PCE STAMP

DYNAMICS of a QUANTUM VORTEX

(ORLANDO, Dec 21st, 2010)

Physics & Astronomy
UBC
Vancouver

Pacific Institute
for
Theoretical Physics

DYNAMICS of a QUANTUM VORTEX

L THOMPSON & PCE STAMP

I WILL TALK ABOUT:

(i) What is the correct equation of motion for a quantum vortex?
(ii) How to understand the vortex-quasiparticle interaction physically
(iii) How to compare theory with experiment

FURTHER INFORMATION:

Email: stamp@physics.ubc.ca
Web: http://www.physics.ubc.ca/~berciu/PHILIP/index.html

SUPPORT FROM:

Fetzer Institute

PiTP

NSERC CRSNG

UBC

25 years cifar

Canadian Institute for Advanced Research

HISTORY of the PROBLEM: FORCES on a QUANTUM VORTEX

The fundamental question of quantum vortex dynamics has been highly controversial. Typically discussed in terms of FORCES:

"Berry" force

Superfluid Magnus force $\mathbf{F}_M = \rho_s \boldsymbol{\kappa}_V \times (\dot{\mathbf{r}}_V(t) - \mathbf{v}_s)$

Magnetic gyrotropic force $\mathbf{F}_g = G M_s q_V p_V \hat{\mathbf{z}} \times \dot{\mathbf{r}}_V$

"Iordanski" force

Superfluid Iordanski force $\mathbf{F}_I = \rho_n \boldsymbol{\kappa}_V \times (\dot{\mathbf{r}}_V(t) - \mathbf{v}_n)$

Drag force

Vortex-phonon drag - superfluid $\mathbf{F}_\parallel = -\gamma(\dot{\mathbf{r}}_V(t) - \mathbf{v}_n)$

Vortex-magnon drag - ferromagnet $\mathbf{F}_\parallel = \eta_m \dot{\mathbf{r}}_V(t)$

Inertial term

Inertial mass term: $M_V \ddot{\mathbf{r}}_V(t)$

HOWEVER: Nobody agrees on Iordanskii force, and estimates of the mass range from zero to infinity (for a macroscopic system)!

S.V. Iordanskii, Ann. Phys (NY) **29**, 335-49 (1964); and JETP **22**, 160-167 (1965)

E.B. Sonin, Phys. Rev. **B55**, 485 (1997)

D.J. Thouless, P. Ao, Q. Niu, Phys. Rev. Lett. **76**, 3758 (1996)

G.E. Volovik, Phys. Rev. Lett. **77**, 4687 (1997)

D. J. Thouless and J. R. Anglin, PRL **99**, 105301 (2007)

PRELIMINARY: analogy with GRAVITATION 1

The simplest approach to superfluids treats them as ideal fluids, with eqns of motion:

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0,$$ (Mass conservation)

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} + (1/\rho)\nabla p = 0$$ (Bernoulli)

where the superfluid velocity is derived from a phase: $\nabla\phi = \mathbf{v}$.

However in reality the fluid is compressible (and so vortices interact with phonons). One can then expand about the incompressible fluid:

$$\rho = \rho_{(0)} + \rho_{(1)} + \cdots \qquad \phi = \phi_{(0)} + \phi_{(1)},$$

The resulting eqtn of motion is now often called the Unruh eqtn, because of the following very nice identity; write an action

$$S = \int d^4x \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi_{(1)}\partial_\nu\phi_{(1)}$$

where

$$\sqrt{-g}g^{\mu\nu} = \frac{\rho_{(0)}}{c^2}\begin{pmatrix} 1, & \mathbf{v}^T \\ \mathbf{v}, & \mathbf{v}\mathbf{v}^T - c^2\mathbf{1} \end{pmatrix}$$

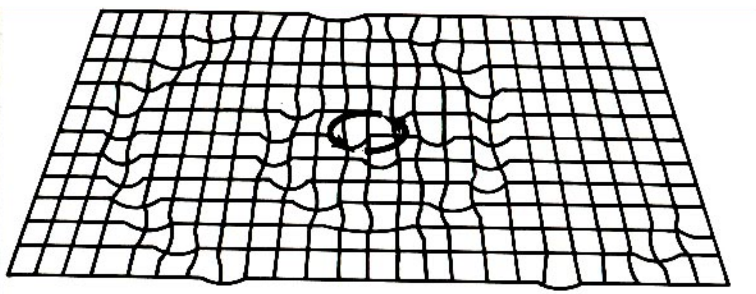
Then: $$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_{(1)} = 0$$

ie.: $$\left(\frac{\partial}{\partial t} + \nabla\cdot\mathbf{v}\right)\frac{\rho_{(0)}}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\phi_{(1)} = \nabla(\rho_{(0)}\nabla\phi_{(1)})$$

This corresponds to an interval: $$ds^2 = \rho(c^2 - v^2)d\tau^2 - \left(\delta_{ij} + \frac{v^i v^j}{c^2 - v^2}\right)dx^i dx^j$$

with 'lapse time': $$d\tau = dt + \frac{v^i dx^i}{c^2 - v^2}$$

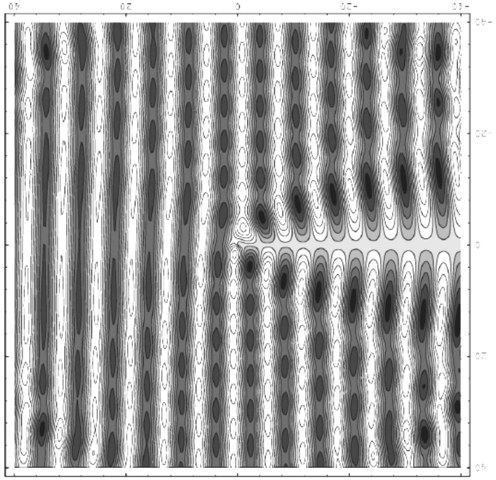
The sound velocity *c* is now the light velocity – nothing goes faster.



Distortion of metric caused by oscillating vortex (in AC driving field) as phonons are emitted

WG Unruh, PRL 46, 1351 (1981)
WG Unruh, Phys Rev D51, 2827 (1995)

PRELIMINARY: analogy with GRAVITATION 2



constant phase ‘eikonal’ plot

If we describe a vortex in this GR language, we get an interesting ‘cosmic string’ situation: spacetime has to be ‘cut’, then repasted together, with a time jump across the cut.

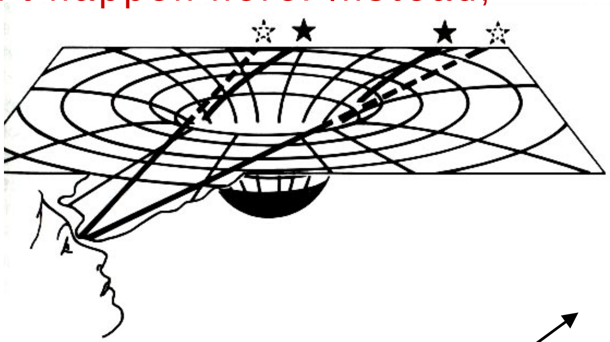
Time jump: $\Delta t = 2\pi/\omega_{cs}$ where $\omega_{cs} = 1/4JG$ for string with angular momentum J

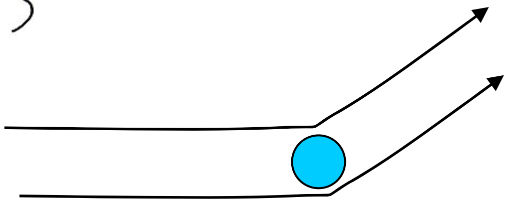
Outside the vortex core, the metric is then flat:

$$ds^2 = c^2\left(dt - \frac{d\phi}{\omega_{cs}}\right)^2 - (dz^2 + dr^2 + r^2 d\phi^2)$$

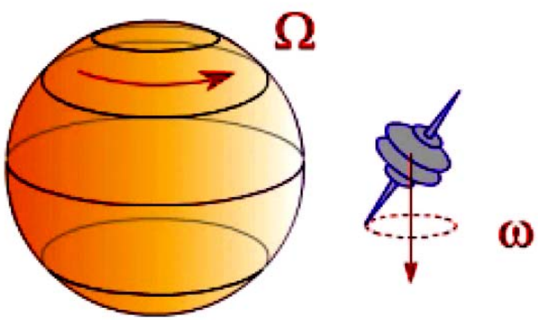
M Stone, Phys Rev B61, 11780 (2000)

There is actually a lot more interesting physics to be gained from this analogy.The usual ‘gravitational bending of light’ (ie., of phonons) doesn’t happen here. Instead, phonons are deflected in the same sense, no matter which side of the vortex core they pass.





One can also ask about the precession of a ‘gyroscope’ attached to a phonon. We expect 2 contributions – a ‘de Sitter’ geodetic contribution, and a ‘Lense-Thirring’ frame dragging contribution.



Both should be present here; the AdS term comes from the acceleration of the phonon, and the ALT term from the interaction with the vortex angular momentum

PRELIMINARY: VORTEX + PHONONS - SCATTERING ANALYSIS

STANDARD PHENOMENOLOGY:

add the Magnus force $\boldsymbol{F}^{\mathrm{M}} = \kappa\rho_{\mathrm{s}}(\boldsymbol{v}_{\mathrm{s}} - \boldsymbol{v}_{\mathrm{L}}) \times \hat{\boldsymbol{z}}$

+ quasiparticle force: $\boldsymbol{F}^{\mathrm{N}} = D(\boldsymbol{v}_{\mathrm{n}} - \boldsymbol{v}_{\mathrm{L}}) + D'\hat{\boldsymbol{z}} \times (\boldsymbol{v}_{\mathrm{n}} - \boldsymbol{v}_{\mathrm{L}})$

let $\boldsymbol{F}^{\mathrm{M}} + \boldsymbol{F}^{\mathrm{N}} = 0$

This gives MUTUAL FRICTION: $\boldsymbol{F}_{\mathrm{sn}} = -n_{\mathrm{L}}\kappa\rho_{\mathrm{s}}\alpha(\boldsymbol{v}_{\mathrm{s}} - \boldsymbol{v}_{\mathrm{n}}) + n_{\mathrm{L}}\kappa\rho_{\mathrm{s}}\alpha'[\hat{\boldsymbol{z}} \times (\boldsymbol{v}_{\mathrm{s}} - \boldsymbol{v}_{\mathrm{n}})]$

where $\alpha = \dfrac{d_{\parallel}}{d_{\parallel}^2 + (1 - d_{\perp})^2}$ $\quad 1 - \alpha' = \dfrac{1 - d_{\perp}}{d_{\parallel}^2 + (1 - d_{\perp})^2}$ and $d_{\perp} = \dfrac{D'}{\kappa\rho_{\mathrm{s}}}$ $\quad d_{\parallel} = \dfrac{D}{\kappa\rho_{\mathrm{s}}}$

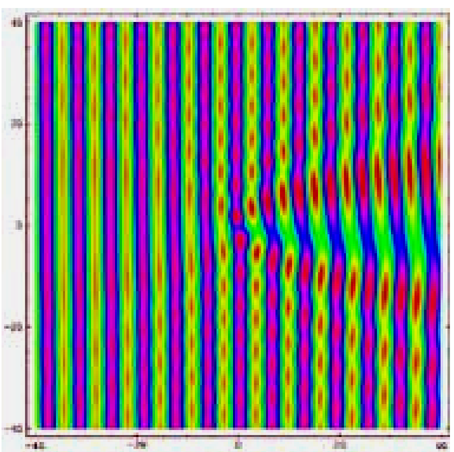
PARTIAL WAVE ANALYSIS

Scattered wave: $\phi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}} e^{-i\omega_k t}\left(e^{i\mathbf{k}\cdot\mathbf{r}} + \dfrac{i a_k(\theta)}{\sqrt{r}} e^{ikr}\right)$

One finds Aharonov-Bohm form:

$$a_k(\theta) = \frac{a_0}{2}\sqrt{2\pi k}e^{i\pi/4}\frac{\sin\theta}{1-\cos\theta}$$

with $\phi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}} e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}}\left(1 + i\pi a_0 k \,\mathrm{erf}\left(\theta\sqrt{\dfrac{kr}{2i}}\right)\right)$



Phonon wave function around moving vortex

in the quasiclassical limit $(kr)^{-\frac{1}{2}} \ll \theta \ll 1$ this has a 'cut':

$$\phi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}} e^{-i\omega_k t}\left(e^{i\mathbf{k}\cdot\mathbf{r}}\left(1 + i\pi a_0 k\frac{\theta}{|\theta|}\right) + i\frac{a_0}{\theta}\sqrt{\frac{2\pi k}{r}}e^{ikr + i\frac{\pi}{4}}\right)$$

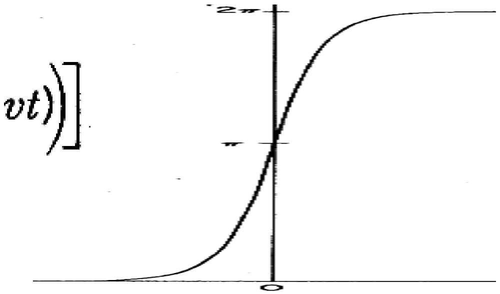
If one calculates the force on the vortex, one gets:

$$F_i^{scatt} = -\sum_{\mathbf{k}}\frac{\hbar^2\rho_s}{m_0^2}\int dS_j\langle\nabla_i\phi_{\mathbf{k}}\nabla_j\phi_{\mathbf{k}}\rangle$$

which contains a Iordanskii term: $\mathbf{F}_{\perp}^{qp} = \kappa \times \mathbf{j}^{qp}$

PRELIMINARY:
QUANTUM SOLITON +
QUASIPARTICLES

1-d Sine-Gordon model: $\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} + \omega_0^2 \sin\psi = 0$



Single 'kink' soliton: $\psi_{\pm}^{v}(x,t) = 4\tan^{-1}\left[\exp\left(\pm \frac{\omega_0}{c_0}\gamma(x - vt)\right)\right]$

where: $\gamma \equiv (1 - v^2/c_0^2)^{-1/2}, \quad |v| < c_0$

Now add small oscillations: $\psi_{\pm}(x,t) = \psi_{\pm}^{v}(x,t) + \phi(x,t)$

Quasiparticle eqtn of motion: $\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} + \omega_0^2\left(1 - 2\,\mathrm{sech}^2 \frac{\omega_0}{c_0}x\right)\phi = 0$

Corresponds to waves, but with 'kink potential': $V(x) = \omega_0^2\left(1 - 2\,\mathrm{sech}^2 \frac{\omega_0}{c_0}x\right)$

These 'quasiparticles' have Lagrangian: $\mathcal{L} = \phi^* \partial_\mu \partial^\mu \phi - \phi^* V(x - Q)\phi$
(for wall at position Q)

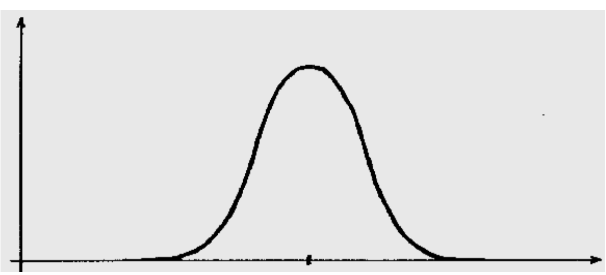
Assume: $\phi(x,t) = f(x)\, e^{-i\omega t}$

Bound QP mode:

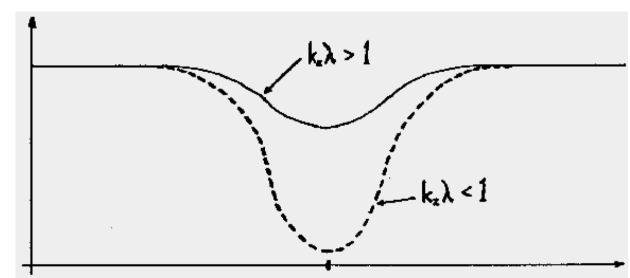
$f_b(x) = \frac{2\omega_0}{c_0}\,\mathrm{sech}\,\frac{\omega_0}{c_0}x$

Extended QP modes: $\omega_\kappa^2 = c_0^2\kappa^2 + \omega_0^2$

$f_\kappa(x) = \frac{1}{(2\pi)^{1/2}}\frac{c_0}{\omega_\kappa}e^{i\kappa x}\left(\kappa + i\frac{\omega_0}{c_0}\tanh\frac{\omega_0}{c_0}x\right)$



This is typical – extended QP modes avoid the soliton and the bound states.



PROPER ANALYSIS of the PROBLEM

DESCRIPTION of a BOSE SUPERFLUID

Assume Action $\mathcal{S} = -\int dt d^2 r \left[\frac{\rho_s + \eta}{m_0} (\hbar \dot{\Phi} + \frac{|\hbar \nabla \Phi|^2}{2m_0}) + \epsilon(\eta) \right]$

$$\rho_s(\mathbf{r}, t) = \rho_s^o + \eta(\mathbf{r}, t)$$

This is a 'long wavelength' action – valid for energy $\ll \; \Lambda_o = \hbar c_o / a_o = m_o / \chi \rho_s$

and for lengths $\gg a_o$ where $a_o^2 \sim \hbar^2 \chi \rho_s / m_o^2$

Here the velocity of sound is $c_o = 1/\sqrt{\chi \rho_s}$, and χ is the compressibility

and the superfluid velocity is $\mathbf{v} = (\hbar / m_o) \nabla \Phi$

Equations of motion; let:

$$\dot{\Phi} + \frac{|\nabla \Phi|^2}{2} + \frac{\delta \epsilon}{\delta \eta} = 0$$

Force (Bernouilli)

$$\dot{\eta} + \nabla \cdot ((1 + \eta) \nabla \Phi) = 0$$

Mass conservation

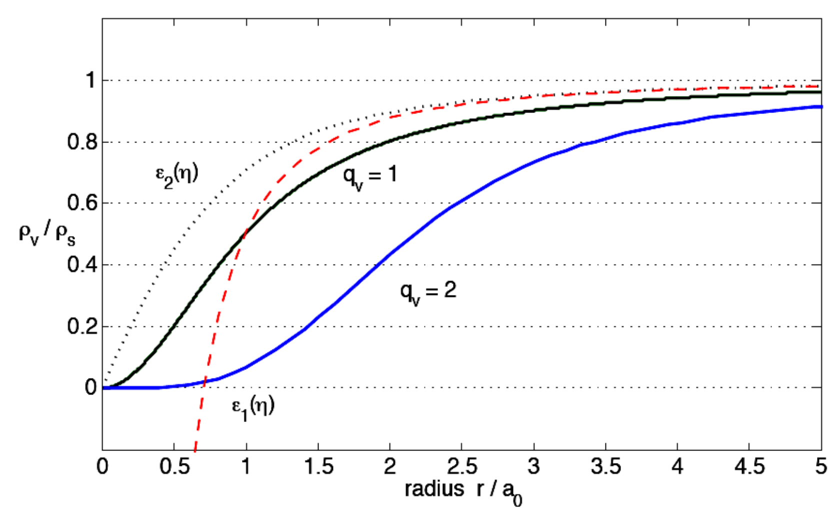
For non-Linear Schrodinger system:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 \psi - \mu \psi + V_0 |\psi|^2 \psi$$

So that:

$$\dot{\Phi} + \frac{|\nabla \Phi|^2}{2} - \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} + \rho - 1 = 0$$

$$\dot{\rho} + \nabla \cdot (\rho \nabla \Phi) = 0$$



VORTEX-PHONON SCATTERING: FORMAL

We assume a phase field: $\Phi(\mathbf{r},t) = \Phi_v(\mathbf{r},t) + \phi(\mathbf{r},t)$

with 'bare' vortex field $\Phi_v(\mathbf{r} - \mathbf{R}(\mathbf{t}))$

Circulation: $\kappa = qh/m_o$

Topological charge: $\tilde{q} = \pm 1$

Writing: $\rho = 1 + \eta_v + \eta$ yields quasiparticle eqtns of motion:

$$\dot{\phi} + \nabla\Phi_v \cdot \nabla\phi + \frac{\delta^2\epsilon}{\delta\eta_v^2}\eta = 0$$

$$\dot{\eta} + \nabla\Phi_v \cdot \nabla\eta + \nabla\cdot((1+\eta_v)\nabla\phi) = 0$$

THE KEY POINT HERE:

It is mistake to use free phonons in any calculation

The true phonons are strongly altered by the vortex. They only couple quadratically to the vortex

In cylindrical coordinates:

$$\eta = -\cos(m\theta + \mu t)\eta_{m\mu}$$
$$\phi = \sin(m\theta + \mu t)\phi_{m\mu}$$

We then have:

$$\mathcal{D}_\phi \phi_{m\mu} + \nabla\Phi_v \cdot \nabla\eta_{m\mu} = -\mu\eta_{m\mu}$$
$$\mathcal{D}_\eta \eta_{m\mu} - \nabla\Phi_v \cdot \nabla\phi_{m\mu} = \mu\phi_{m\mu}$$

where:

$$\mathcal{D}_\phi \phi = r\nabla\cdot((1+\eta_v)\nabla\phi) = \partial_r\left(r(1+\eta_v)\partial_r\phi\right) - (1+\eta_v)\frac{m^2}{r}\phi$$

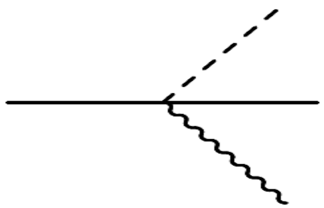
$$\mathcal{D}_\eta \eta = \begin{cases} \frac{1}{4}r\nabla\cdot\left(\frac{\nabla\eta}{1+\eta_v}\right) - r\left(1 + \frac{1}{2}\nabla\cdot\left(\frac{\nabla f_v}{f_v}\right)\right)\eta & \text{(for the NLSE)} \\ -\frac{\delta^2\epsilon}{\delta\eta_v^2} r\eta & \text{(for Popov's action)} \end{cases}$$

And where for the NLSE we have defined: $f = \sqrt{\rho}$

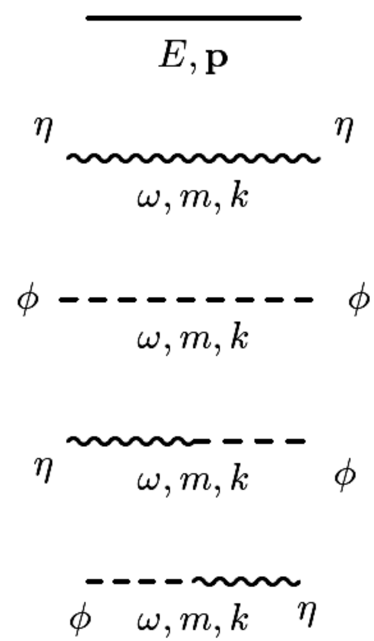
So that: $\nabla^2 f - \frac{q_v^2}{r^2} f - f^3 + f = 0$ with approx. static solution: $f_v = r/(2+r^2)^{1/2}$

INTERLUDE: DIAGRAMMAR

We have a system in which a vortex couples to a system of vortex-renormalized fluctuations, divided into density fluctuations & phase fluctuations. All the complexity goes into the vertices. For example:



which for large cylinder becomes:



$$G_V(E,\mathbf{p}) = \frac{\hbar}{E - E_p + i\delta}$$

$$G_{\eta\eta}(\omega, m, k) = \frac{-\hbar\rho_s k^2}{\omega^2 - \omega_k^2 + i\delta}$$

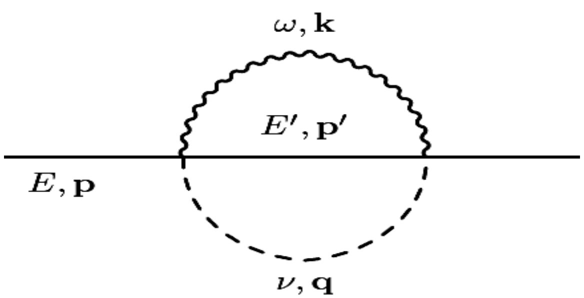
$$G_{\phi\phi}(\omega, n, k) = \frac{-\frac{m_0^2}{\hbar\rho_s^2\chi}}{\omega^2 - \omega_k^2 + i\delta}$$

$$G_{\eta\phi}(\omega, n, k) = \frac{i m_0 \omega}{\omega^2 - \omega_k^2 + i\delta}$$

$$G_{\phi\eta}(\omega, n, k) = -G_{\eta\phi}(\omega, n, k)$$

$$\delta(\mathbf{p}' - \mathbf{p} + \hbar[\mathbf{k} - \mathbf{q}])\delta(E' - E + \hbar[\omega - \nu])$$
$$\times 2\hbar^2 \pi q_V \frac{kq}{|\mathbf{k} - \mathbf{q}|^2} \sin\theta_{kq}$$

Note the unusual form of this vertex



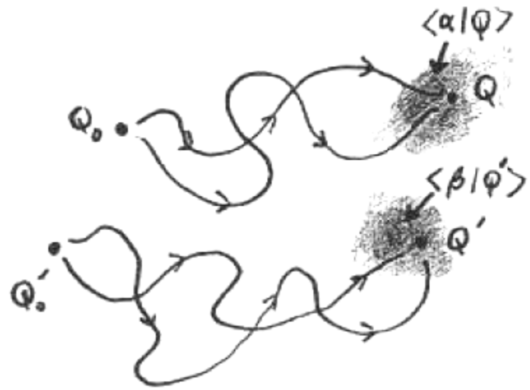
We can form low-order diagram with this. But what we really want to do is go beyond lowest-order diagrams and do a non-perturbative calculation.

This is done using a path integral formulation...

PATH INTEGRAL FORMULATION – VELOCITY EXPANSION

Let us write a path integral expression for the propagator of a density matrix. One has

$$\rho_o(Q,Q';t) = \langle Q|\psi(t)\rangle\langle\psi(t)|Q'\rangle$$



with matrix elements between states

$$\rho_{\alpha\beta}(t) = \int dQ \int dQ' \langle\alpha|Q\rangle \rho_{QQ'}(t) \langle Q'|\beta\rangle$$

We can then always formulate the dynamics for the reduced density matrix as:

Density matrix propagator: $\rho(Q_1,Q_2;t) \;=\; K(Q_1,Q_2;Q_1',Q_2';t,t')\rho(Q_1',Q_2';t,t')$

where $K(Q_2,Q_2';Q_1,Q_1';t,t') \;=\; \int_{Q_1}^{Q_2}\mathcal{D}q\int_{Q_1'}^{Q_2'}\mathcal{D}q'\; e^{-i/\hbar(S_o[q]\;-\;S_o[q'])}\mathcal{F}[q,q']$

Path integral theory calculates the influence functional NON-PERTURBATIVELY

For the vortex we define: $\rho_V(\mathbf{X},\mathbf{Y},t) = \mathrm{Tr}_{\mathbf{r}_i}\langle\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots;\mathbf{X},t)\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots;\mathbf{Y},t)\rangle$

where *X* and *Y* are vortex coordinates.

We define $\mathbf{R} = \dfrac{\mathbf{X}+\mathbf{Y}}{2}$ "centre of mass" coordinate

$\xi = \mathbf{X}-\mathbf{Y}$ "Q Fluctuation" coordinate

VORTEX ACTION

We now have a bare vortex action of form: $\mathcal{S}_V = \frac{1}{2}\int dt \left(M_V \dot{\mathbf{X}}^2 - \rho_s q_V \kappa \times \mathbf{X} \cdot \dot{\mathbf{X}} \right)$

Where we define a hydrodynamic mass: $M_V = -\frac{\partial^2 E_v}{\partial v^2}$

For example, in a large cylinder: $M_V = \frac{1}{4}\rho_s^2 \chi \kappa^2 \log R_s / a_0$

The quasiparticle terms are:

$$\mathcal{S}_0^{qp} = -\frac{1}{2}\int dt \int d^2 r \left(2\eta\dot{\phi} + \eta \nabla \Phi_v \cdot \nabla \phi - \phi \nabla \Phi_v \cdot \nabla \eta - \phi \mathcal{D}_\phi \phi - \eta \mathcal{D}_\eta \eta \right)$$

$$\mathcal{S}_{int}^{qp} = \frac{1}{2}\int dt \int d^2 r \left(\eta \nabla \phi \cdot \dot{\mathbf{X}} - \phi \nabla \eta \cdot \dot{\mathbf{X}} \right)$$

Integrating them out gives:

$$\begin{aligned}
\mathcal{F}[\mathbf{X},\mathbf{Y}] = \exp\Bigg(& -\frac{i}{2}\sum_{\mu\nu m\sigma} (g_{\mu\nu}^{\sigma m})^2 (n_\mu - n_\nu) \\
& \times \int_0^T dt \int_0^t ds \Big[\Big(\dot{\mathbf{X}}(t) - \dot{\mathbf{Y}}(t)\Big) \cdot \Big(\dot{\mathbf{X}}(s) + \dot{\mathbf{Y}}(s)\Big) \sin(\mu - \nu)(t - s) \\
& \qquad + \sigma \hat{k} \cdot \Big(\dot{\mathbf{X}}(t) - \dot{\mathbf{Y}}(t)\Big) \times \Big(\dot{\mathbf{X}}(s) + \dot{\mathbf{Y}}(s)\Big) \cos(\mu - \nu)(t - s) \Big] \\
& - \sum_{\mu\nu m\sigma} n_\mu (1 + n_\nu) (g_{\mu\nu}^{\sigma m})^2 \\
& \times \int_0^T dt \int_0^t ds \Big[\Big(\dot{\mathbf{X}}(t) - \dot{\mathbf{Y}}(t)\Big) \cdot \Big(\dot{\mathbf{X}}(s) - \dot{\mathbf{Y}}(s)\Big) \cos(\mu - \nu)(t - s) \\
& \qquad - \sigma \hat{k} \cdot \Big(\dot{\mathbf{X}}(t) - \dot{\mathbf{Y}}(t)\Big) \times \Big(\dot{\mathbf{X}}(s) - \dot{\mathbf{Y}}(s)\Big) \sin(\mu - \nu)(t - s) \Big] \Bigg)
\end{aligned}$$

VORTEX EQUATION OF MOTION

Define the c.o.m. variable $\mathbf{R}(t) = \frac{1}{2}(\mathbf{X}(t) + \mathbf{Y}(t))$ & the relative coordinate $\boldsymbol{\xi} = \mathbf{X} - \mathbf{Y}$

Then we find:

$$M_V \ddot{\mathbf{R}}(t) - \rho_s q_v \kappa \times \dot{\mathbf{R}}(t) + \sum_{m\mu\nu\sigma} (g_{\mu\nu}^{\sigma m})^2 (n_\mu - n_\nu) \Big(\sigma \hat{k} \times \dot{\mathbf{R}}(t) - (\mu - \nu) \int_0^t ds \left(\dot{\mathbf{R}}(s) \cos(\mu - \nu)(t - s) + \sigma \hat{k} \times \dot{\mathbf{R}}(s) \sin(\mu - \nu)(t - s) \right) \Big) = 0$$

and also:

$$M_V \ddot{\boldsymbol{\xi}}(t) - \rho_s q_v \kappa \times \dot{\boldsymbol{\xi}}(t) + \sum_{m\mu\nu\sigma} (g_{\mu\nu}^{\sigma m})^2 (n_\mu - n_\nu) \Big(\sigma \hat{k} \times \dot{\boldsymbol{\xi}}(t) + (\mu - \nu) \int_t^T ds \left(\dot{\boldsymbol{\xi}}(s) \cos(\mu - \nu)(t - s) + \sigma \hat{k} \times \dot{\boldsymbol{\xi}}(s) \sin(\mu - \nu)(t - s) \right) \Big) = 0$$

Where M_V is the superfluid hydrodynamic mass, and we find that all dissipative forces are non-local in time, integrating over the previous trajectory of the vortex

These equations look more elegant in Fourier space:

$$-\omega^2 M_V \mathbf{R}_\omega - i\omega D_\perp \hat{k} \times \mathbf{R}_\omega - i\omega D_\parallel(\omega) \mathbf{R}_\omega - \omega m_\perp(\omega) \hat{k} \times \mathbf{R}_\omega = 0$$

$$-\omega^2 M_V \boldsymbol{\xi}_\omega - i\omega D_\perp \hat{k} \times \boldsymbol{\xi}_\omega + i\omega D_\parallel(\omega) \boldsymbol{\xi}_\omega + \omega m_\perp(\omega) \hat{k} \times \boldsymbol{\xi}_\omega = 0$$

where:

$$D_\parallel(\omega) = \sum_{m\mu\nu\sigma} (g_{\mu\nu}^{\sigma m})^2 (n_\mu - n_\nu)(\mu - \nu) \frac{i\omega - \gamma}{(i\omega - \gamma)^2 + (\mu - \nu)^2}$$

$$m_\perp(\omega) = \sum_{m\mu\nu\sigma} \sigma (g_{\mu\nu}^{\sigma m})^2 (n_\mu - n_\nu)(\mu - \nu) \frac{\mu - \nu}{(i\omega - \gamma)^2 + (\mu - \nu)^2}$$

SUMMARY of RESULTS

The standard vortex eqtn of motion: $M_V \ddot{\mathbf{r}}_V(t) - \mathbf{F}_M - \mathbf{F}_I - \mathbf{F}_\parallel - \mathbf{F}_{pinning} = 0$

The true equations of motion that we find are highly non-local in spacetime:

$$M_V \ddot{\mathbf{R}}(t) - \rho_s \kappa_V \times (\dot{\mathbf{R}}(t) - \mathbf{v}_s) - \mathbf{F}_I[\dot{\mathbf{R}}] - \mathbf{F}_\parallel[\dot{\mathbf{R}}] - \mathbf{F}_\perp[\dot{\mathbf{R}}] = \mathbf{F}_{fluc}(t)$$

$$M_V \ddot{\boldsymbol{\xi}}(t) - \rho_s \kappa_V \times (\dot{\boldsymbol{\xi}}(t) - \mathbf{v}_s) - \mathbf{F}_I[\dot{\boldsymbol{\xi}}] + \mathbf{F}_\parallel[\dot{\boldsymbol{\xi}}] + \mathbf{F}_\perp[\dot{\boldsymbol{\xi}}] = 0$$

$$\mathbf{F}_\parallel[\dot{\mathbf{R}}] = \frac{\hbar}{L_z} \sum_{m\sigma kq} (\Lambda_{kq}^{\sigma m})^2 (n_k - n_q)(\omega_k - \omega_q)$$

$$\times \int_0^t ds\, (\dot{\boldsymbol{R}}(s) - \mathbf{v}_n) \cos[(\omega_k - \omega_q)(t-s)]$$ ...... Non-local drag force

$$\mathbf{F}_I[\dot{\mathbf{R}}] = -\frac{\hbar}{L_z} \sum_{m\sigma kq} (\Lambda_{kq}^{\sigma m})^2 (n_k - n_q) \sigma \hat{\mathbf{z}} \times (\dot{\boldsymbol{R}}(t) - \mathbf{v}_n)$$ .... Local Iordanskii force

$$\mathbf{F}_\perp[\dot{\mathbf{R}}] = \frac{\hbar}{L_z} \sum_{m\sigma kq} (\Lambda_{kq}^{\sigma m})^2 (n_k - n_q)(\omega_k - \omega_q)$$

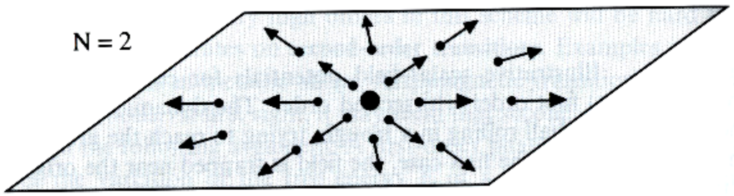
$$\times \int_0^t ds\, \sigma \hat{\mathbf{z}} \times (\dot{\boldsymbol{\xi}}(s) - \mathbf{v}_n) \sin[(\omega_k - \omega_q)(t-s)]$$ Non-local transverse memory term

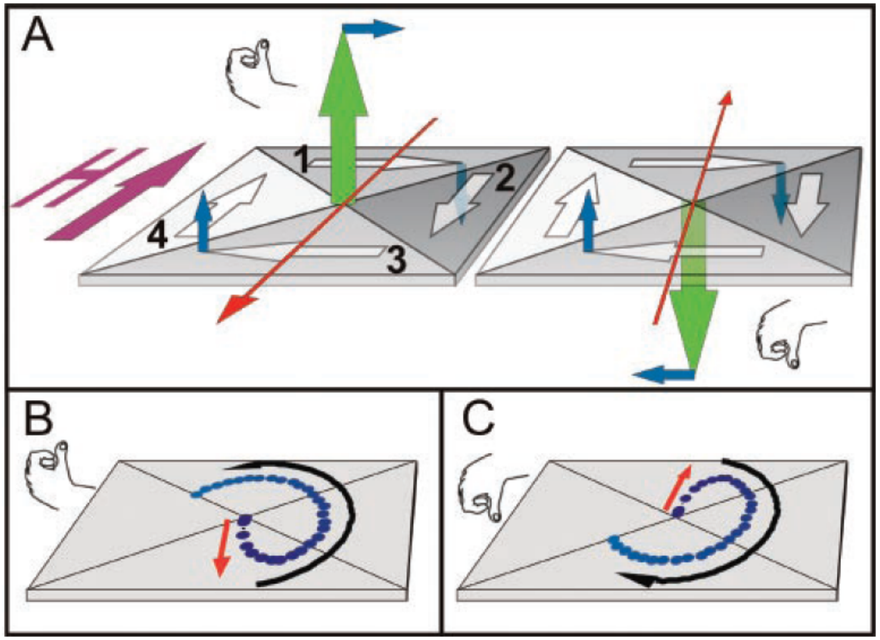
$$\langle F_{fluc}^i(t) F_{fluc}^j(s) \rangle = \frac{\hbar^2}{L_z^2} \sum_{m\sigma kq} (\omega_k - \omega_q)^2 n_k (1 + n_q)(\Lambda_{kq}^{\sigma m})^2$$

$$\times \left(\delta_{ij} \cos[(\omega_k - \omega_q)(t-s)] - \epsilon_{ijz} \sigma \sin[(\omega_k - \omega_q)(t-s)]\right)$$ Non-local Langevin term

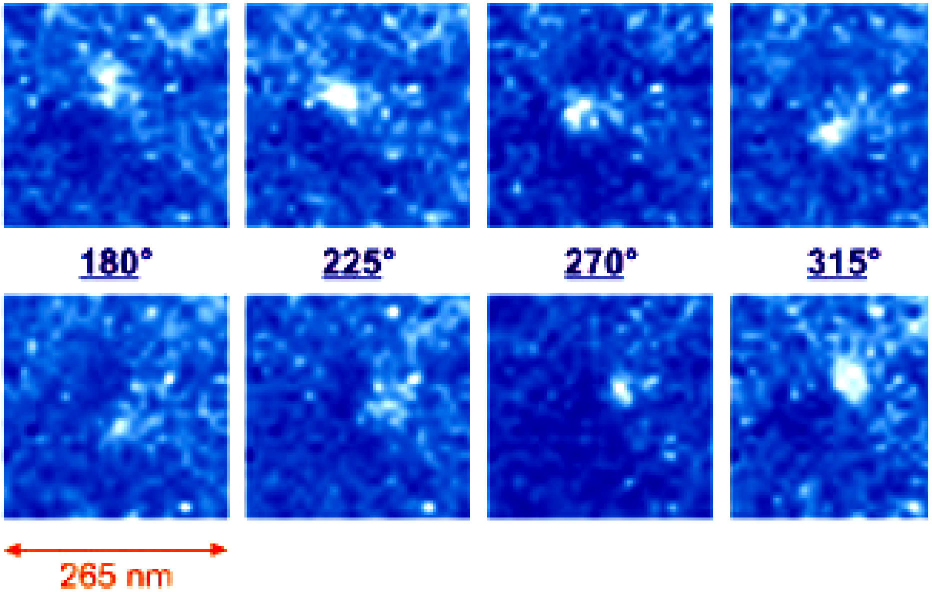
MAGNETIC VORTICES

Q. VORTICES in MAGNETS





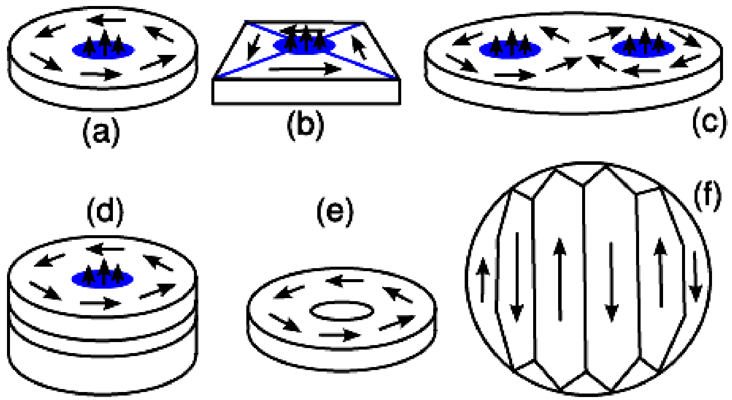
A vortex-like topological excitation can exist in a 2-d ordered spin system, or in a thin film.



These vortices can be imaged in a variety of ways (magneto-optical, or using MFM techniques.

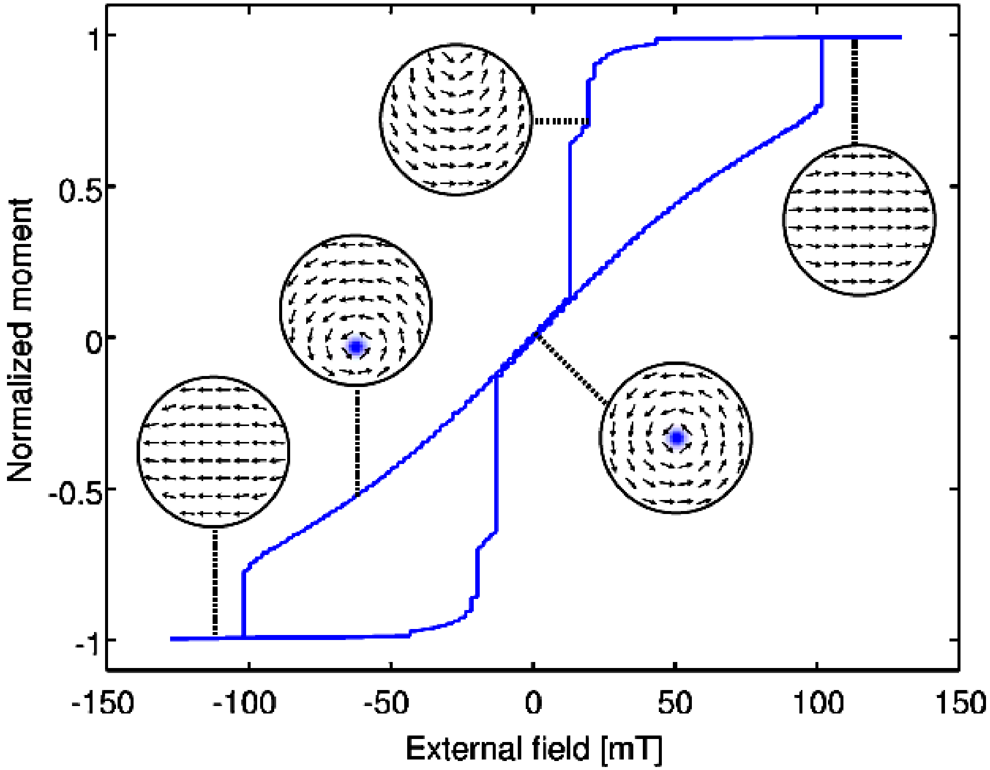
In this way one can also watch the dynamics of individual magnetic vortices under external influences.

TYPICAL EXPERIMENTAL SYSTEM: PERMALLOY



A well-controlled system where many experiments on vortices have been done is permalloy, $Ni_{80}Fe_{20}$, which has very small (quartic) magnetoelastic coupling. The vortex dynamics is very important for hard drive technology, and is described using the eqtn:

$$M_V \ddot{\mathbf{r}}_V(t) + \eta_m \dot{\mathbf{r}}_V(t) = \mathbf{F}_g - \nabla V_{bc}$$



We will be interested in a disc geometry (see above for others). Key parameters:

Magnon velocity: $c_m = \dfrac{2A}{a_m \mu_0 M_s / \gamma}$

Exchange length: $a_m = \sqrt{\dfrac{2A}{\mu_0 M_s^2}}$

where:

$$\begin{aligned} M_s &= 8.6 \times 10^5 \ \mathrm{A/m} \\ A &= 1.3 \times 10^{-11} \ \mathrm{J/m} \\ \gamma &= 2.2 \times 10^5 \ \mathrm{m/As} \end{aligned}$$

so that:

$$c_m = 500 \ \mathrm{m/s}$$

$$a_m = 5.3 \ \mathrm{nm}$$

Quantum Vortex in 2D Easy-plane Ferromagnet

Hamiltonian: $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j + \sum_i \sum_{a,b=x,y,z} K_{ab}S_{ia}S_{jb} - \frac{\mu_0\gamma^2}{4\pi}\sum_{i,j}\frac{3(\mathbf{S}_i\cdot\hat{\mathbf{e}}_{ij})(\mathbf{S}_j\cdot\hat{\mathbf{e}}_{ij}) - \mathbf{S}_i\cdot\mathbf{S}_j}{r_{ij}^3}$

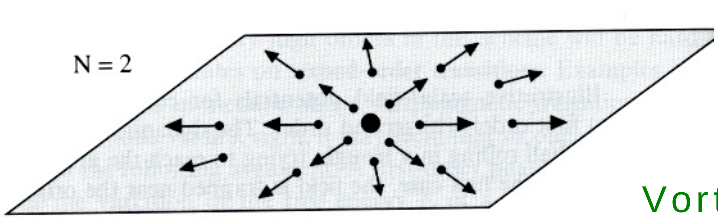
$\longrightarrow$ **Continuum Limit** $\frac{L_z}{M_s^2}\int d^2r A(\nabla\mathbf{M})^2 \; + \; \frac{\mu_0}{4\pi}\int d^3r\,\mathbf{M}(\mathbf{r})\cdot\nabla\int d^3r'\frac{\mathbf{M}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

The action is: $\mathcal{S} = \omega_B - \int dt\mathcal{H}$ where $\omega_B = -S\int\frac{d^2r}{a^2}\cos\theta\dot{\phi}$ **(Berry phase)**

The vortex is a 'skyrmion', with profile:

$$\phi_v(\mathbf{r}) = q\xi + \delta$$

$$\cos\theta_v(\mathbf{r}) = p\begin{cases} 1 - c_1\left(\frac{r}{r_v}\right)^2, & r\to 0; \\ c_2\sqrt{\frac{r_v}{r}}\exp(-\frac{r}{r_v}), & r\to\infty. \end{cases}$$

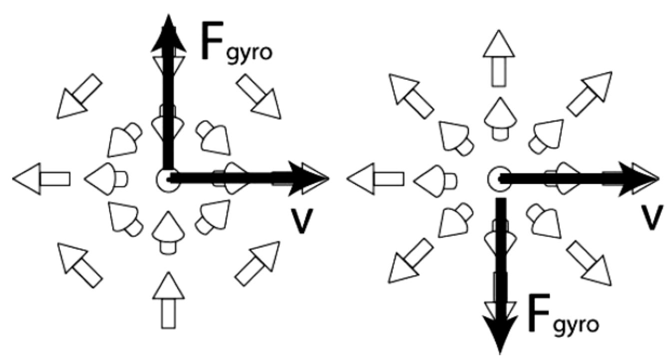


Vortex core radius $r_v^2 = J/2K$

MAGNON SPECTRUM

$\omega = ckQ$ with Spin Wave velocity: $c = SJa^2$

$Q^2 = k^2 + k_o^2$ with $k_o^2 = 2K/J$



We end up with:

$$\mathcal{L} = \frac{L_z \mu_0 M_s}{\gamma} \int d^2r \left(\cos\Theta\dot{\Phi} - \frac{c_m a_m}{2} \left((\nabla\Theta)^2 + \sin^2\Theta \left((\nabla\Phi)^2 - \frac{1}{a_m^2} \right) \right) \right)$$

The Berry phase gives a gyrotropic "Magnus" force:

$$F_M^{\perp}(t) \;=\; \partial_{\mathbf{R}} \int \frac{d^2r}{a^2} \left(-S\partial_t \mathrm{R} \cdot \nabla\phi_V \cos\theta_V \right) \;=\; \frac{\pi Spq}{a^2}\, \hat{\mathbf{z}} \times \partial_t \mathrm{R}(t)$$

Get effective mass tensor:

$$M_{ij} = \frac{\pi L_z a_m^2}{2A} \left(\frac{\mu_0 M_s}{\gamma} \right)^2 \begin{cases} q_i^2 \ln \frac{R_S}{a_m} & \text{for } i = j \\ q_i q_j \big(\ln \frac{R_S}{r_{ij}} & \\ \quad + \frac{1}{2} \cos(\theta_{ij} - \theta_{\dot{r}_i}) \cos(\theta_{ij} - \theta_{\dot{r}_j}) & \text{for } i \neq j \\ \quad + \frac{1}{2} \sin(\theta_{ij} - \theta_{\dot{r}_i}) \sin(\theta_{ij} - \theta_{\dot{r}_j}) \big) & \end{cases}$$

The vortex motion generates additional velocity terms:

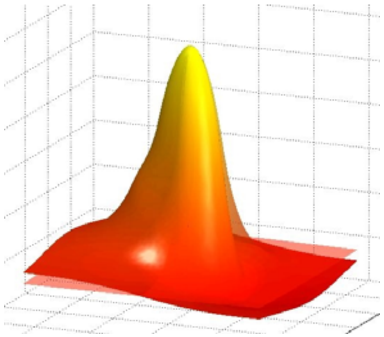
$$\tilde{\mathcal{S}}_1^{int} = \frac{L_z \mu_0 M_s}{\gamma} \int dt \int d^2r \left(\phi \nabla(\cos\Theta_V) \cdot \dot{\mathbf{r}}_V + \eta \nabla\Phi_V \cdot \dot{\mathbf{r}}_V \right)$$

$$\tilde{\mathcal{S}}_2^{int} = \frac{L_z \mu_0 M_s}{\gamma} \int dt \int d^2r \left(\eta \nabla\phi \cdot \dot{\mathbf{r}}_V \right)$$

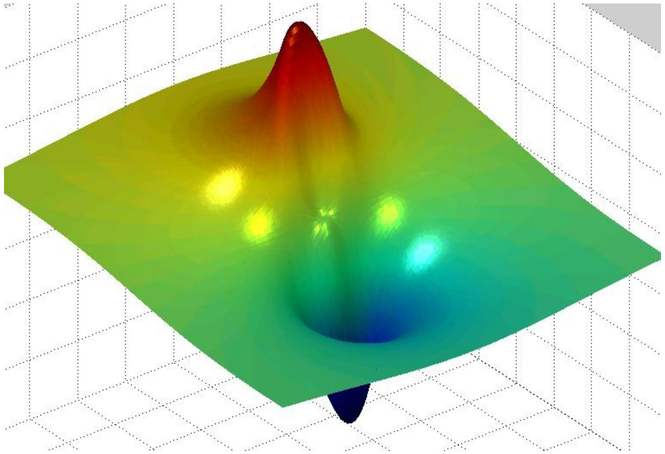
MAGNETIC VORTICES
INTERACTION WITH MAGNONS

At first this problem looks almost identical to He-4.
Define:

$$\Phi = \Phi_V(\mathbf{r} - \mathbf{r}_V) + \phi(\mathbf{r} - \mathbf{r}_V)$$
$$\Theta = \Theta_V(\mathbf{r} - \mathbf{r}_V) + \eta(\mathbf{r} - \mathbf{r}_V)$$



Profile of moving Vortex



Metric Curvature

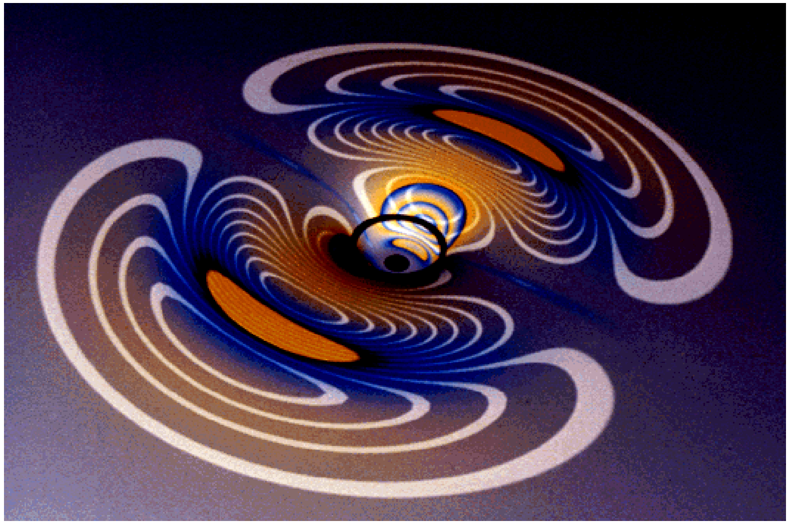
Magnon eqtns of motion:

$$\frac{\sin\Theta_V}{c_m a_m}\frac{\partial\phi}{\partial t} = \nabla^2\eta + \cos 2\Theta_V\left(\frac{1}{a_m^2} - (\nabla\Phi_V)^2\right)\eta - \sin 2\Theta_V \nabla\phi\cdot\nabla\Phi_V$$
$$\frac{1}{c_m a_m}\frac{\partial\eta}{\partial t} = -\sin\Theta_V\nabla^2\phi - 2\cos\Theta_V\left(\nabla\Theta_V\cdot\nabla\phi + \nabla\eta\cdot\nabla\Phi_V\right)$$

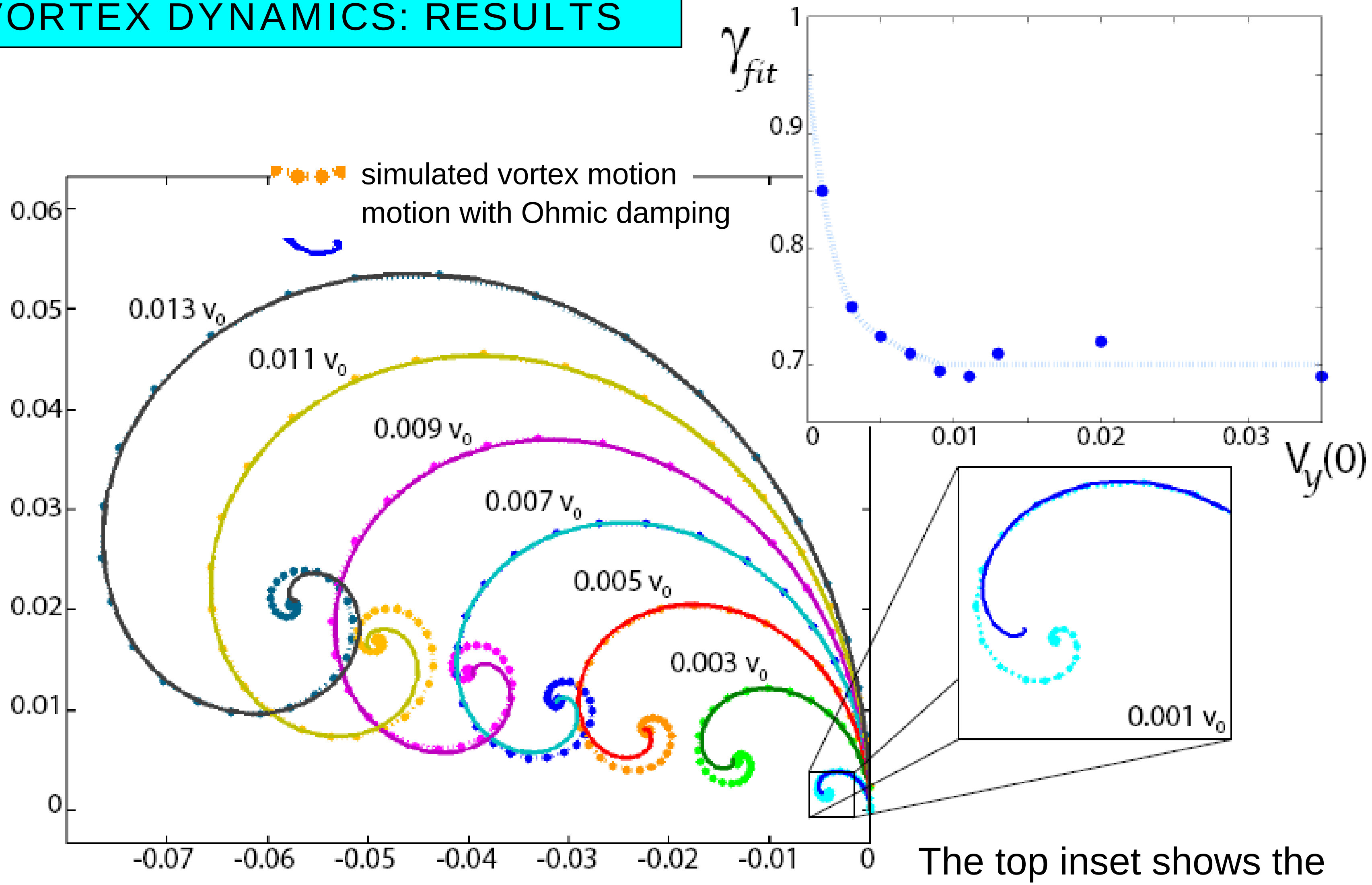
However these are not exactly the same.

In the spacetime metric picture, we find that the vortex has an unusul effect on a precessing spin \(ie., the magnon spin) moving past it.

The AdS and the ALT forces are equal and opposite: there is NO transverse force in a bulk system! At right we see the resulting scattering waves of magnons

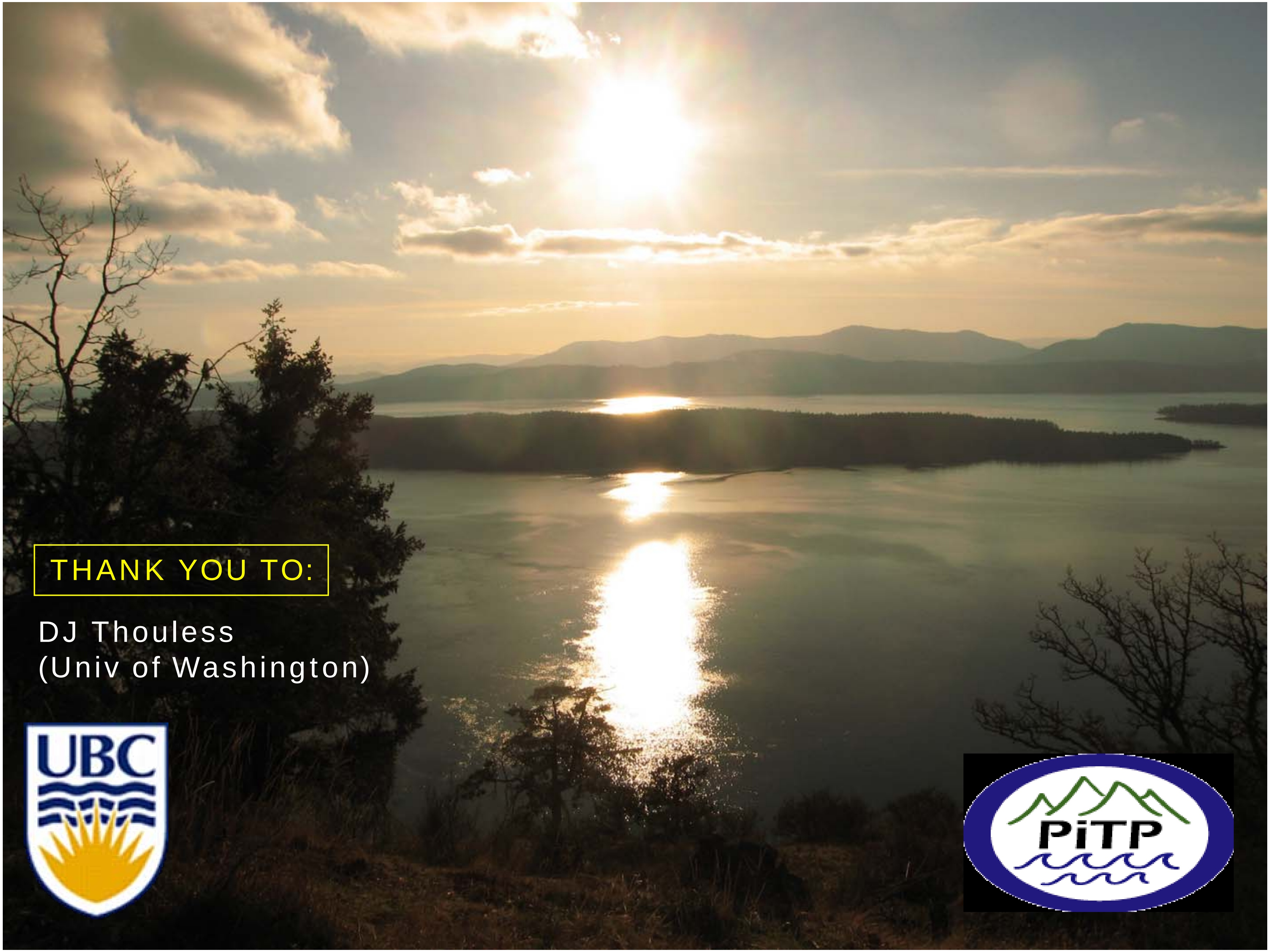


VORTEX DYNAMICS: RESULTS



If we set the vortex into motion with a δ-kick, we find decaying spiral motion dependent on the initial vortex speed (shown in fractions of $v_0 = c/r_v$)

The top inset shows the necessary γ of Ohmic damping to fit full simulated motion. *Note the strong upturn at low speeds!*


THANK YOU TO:
DJ Thouless
(Univ of Washington)