B$

- It appears in string theory as an action for the open string in the background of the magnetic field and the tachyon potential

- String Theory

$$
\begin{aligned}
S &= \frac{1}{4\pi\alpha'}\int d^2\xi E_{ij}(\partial_\tau+\partial_\sigma)X^i(\partial_\tau-\partial_\sigma)X^j \\
&\quad -i\frac{g\pi}{2}\int_{\partial M}\frac{d\sigma}{2\pi}\sum_i\left(e^{iX^i}+e^{-iX^i}\right)
\end{aligned}
$$

where $E_{ij}=(g+2\pi\alpha' B)_{ij}$.

- Boundary State

$$
|B\rangle = \int D[x,\bar{x}]\exp\left\{-S_{DHM}[x,\bar{x}]-a_n^\dagger\tilde{a}_n^\dagger+a_n^\dagger x_n+\bar{x}_n\tilde{a}_n^\dagger\right\}|0\rangle
$$

where $X(\sigma)=x+\sqrt{\frac{\alpha'}{2}}\sum_{n=1}\frac{1}{\sqrt{n}}\left(x_n e^{in\sigma}+\bar{x}_n e^{-in\sigma}\right)$.

- In the absence of the periodic potentials $|B\rangle$ reduces to $|B_E\rangle$ which satisfies

$$\left(\delta_{ij}\partial_\tau X^j - \frac{\beta}{\alpha}\epsilon_{ij}\partial_\sigma X^j\right)|B_E\rangle = 0.$$

$$|B_E\rangle = \sqrt{\det E}\exp\left(-\sum_{n=1}\frac{1}{n}\alpha^i_{-n}\left(g(E)^{-1}E^T\right)_{ij}\tilde{\alpha}^j_{-n}\right)|0\rangle$$

- Boundary state in the presence of the tachyon potential

$$\begin{aligned}|B\rangle &= \exp\left[g\pi\int_{\partial M}\frac{d\sigma}{2\pi}\left(e^{iX^1}+e^{-iX^1}+e^{iX^2}+e^{-iX^2}\right)\right]|B_E\rangle \\ &= \sum_{n=0}^{\infty}\int d\sigma_1\ldots d\sigma_n\left(\frac{g}{2}\right)^n\frac{1}{n!}\sum_{\mathbf{q}_j=\pm\mathbf{i},\pm\mathbf{j}}\prod_{j=1}^{n}\exp\left[i\mathbf{q}_j\cdot\mathbf{X}(\sigma_j)\right]|B_E\rangle\end{aligned}$$

IX. Magic Circles

- $O(2,2,R)$ Transformation: T-dual Transformation

$$\alpha^i_n = \left(G(E)^{-1}\right)^i{}_j \beta^j_n, \quad \tilde{\alpha}^i_n = \left(G(E^T)^{-1}\right)^i{}_j \tilde{\beta}^j_n$$

In terms of the new oscillator basis the boundary condition for $|B_E\rangle$ is transcribed into the Neumann condition

$$\left(\beta^i_{-n} + \tilde{\beta}^i_n\right)|B_E\rangle = 0.$$

The oscillators $\{\beta, \bar{\beta}\}$ respect the world-sheet metric G

$$G = E^T E = \begin{pmatrix} 1 + \left(\frac{\beta}{\alpha}\right)^2 & 0 \\ 0 & 1 + \left(\frac{\beta}{\alpha}\right)^2 \end{pmatrix}$$

$$\left[\beta^i_n, \beta^j_m\right] = (G^{-1})^{ij} n\delta(n+m), \ \left[\tilde{\beta}^i_n, \tilde{\beta}^j_m\right] = (G^{-1})^{ij} n\delta(n+m)$$

- $$\begin{aligned} L &= \frac{1}{4\pi\alpha'} E_{ij} \left(\partial_\tau + \partial_\sigma\right) X^i \left(\partial_\tau - \partial_\sigma\right) X^j \\ &= \frac{1}{4\pi\alpha'} G_{ij} \left(\partial_\tau + \partial_\sigma\right) Z^i \left(\partial_\tau - \partial_\sigma\right) Z^j \end{aligned}$$

  where

$$Z^i(0,\sigma) = x^i + \omega^i \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\beta^i_n - \tilde{\beta}^i_{-n}\right) e^{-in\sigma}$$

- Relation between two oscillator bases is summarized as

$$X^i(\sigma) = \left(\delta^{ij} - \frac{\beta}{\alpha}\epsilon^{ij}\right) Z^j_L(\sigma) + \left(\delta^{ij} + \frac{\beta}{\alpha}\epsilon^{ij}\right) Z^j_R(\sigma)$$

- Boundary state $|B_E\rangle$

$$|B_E\rangle = \sqrt{\det E} \prod_{n=1} \exp\left(-\frac{1}{n}\beta^i_{-n} G_{ij} \tilde{\beta}^j_{-n}\right) |0\rangle.$$

- Note

$$|B\rangle = \sum_n \frac{1}{n!}\left(\frac{g}{2}\right)^n \prod_{j=1}^{n} \int d\sigma_j \exp\left[i\mathbf{q}_j \cdot \mathbf{X}(\sigma_j)\right] |B_E\rangle$$

- Using the Baker-Hausdorff Lemma, $e^A e^B = e^B e^A e^{[A,B]}$,

$$\begin{aligned}
&\prod_{j=1}^{n} \exp\left[i\mathbf{q}_j \cdot \mathbf{X}(\sigma_j)\right] |B_E\rangle \\
&\quad = \exp\left[i\mathbf{q}_n \cdot \mathbf{X}(\sigma_n)\right] \ldots \exp\left[i\mathbf{q}_1 \cdot \mathbf{X}(\sigma_1)\right] |B_E\rangle \\
&\quad = \exp\left[\sum_{i>j} i\pi\mathbf{q}_i \cdot (2\pi\alpha'^2 \mathbf{B} G^{-1}) \cdot \mathbf{q}_j \,\mathrm{sign}(\sigma_i - \sigma_j)\right] \\
&\qquad \prod_{j=1}^{n} \exp\left[i\mathbf{q}_j \cdot \mathbf{Z}(\sigma_j)\right] |B_E\rangle
\end{aligned}$$

- By some algebra

$$\exp\left[\sum_{i>j} i\pi \mathbf{q}_i \cdot (2\pi\alpha'^2 \mathbf{B} G^{-1}) \cdot \mathbf{q}_j \,\mathrm{sign}(\sigma_i - \sigma_j)\right]$$
$$= \exp\left[\sum_{i>j} i\pi \frac{\beta}{\alpha^2+\beta^2}\left(q_i^1 q_j^2 - q_i^2 q_j^1\right)\,\mathrm{sign}(\sigma_i - \sigma_j)\right]$$
$$= \exp\left[\sum_{i} 2\pi i \frac{\beta}{\alpha^2+\beta^2} q_i^1 \left(\sum_{\sigma_i>\sigma_j} q_j^2\right)\right].$$

- Magic circles: Since $q_i^1, q_i^2 = 0, \pm 1$ for $i = 1, 2$, if $\frac{\beta}{\alpha^2+\beta^2}$ is an integer, this phase due to the magnetic field reduces to 1.

$$\alpha^2 + \left(\beta - \frac{1}{2n}\right)^2 = \left(\frac{1}{2n}\right)^2, \quad n \in Z$$

- On the magic circles, the dissipative Hofstadter model can be mapped into the boundary sine-Gordon model.

$$\exp\left(-S_{DHM}\right) = \int D[Z]$$
$$\exp\left[-\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma G_{ij}\left(\partial_\tau + \partial_\sigma\right) Z^i \left(\partial_\tau - \partial_\sigma\right) Z^j\right.$$
$$\left.+\frac{g}{2}\int_{\partial M} d\sigma \left(e^{iZ^1} + e^{-iZ^1} + e^{iZ^2} + e^{-iZ^2}\right)\right].$$

X. Critical Circle

- It may be convenient to scale $\beta^i, \tilde{\beta}^i$ and G such that

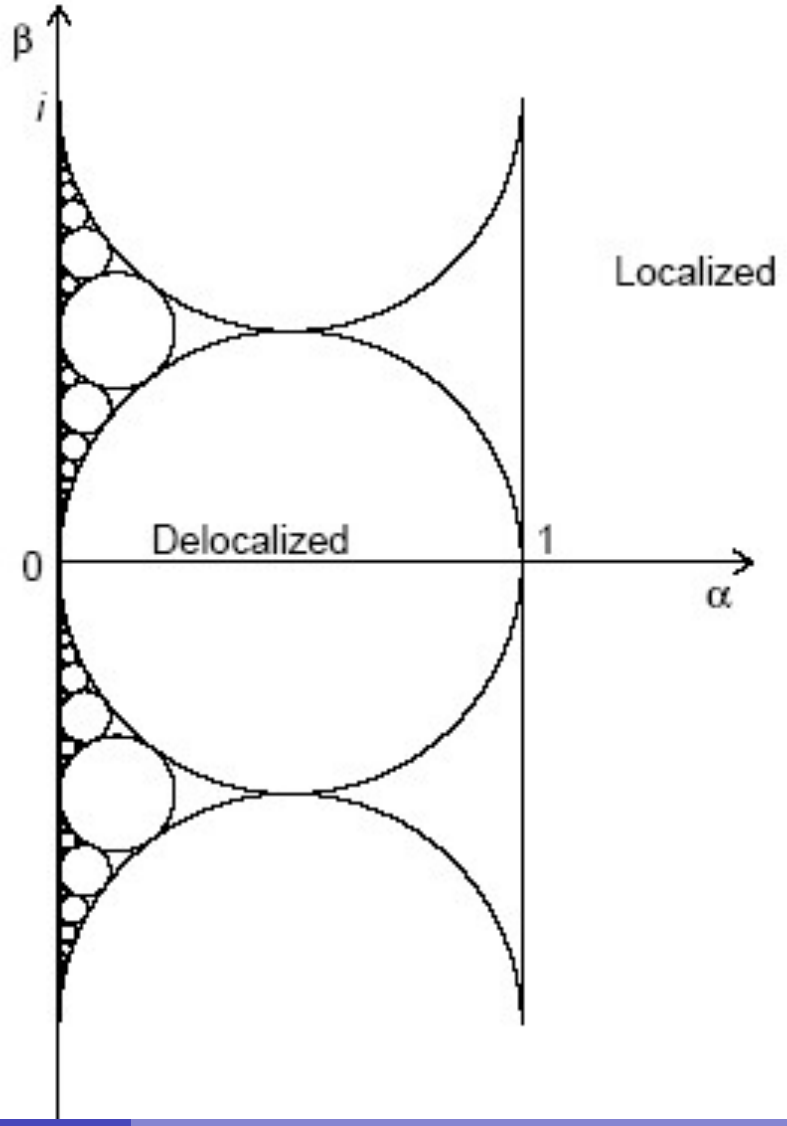
$$\begin{aligned} \beta^i_n &\to \sqrt{\alpha'}\beta^i_n, \quad \tilde{\beta}^i \to \sqrt{\alpha'}\tilde{\beta}^i, \\ G &\to \frac{1}{\alpha'}G = \begin{pmatrix} \frac{\alpha^2+\beta^2}{\alpha} & 0 \\ 0 & \frac{\alpha^2+\beta^2}{\alpha} \end{pmatrix}. \end{aligned}$$

- The points where the effective world-sheet metric becomes a unit metric forms a circle called the “critical circle";

$$\frac{\alpha^2+\beta^2}{\alpha} = 1.$$

- Magic Points: The points where the magic circles meet the critical circle, are magic points. At the magic points the DHM is equivalent to a set of two independent critical boundary sine-Gordon models.

Figure: Critical Circles



XI. Exact Duality

- Two DH models are equivalent if the following conditions are satisfied:
  - $$\exp\left[\sum_i 2\pi i \frac{\beta}{\alpha^2+\beta^2} q_i^1 \left(\sum_{\sigma_i>\sigma_j} q_j^2\right)\right]$$
    $$= \exp\left[\sum_i 2\pi i \frac{\bar{\beta}}{\bar{\alpha}^2+\bar{\beta}^2} q_i^1 \left(\sum_{\sigma_i>\sigma_j} q_j^2\right)\right]$$

    or

    $$\frac{\beta}{\alpha^2+\beta^2} - \frac{\bar{\beta}}{\bar{\alpha}^2+\bar{\beta}^2} = n, \quad n \in Z$$
  - Same metric

    $$\alpha G(\alpha,\beta)_{ij} = \bar{\alpha} G(\bar{\alpha},\bar{\beta})_{ij} \quad \text{or} \quad \frac{\alpha^2+\beta^2}{\alpha} = \frac{\bar{\alpha}^2+\bar{\beta}^2}{\bar{\alpha}}$$

T-Dual Transformation

- Both conditions can be summarized as

$$\frac{1}{\bar{z}} = \frac{1}{z} + ni, \quad \text{where} \quad z = \alpha + \beta i, \quad \bar{z} = \bar{\alpha} + \bar{\beta} i$$

- $O(2,2,R)$ Transformation

$$E' = (aE + b)(cE + d)^{-1}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The left and right movers transform as

$$\begin{aligned} \alpha_n(E) &\rightarrow (d - cE^T)^{-1}\alpha_n(E'), \quad \alpha_n^\dagger(E) \rightarrow \alpha_n^\dagger(E')(d^T - Ec^T)^{-1}, \\ \tilde{\alpha}_n(E) &\rightarrow (d + cE)^{-1}\tilde{\alpha}_n(E'), \quad \tilde{\alpha}_n^\dagger(E) \rightarrow \tilde{\alpha}_n^\dagger(E')(d^T + E^T c^T)^{-1} \end{aligned}$$

- T-dual transformation from the DH model with (α, β) and that with $(\bar{\alpha}, \bar{\beta})$:

$$
\begin{aligned}
T^{-1}(\bar{\alpha}, \bar{\beta}) T(\alpha, \beta) &= \begin{pmatrix} I & 0 \\ -\frac{\bar{\alpha}\bar{\beta}}{\bar{\alpha}^2+\bar{\beta}^2}\epsilon & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \frac{\alpha\beta}{\alpha^2+\beta^2}\epsilon & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ \left(\frac{\alpha\beta}{\alpha^2+\beta^2} - \frac{\bar{\alpha}\bar{\beta}}{\bar{\alpha}^2+\bar{\beta}^2}\right)\epsilon & I \end{pmatrix}
\end{aligned}
$$

- The periodic boundary potential is invariant under this T-dual transformation.

XII. Summary

- The dissipative Hofstadter model describes quantum particles moving in two dimensions subject to a uniform magnetic field, a periodic potential and a dissipative force.
- We discuss the dissipative Hofstadter model in the framework of the boundary state formulation in string theory and construct exact boundary states for the model at the magic points using the fermion representation.
- The exact duality of the dissipative Hofstadter model is shown to be equivalent to the subgroup of T-duality symmetry group in string theory unbroken by the boundary periodic potential.