Introduction to Tensor Networks: PEPS, Fermions, and More

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School on computational methods in quantum materials

Jouvence, June 5 2014

www.romanorus.com/JouvenceTutorial2014.pdf

Some reviews

J. Eisert, Modeling and Simulation 3, 520 (2013), arXiv:1308.3318

N. Schuch, QIP, Lecture Notes of the 44th IFF Spring School 2013, arXiv:1306.5551

R. Orus, arXiv:1306.2164

J. I. Cirac, F. Verstraete, J. Phys. A: Math. Theor. 42, 504004 (2009)

F. Verstratete, J. I. Cirac, V. Murg, Adv. Phys. 57,143 (2008)

J. Jordan, PhD thesis, www.romanorus.com/JordanThesis.pdf

G. Evenbly, PhD thesis, arXiv:1109.5424

U. Schollwöck, RMP 77, 259 (2005)

U. Schollwöck, Annals of Physics 326, 96 (2011)

Outline

1) Generalities

2) 2d PEPS

- 2.1) Basics
- 2.2) Computing
- 2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

Outline

1) Generalities

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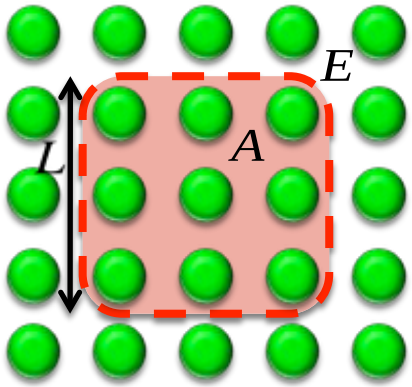
Entanglement
obeys area-law

Entanglement → **key resource in quantum information**

teleportation, quantum algorithms,
quantum error correction, quantum cryptography…

2d system



$$\rho_A = \mathrm{tr}_E\left(|\Psi\rangle\langle\Psi|\right)$$

Reduced density matrix of subsystem A

$$S(A) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

Entanglement entropy (von Neumann entropy)

For many ground states → $S(A) \sim L$ $(L > \xi)$

In d dimensions

Generic state $S(A) \sim L^d$ (volume)

Ground states of (most) local Hamiltonians $S(A) \sim L^{d-1}$ (area)

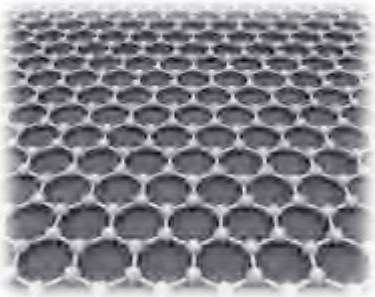
Srednicki, Plenio, Eisert, Dreißig, Cramer, Wolf…

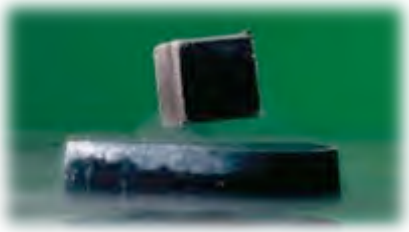
Locality of interactions ⇔ area-law

Many-body Hilbert space
is far too large

System of N spins 1/2

Everyday material, N ~ Avogadro number ~ $O(10^{23})$

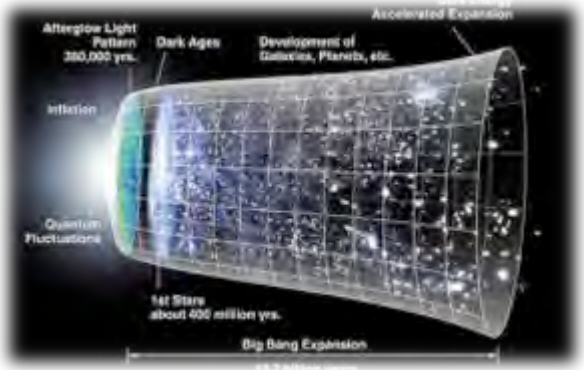




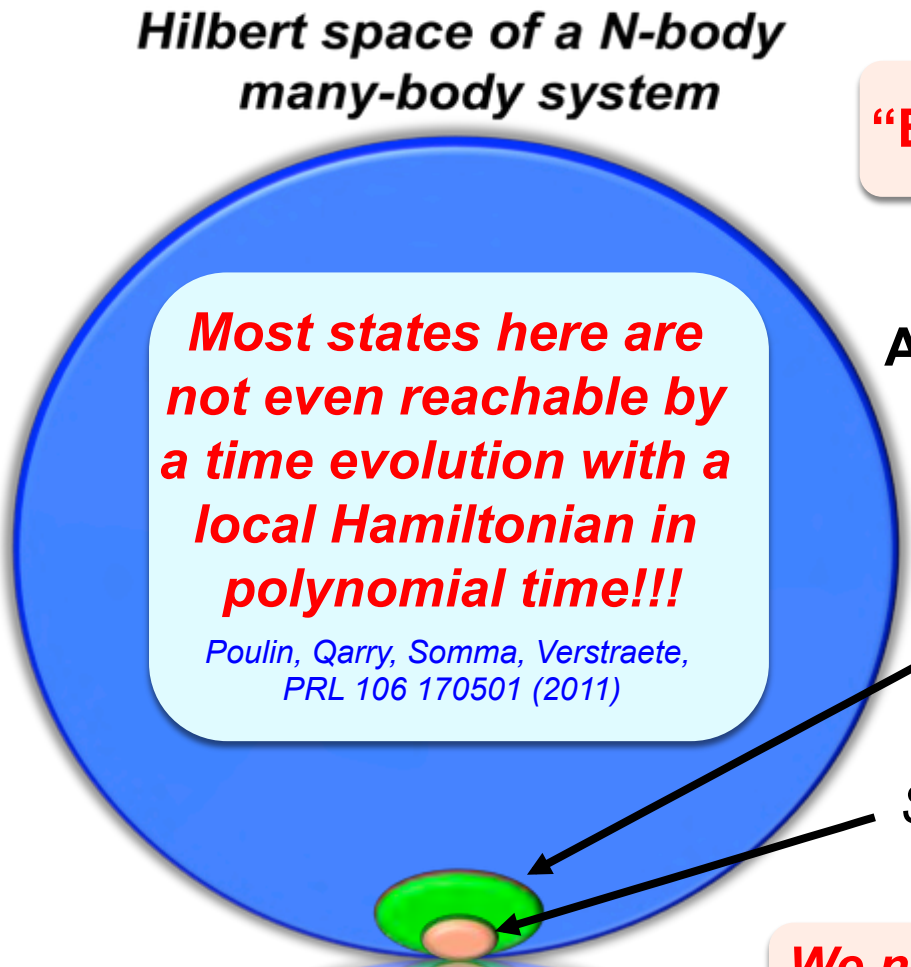
Number of basis states in the Hilbert space ~ $O(10^{10^{23}})$

Compare to…

Number atoms in the observable universe ~ $O(10^{80})$



Hilbert space is a convenient illusion



"Exploration" time ~ $O(10^{10^{23}})$ sec.

Compare to…

Age of the universe ~ $O(10^{17})$ sec.

Set of relevant states (area-law, ground states of local Hamiltonians)

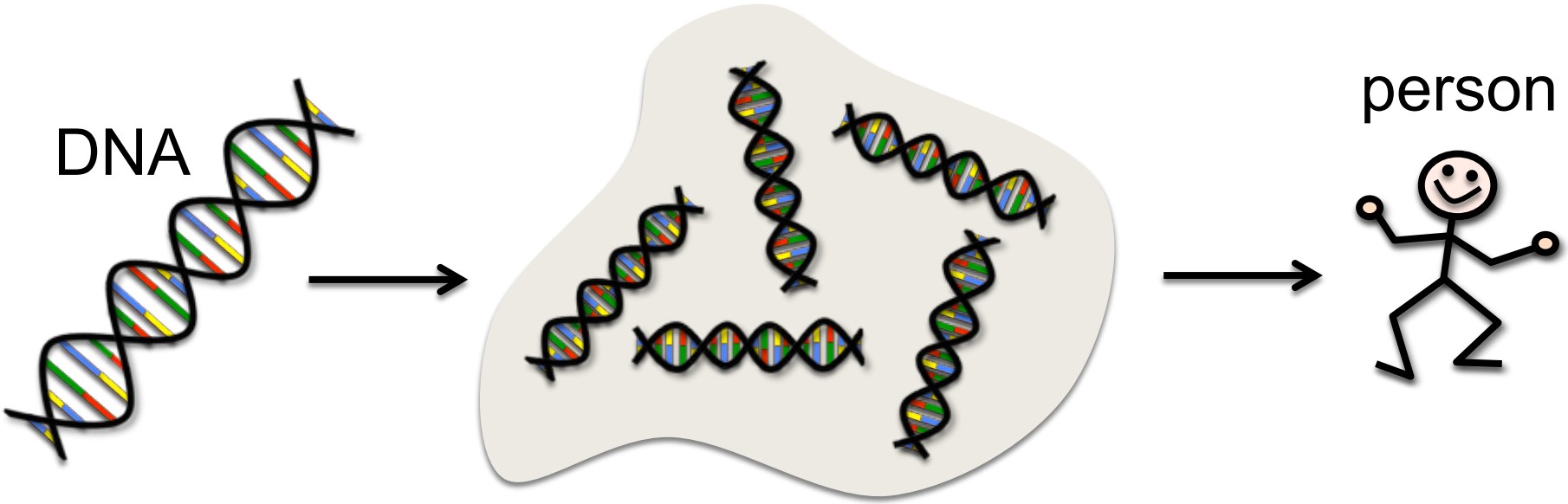
Set of product states (mean field)

We need a language to target the relevant corner of quantum states directly

Tensor Networks

A new language

$$|\Psi\rangle = \sum_{i's} \Psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$



A new language

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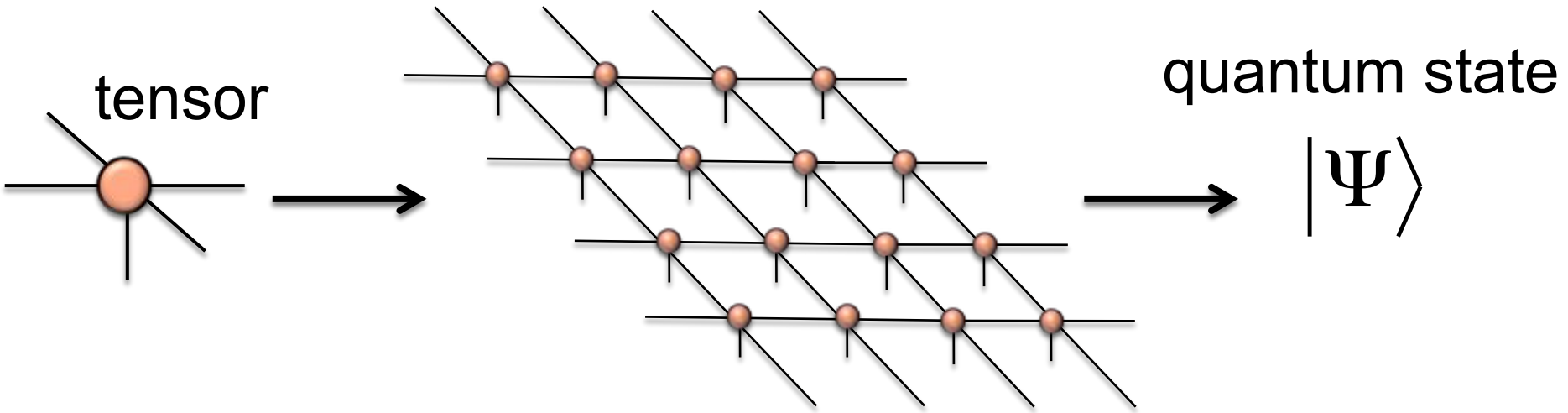
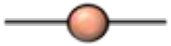


Tensors are local building blocks for the quantum state (like a DNA, or LEGO)

Tensor network diagrams

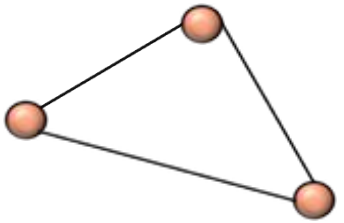
vector $\vec{v}$

matrix A

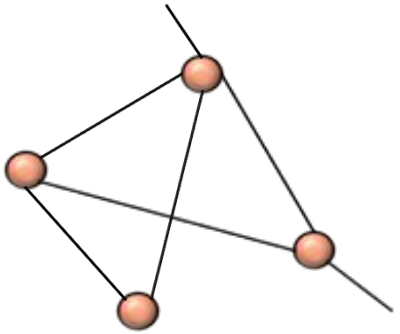


matrix product AB

trace of matrix product $\mathrm{tr}(ABC)$



tensor contraction $f(A,B,C,D)$



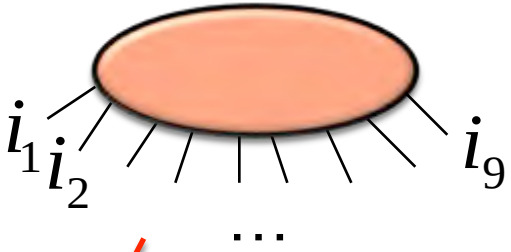
Break the wavefunction locally

$$|\Psi\rangle = \sum_{i's} \Psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

p-level systems

Tensor (multidimensional array of complex numbers)

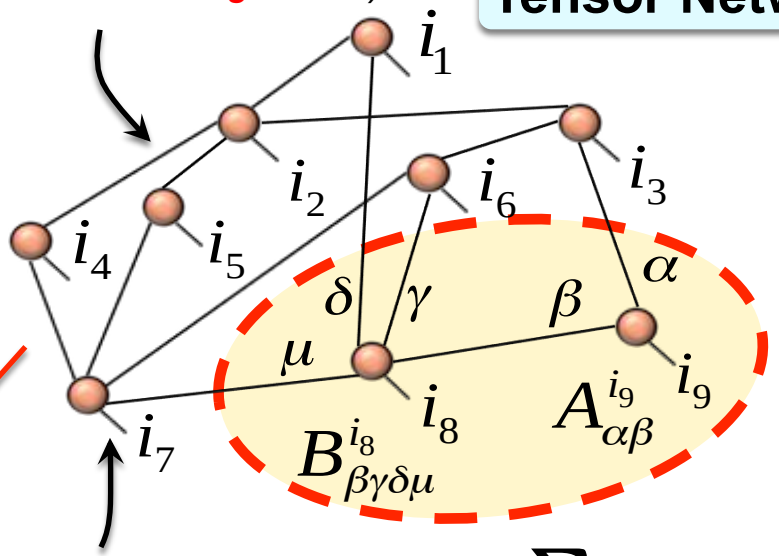
$\Psi_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9}$



replacement

summed ancillary bond index
1…D *(carries entanglement)*

Tensor Network



physical index *(local basis)*

$$C^{i_8 i_9}_{\alpha\gamma\delta\mu} = \sum_{\beta} A^{i_9}_{\alpha\beta} B^{i_8}_{\beta\gamma\delta\mu}$$

$$\beta = 1, 2, \ldots, D$$

$O(p^N)$ → $O(poly(N, p, D))$

Efficient representation, satisfies area-law, and targets low-energy eigenstates of local Hamiltonians

Important families of Tensor Network states

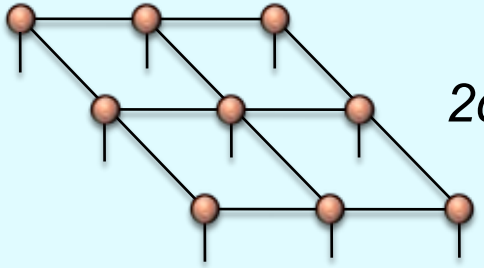
Matrix Product States (MPS)

1d systems

DMRG, Power Wave Function Renormalization Group, Time Evolving Block Decimation…

c.f., Miles Stoudenmire & Ulrich Schollwöck's talk

Projected Entangled Pair States (PEPS), Tensor Product States (TPS)

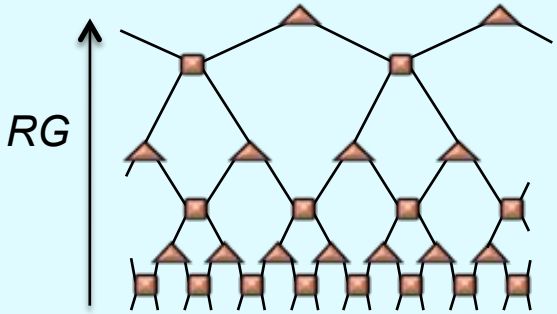


2d,3d… systems

Tensor Product Variational Approach, PEPS, Infinite-PEPS algorithm, Tensor-Entanglement Renormalization, TRG/SRG/HOTRG/HOSRG…

Multiscale Entanglement Renormalization Ansatz (MERA)

Entanglement Renormalization

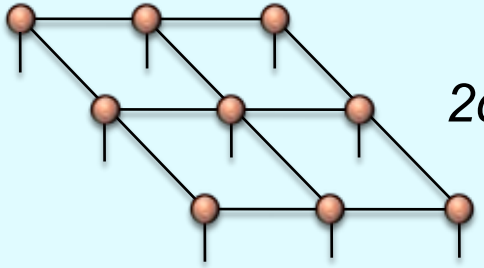


any-d scale invariant systems

Other: MPO, MPDO, TTN, PEPO…

This lecture

Projected Entangled Pair States (PEPS), Tensor Product States (TPS)



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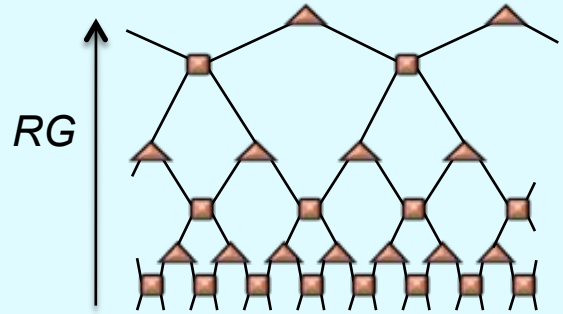

any-d scale invariant systems

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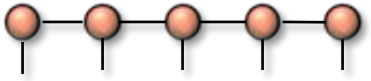
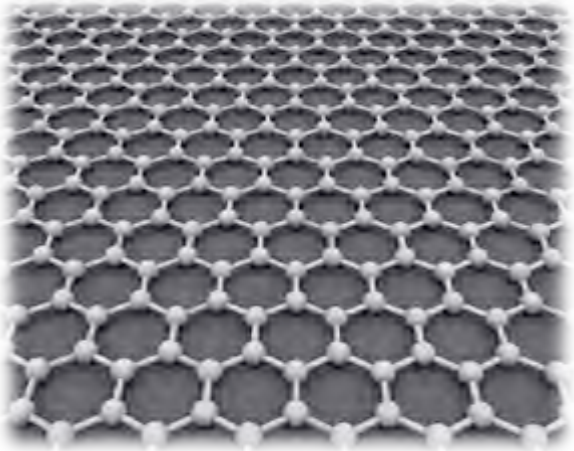
From MPS to PEPS

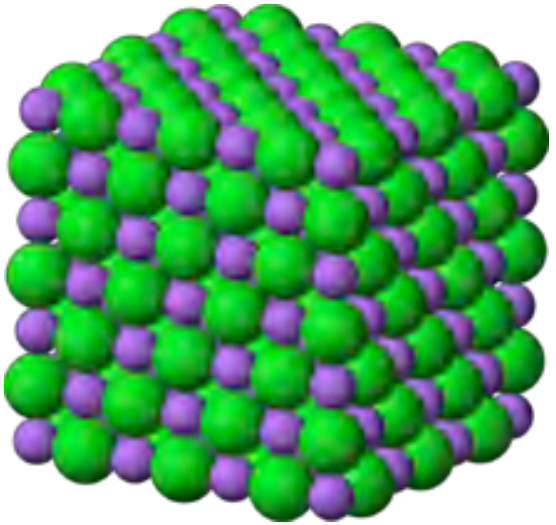
Matrix Product States (MPS)



1d systems

But we want to go beyond 1d systems!!!

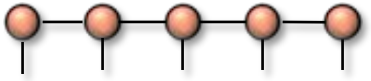




Very painful for DMRG...

From MPS to PEPS

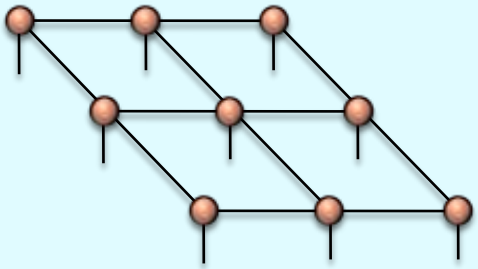
Matrix Product States (MPS)



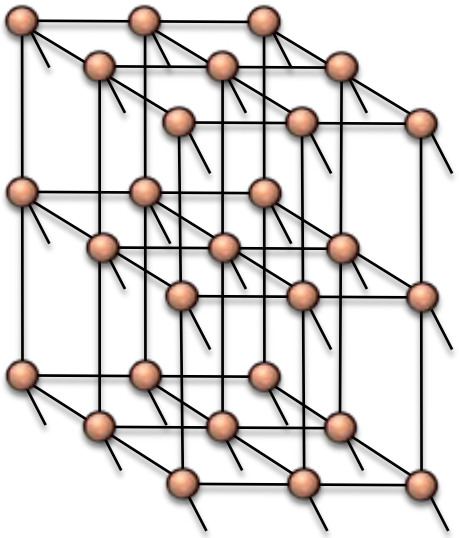
1d systems

Projected Entangled Pair States (PEPS),
Tensor Product States (TPS)

This lecture



2d systems



3d systems

and so on…

Two exact examples

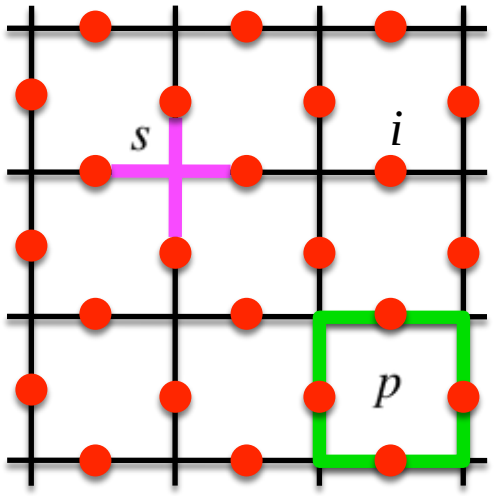
An exact example: Kitaev's Toric Code

Kitaev, 1997

$$H = -J\sum_s A_s - J\sum_p B_p$$

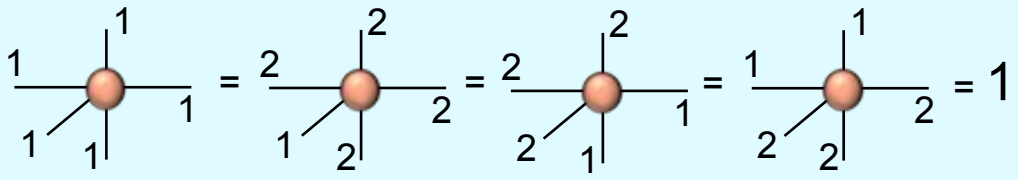
$A_s = \prod_{i\in s}\sigma_i^x$ *star operator*

$B_p = \prod_{i\in p}\sigma_i^z$ *plaquette operator*

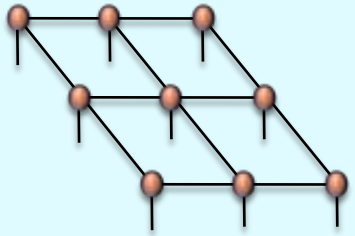


Simplest known model with "topological order"

Ground state (and in fact all eigenstates) are PEPS with D=2



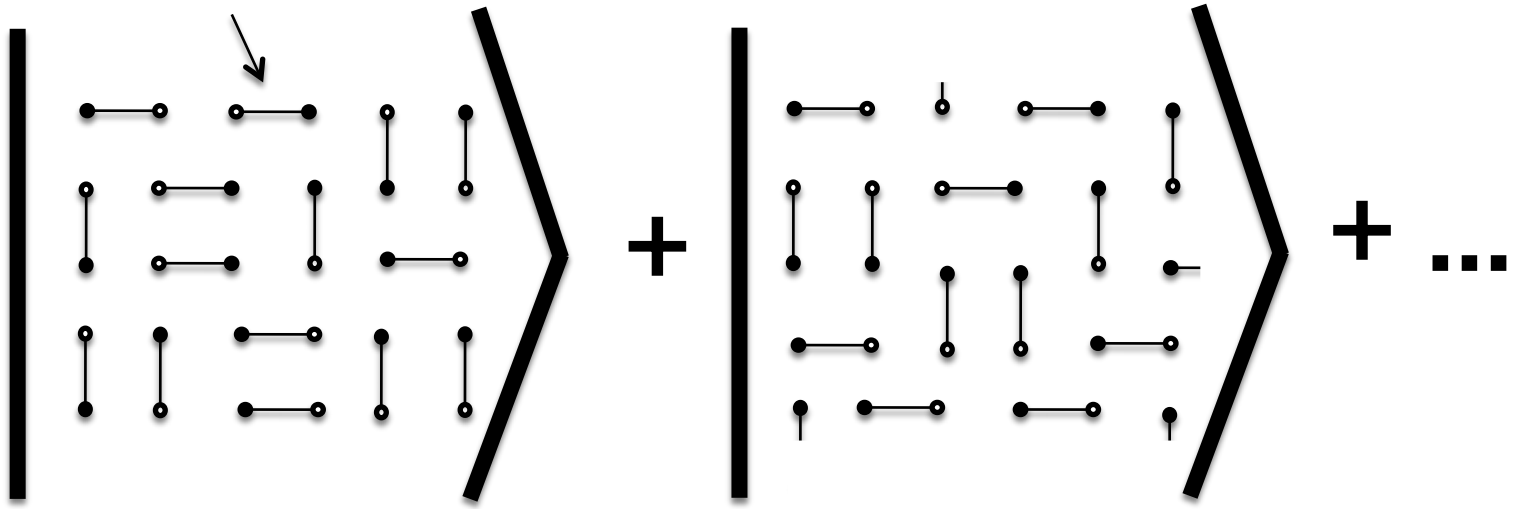
And another tensor rotated 90˚



Resonating Valence Bond State

Anderson, 1987

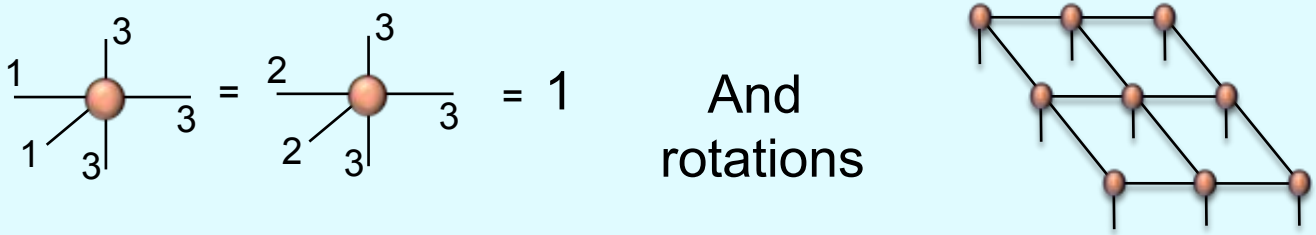
SU(2) singlets



Equal superposition of all possible nearest-neighbor singlet coverings of a lattice (spin liquid)

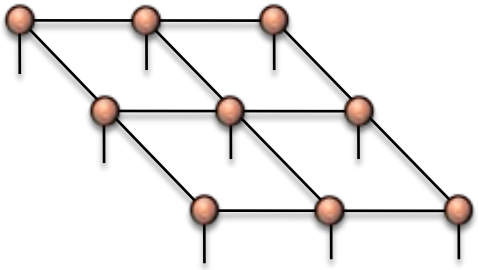
Proposed to understand high-T_C superconductivity

It is a PEPS with D=3



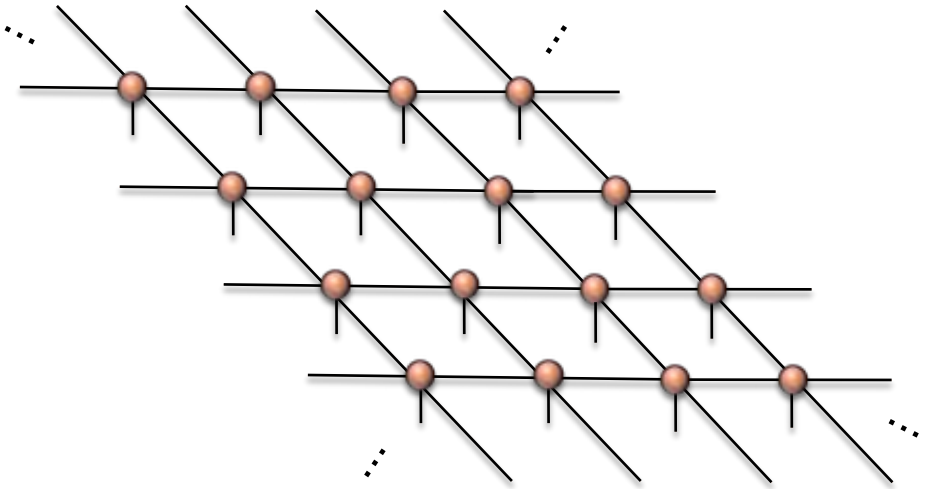
And rotations

PEPS...



F. Verstraete, I. Cirac, cond-mat/0407066

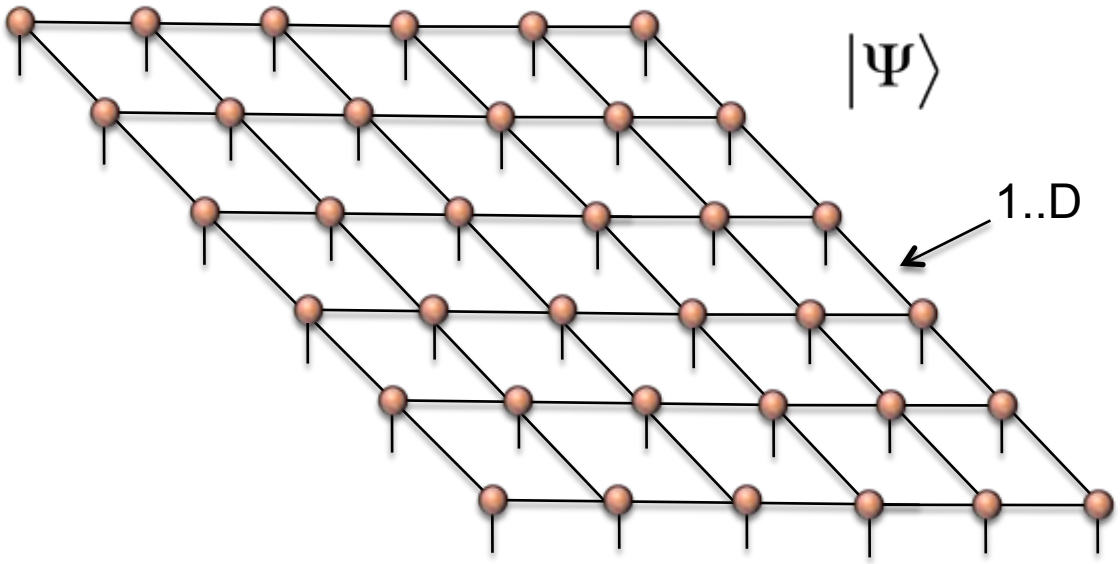
and infinite PEPS (iPEPS)

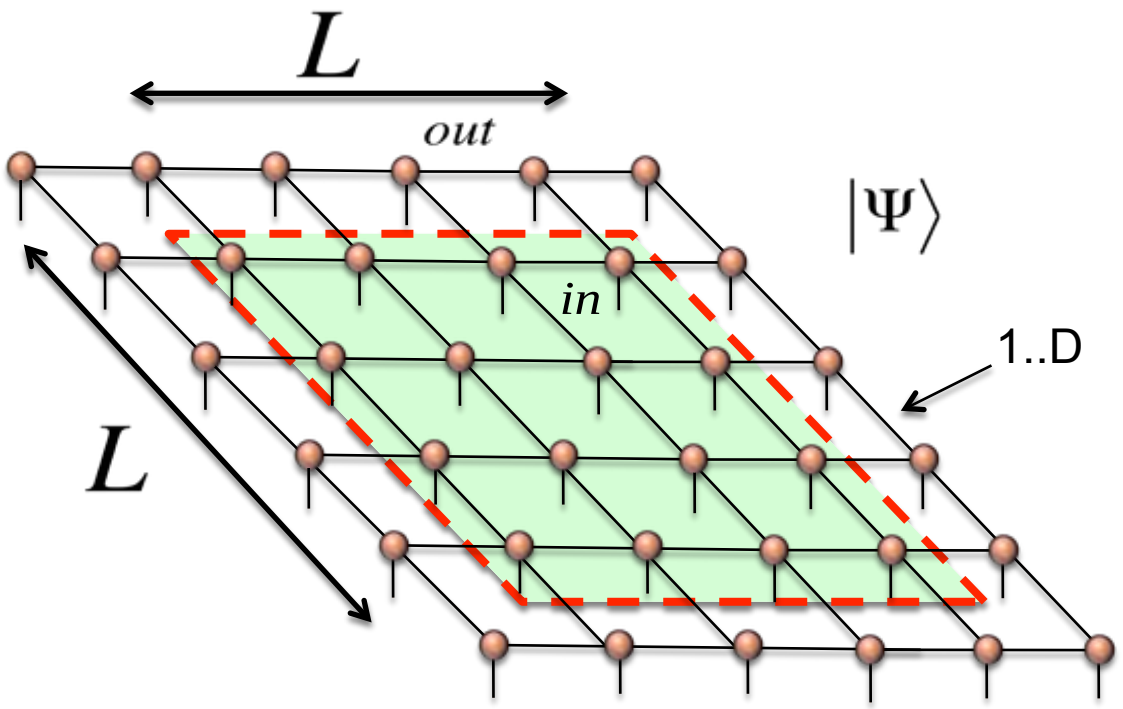


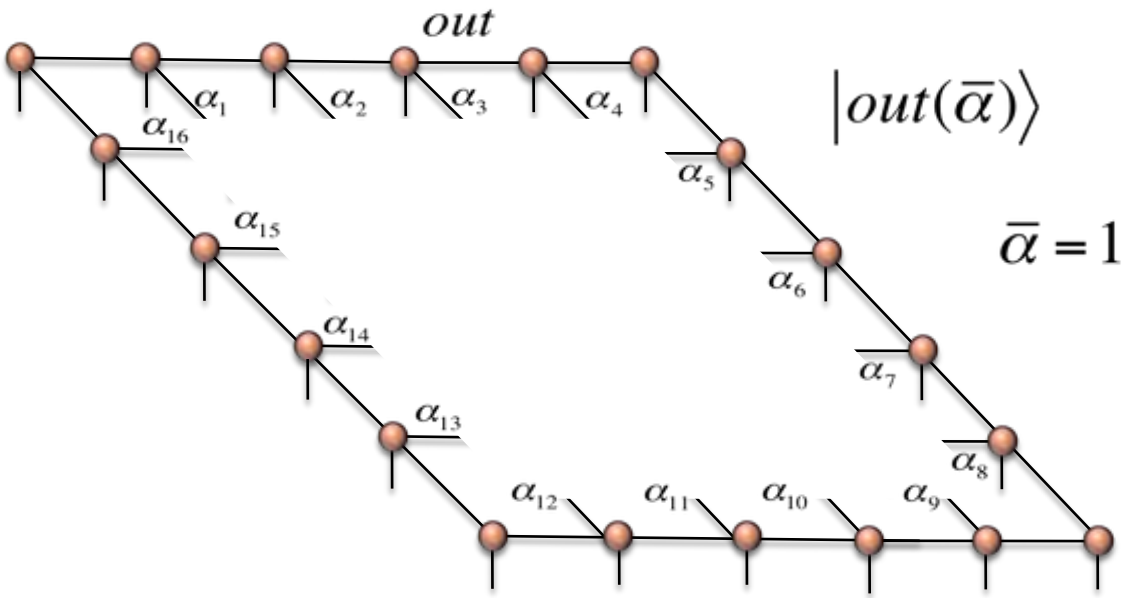
assuming translation invariance

J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, Phys. Rev. Lett. 101, 250602 (2008)

PEPS obey 2d area-law

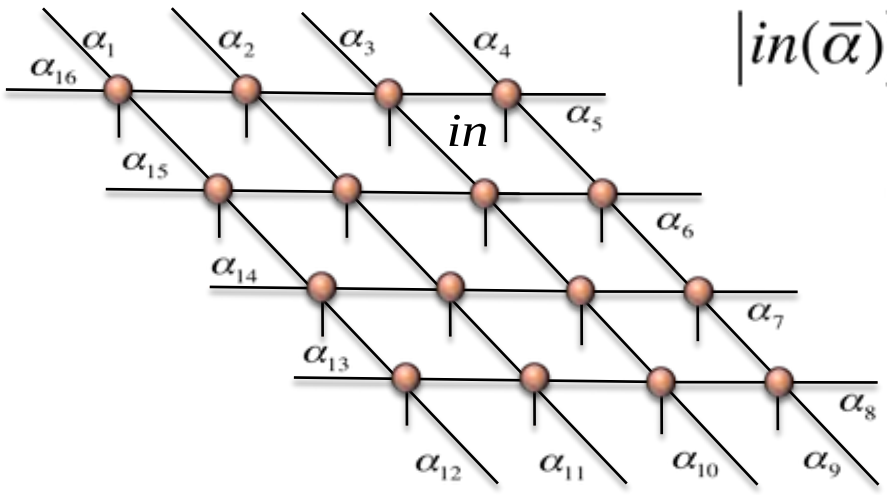

$|\Psi\rangle$
1..D


L
out
$|\Psi\rangle$
in
1..D
L



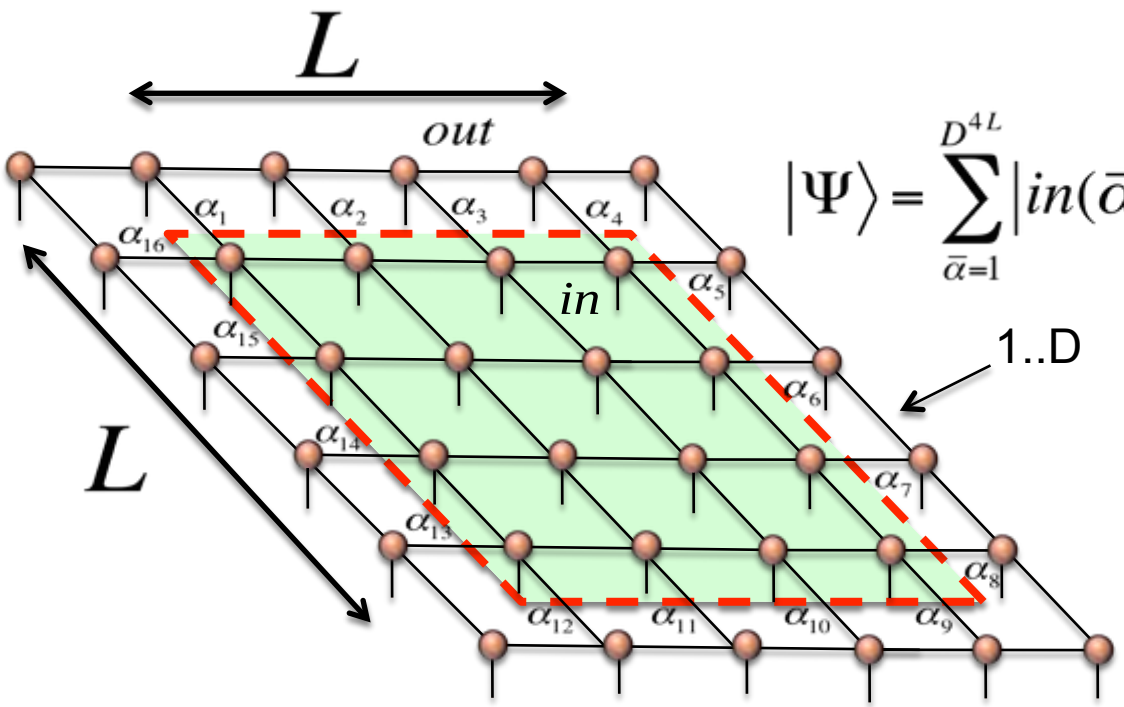
$$|out(\bar{\alpha})\rangle$$

$$\bar{\alpha} = 1,2,\ldots,D^{4L}$$



$$\left|in(\bar{\alpha})\right\rangle$$

$$\bar{\alpha} = 1,2,\ldots,D^{4L}$$



$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$

$$\rho_{in} = \mathrm{tr}_{out}\left(|\Psi\rangle\langle\Psi|\right) = \sum_{\bar{\alpha},\bar{\alpha}'} X_{\bar{\alpha},\bar{\alpha}'} |in(\bar{\alpha})\rangle\langle in(\bar{\alpha}')|$$

$$X_{\bar{\alpha},\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

$$rank(\rho_{in}) \le D^{4L}$$

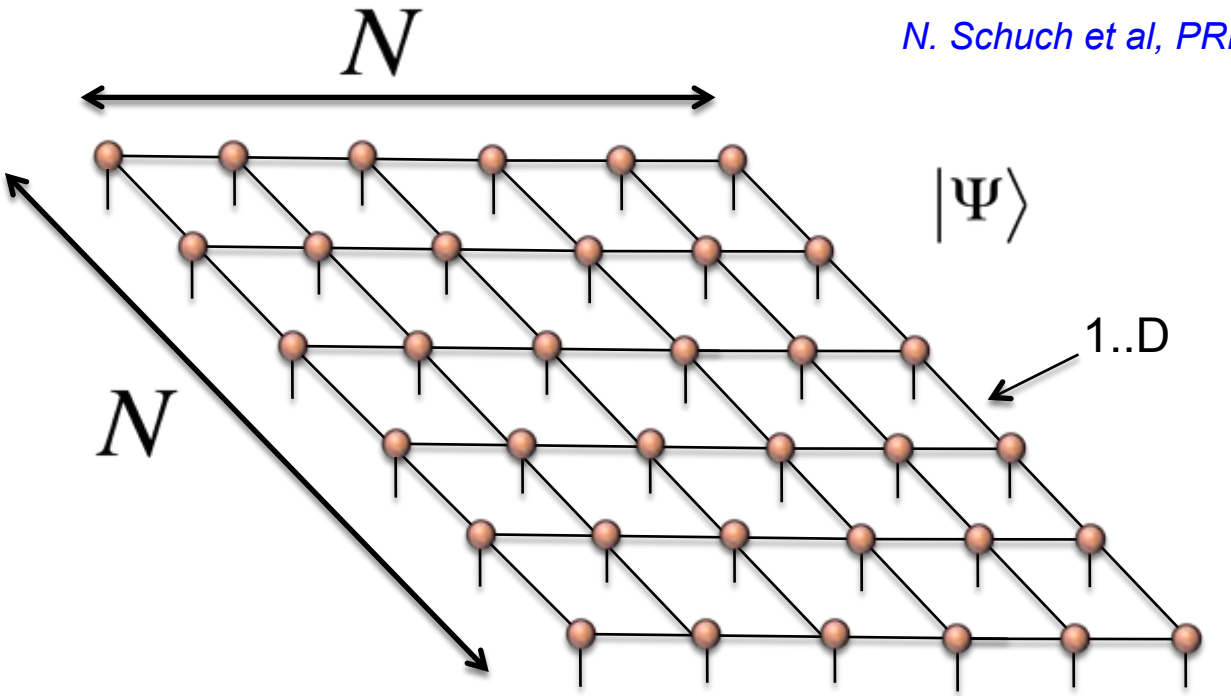
$$S(L) = -\mathrm{tr}\left(\rho_{in}\log\rho_{in}\right) \le \log(D)\,4L$$

prefactor

size of the boundary

Exact contraction is inneficient

N. Schuch et al, PRL 98, 140506 (2007)



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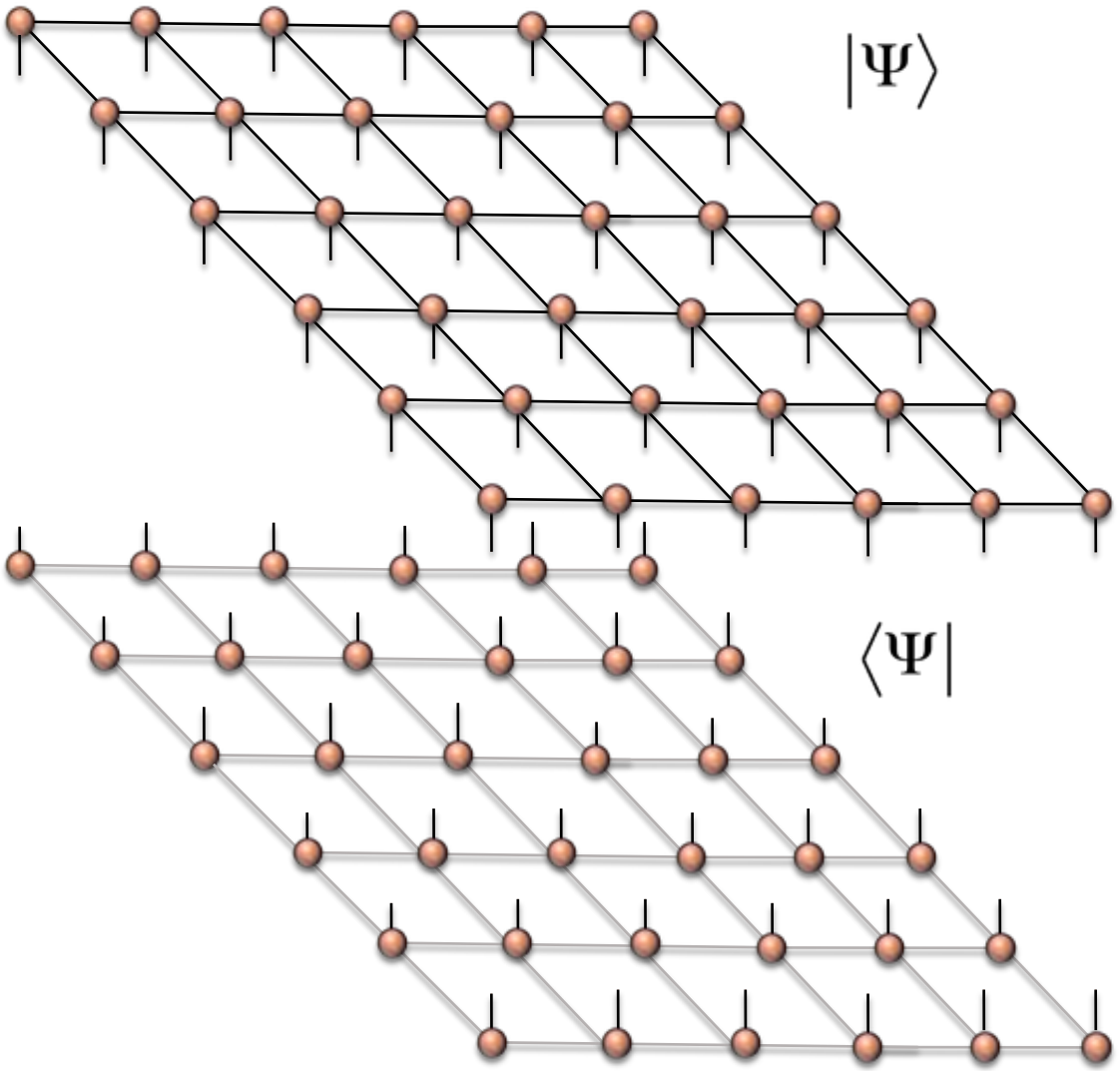

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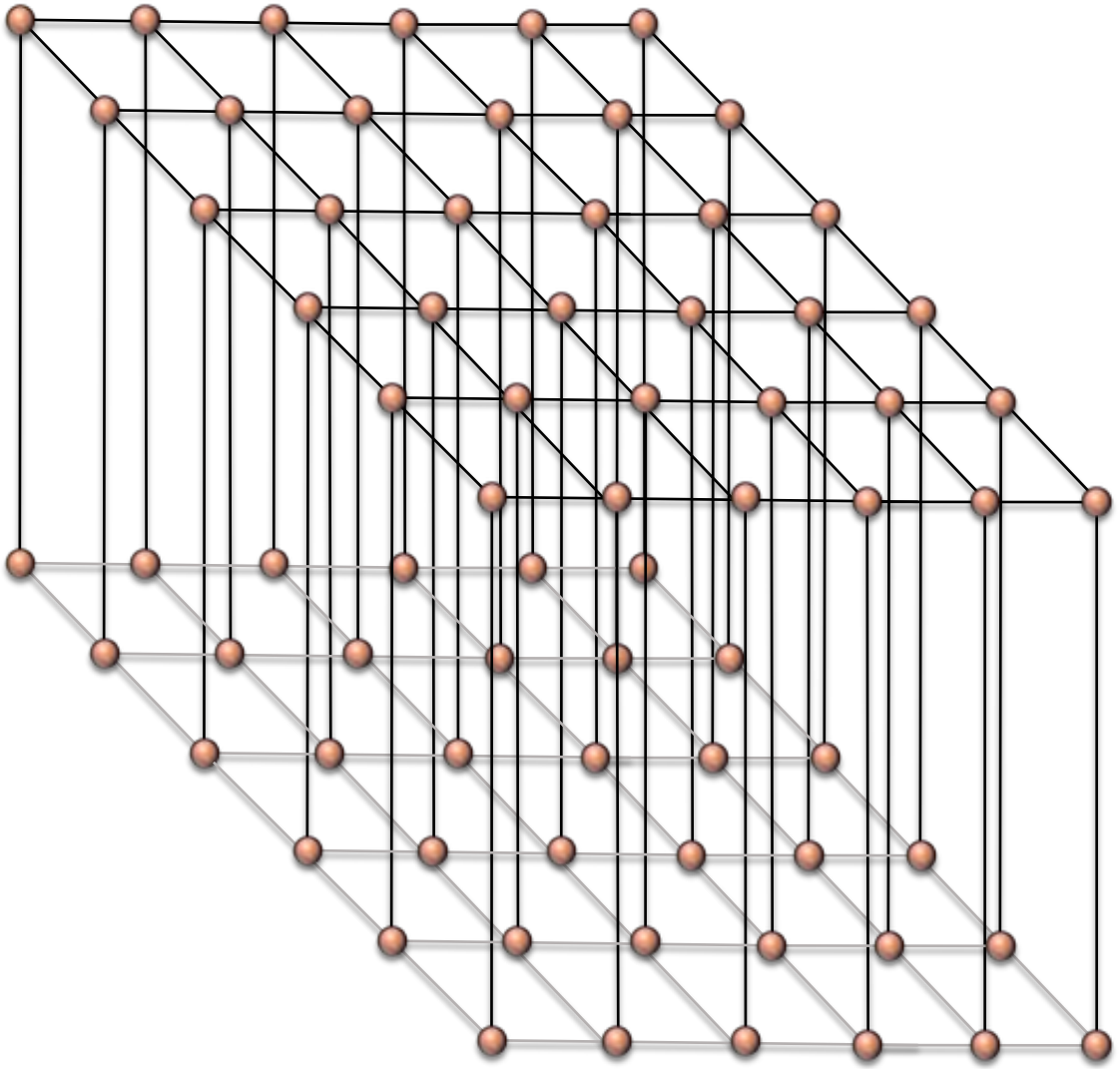

$\langle \Psi | \Psi \rangle$

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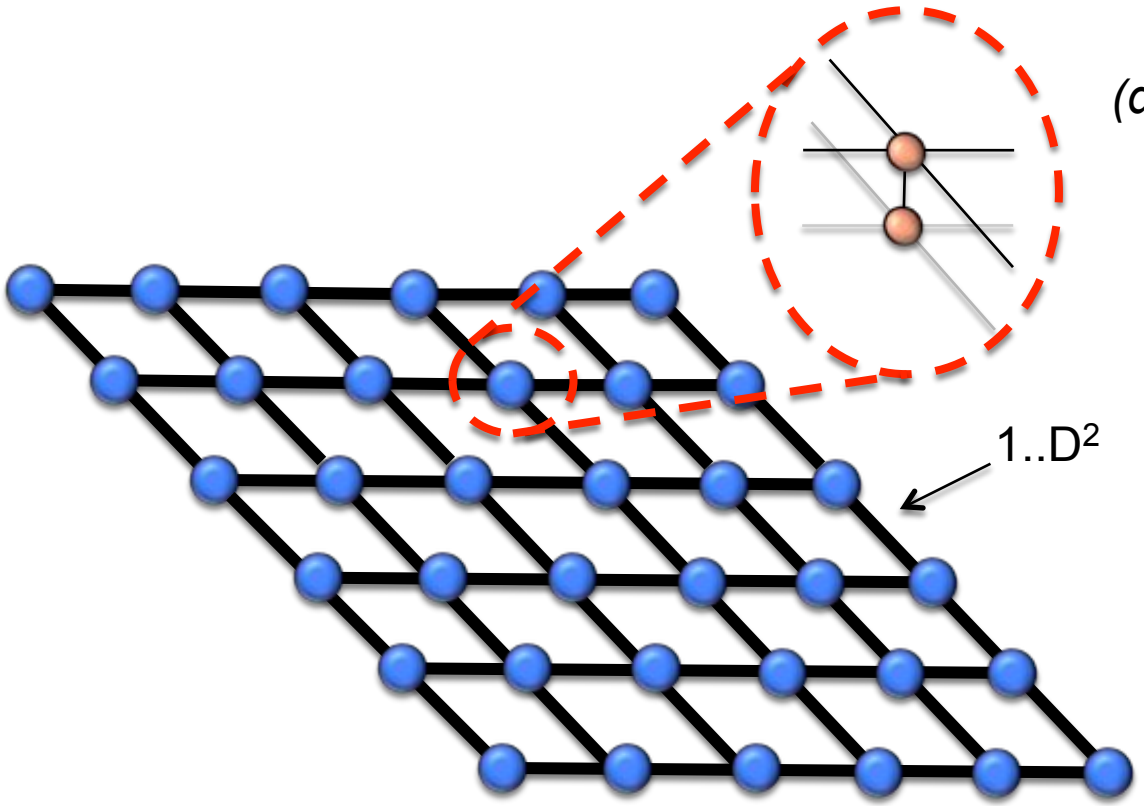

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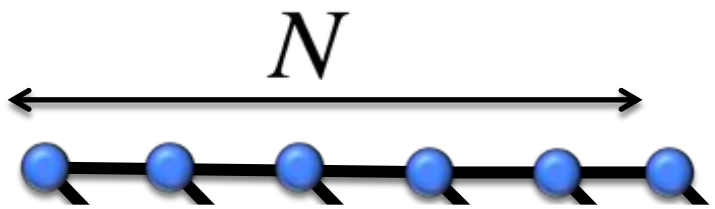

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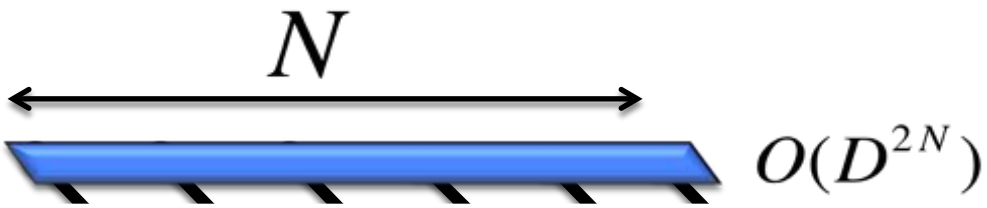

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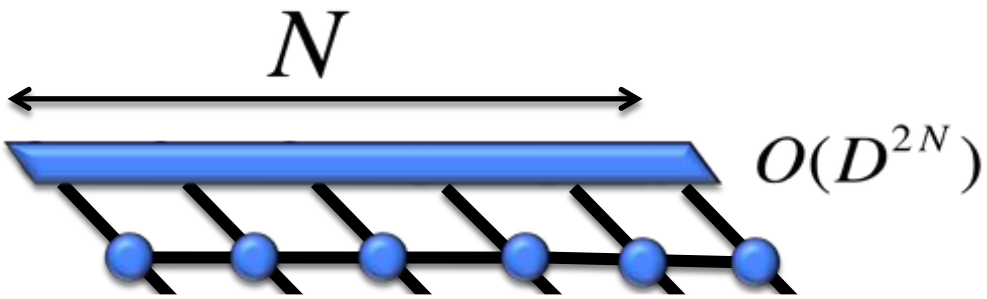

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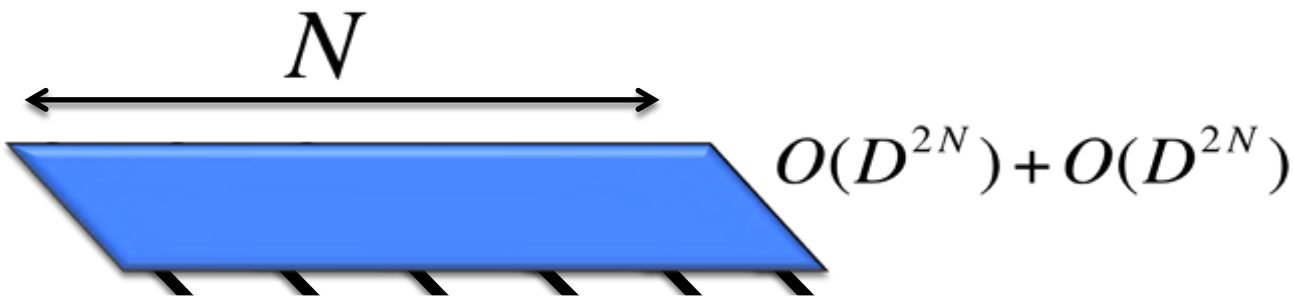

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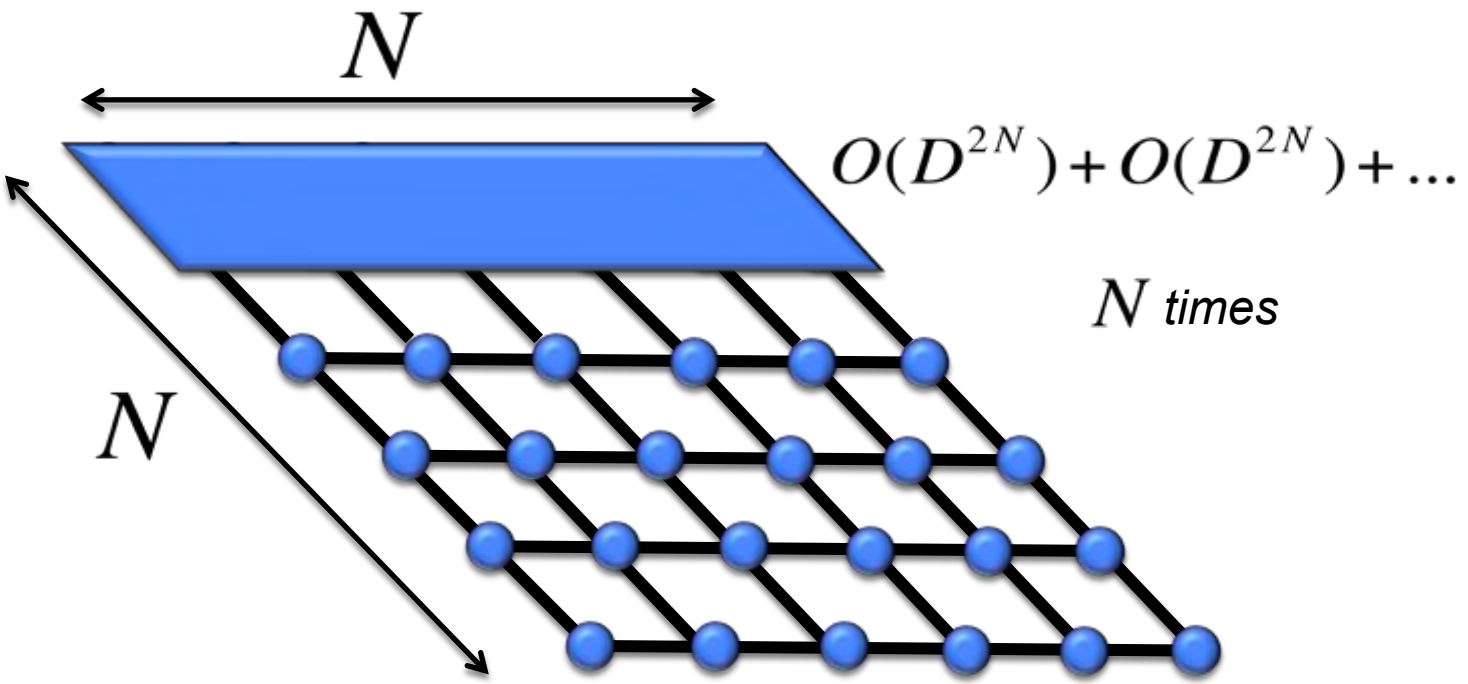


computing time ~ $O(ND^{2N})$ **Exponential amount of time!**

Mathematical statement: *exact contraction of a PEPS is a #P-Hard problem (harder than NP-Complete)*

Applies also to expectation values of observables

Critical correlation functions

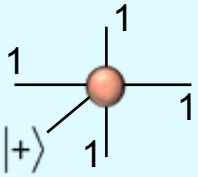
F. Verstraete et al, PRL 96, 220601 (2006)

$$|\Psi(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \exp\left(\frac{\beta}{2}\sum_{\langle i,j\rangle}\sigma_z^i\sigma_z^j\right)|+,+\ldots+\rangle$$

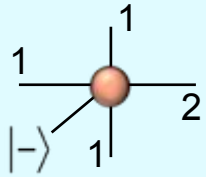
Expectation values are those of the classical 2d Ising model

$$\langle\sigma_z^r\sigma_z^{r'}\rangle_\beta = \frac{1}{Z(\beta)}\sum_{\{s\}} s^r s^{r'} \exp\left(\beta\sum_{\langle i,j\rangle} s^i s^j\right) \qquad s=\pm 1$$

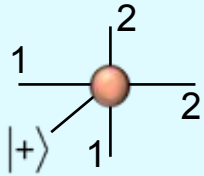
It is a PEPS with D=2 *(left as exercise):*



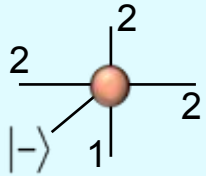
$= (\cosh(\beta/2))^4$



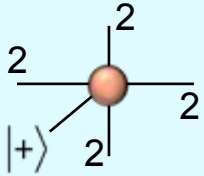
$= (\cosh(\beta/2))^3(\sinh(\beta/2))$



$= (\cosh(\beta/2))^2(\sinh(\beta/2))^2$



$= (\cosh(\beta/2))(\sinh(\beta/2))^3$



$= (\sinh(\beta/2))^4$ + permutations

At $\beta_c = \left(\log(1+\sqrt{2})\right)/2$ ***the correlation length is infinite:*** $\langle\sigma_z^r\sigma_z^{r'}\rangle_{\beta_c} \approx \frac{a}{|r-r'|^{1/4}}$

PEPS to/from Hamiltonians

World of Hamiltonians

Ground state of a gapped Parent Hamiltonian with local interactions

$$H_{parent} = \sum h_{local}$$

Hamiltonian with local interactions

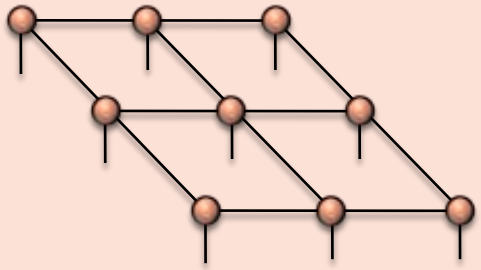
$$H' = \sum h'_{local}$$

e.g. D. Perez-Garcia et al, QIC 8, 0650 (2008)

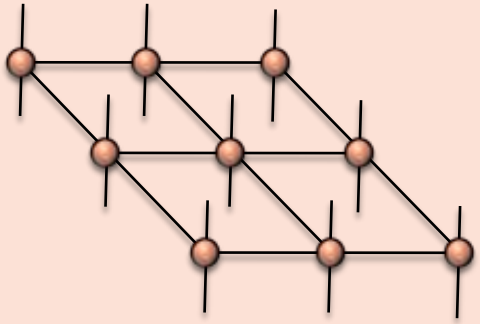
M. Hastings, PRB 73, 085115 (2006)

World of states

Generic PEPS with finite D

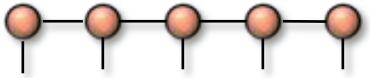


Thermal states can be approx. by a PEPS with finite D



PEPS target the relevant corner of the Hilbert space (area-law)

MPS vs PEPS



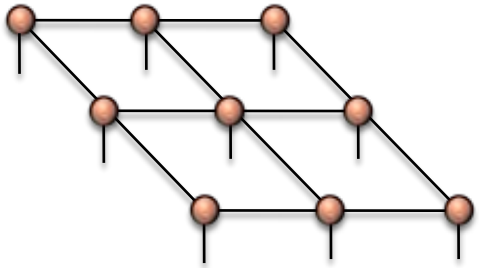
MPS in 1d

1d area law $S(L) = O(1)$

Exact contraction is efficient

Finite correlation length

to/from 1d Hamiltonians



PEPS in 2d

2d area law $S(L) = O(L)$

Exact contraction is inefficient

Finite and infinite correlation lengths

to/from 2d Hamiltonians

Outline

1) Generalities

2) 2d PEPS

- 2.1) Basics
- 2.2) Computing
- 2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

Outline

1) Generalities

2) 2d PEPS

- 2.1) Basics
- 2.2) Computing
- 2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

PEPS as ansatz: variational optimization

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \cdots \xrightarrow{\min_{A^N}} |\Psi^N\rangle \xrightarrow{\min_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min_{A^{N-2}}} \cdots$$

$$E \geq E^1 \geq \cdots \geq E^N \geq E^{N+1} \geq \cdots$$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0$$

(A^{*i} and A^i)

Minimization of quadratic function

$$\mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

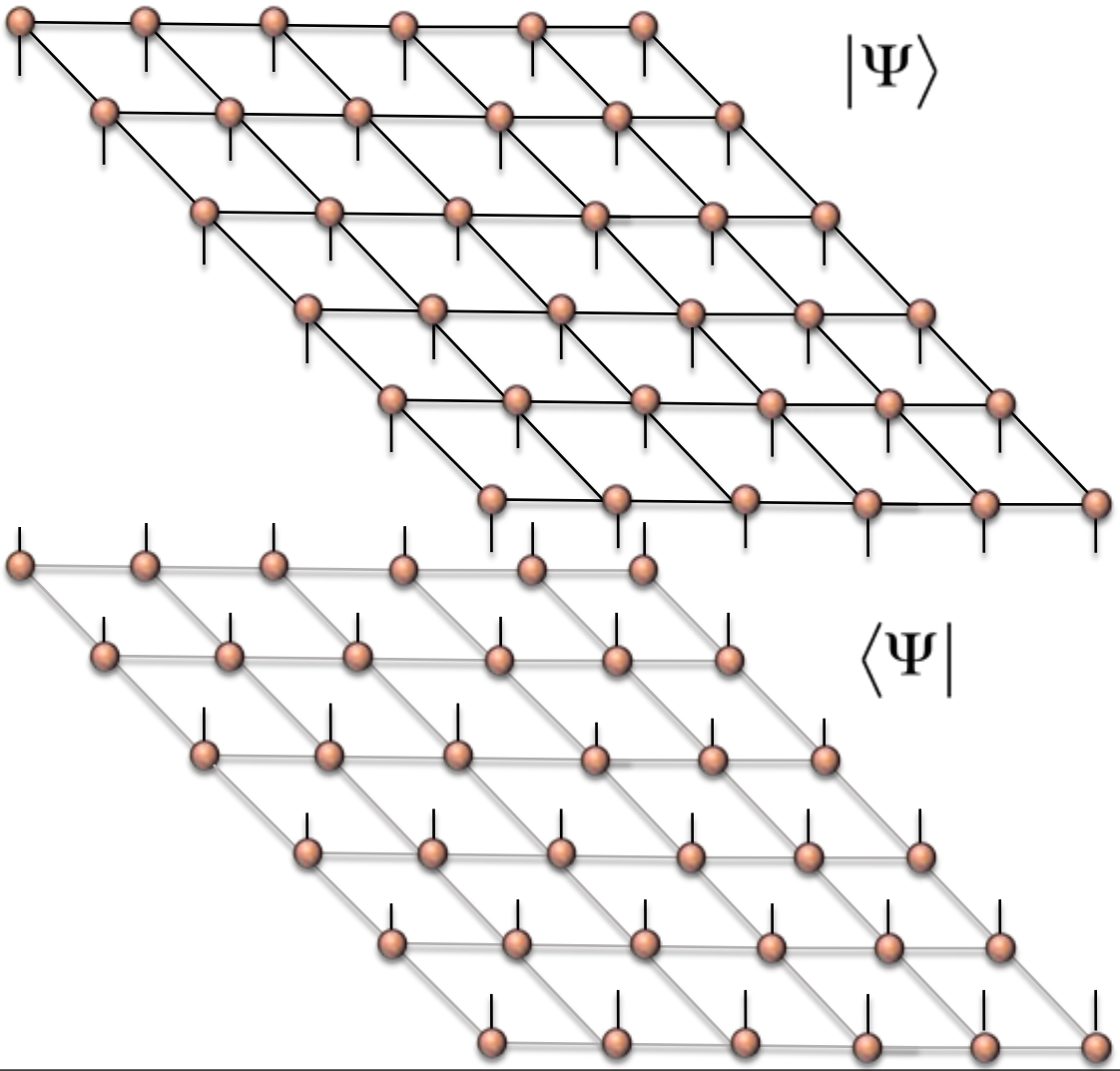
Generalized eigenvalue problem

Once $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$ are known, we can solve this problem efficiently

Approximate calculation of $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$



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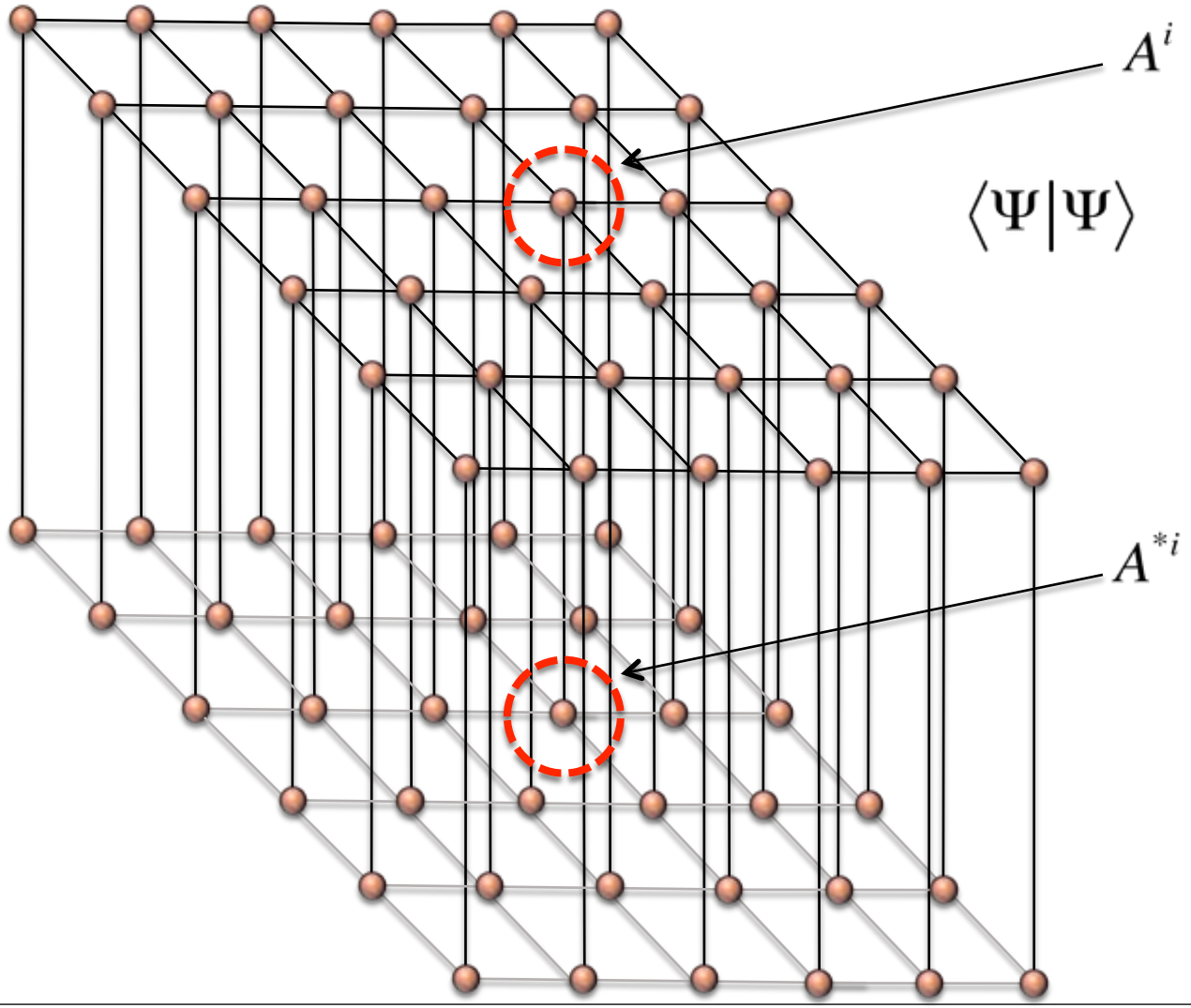

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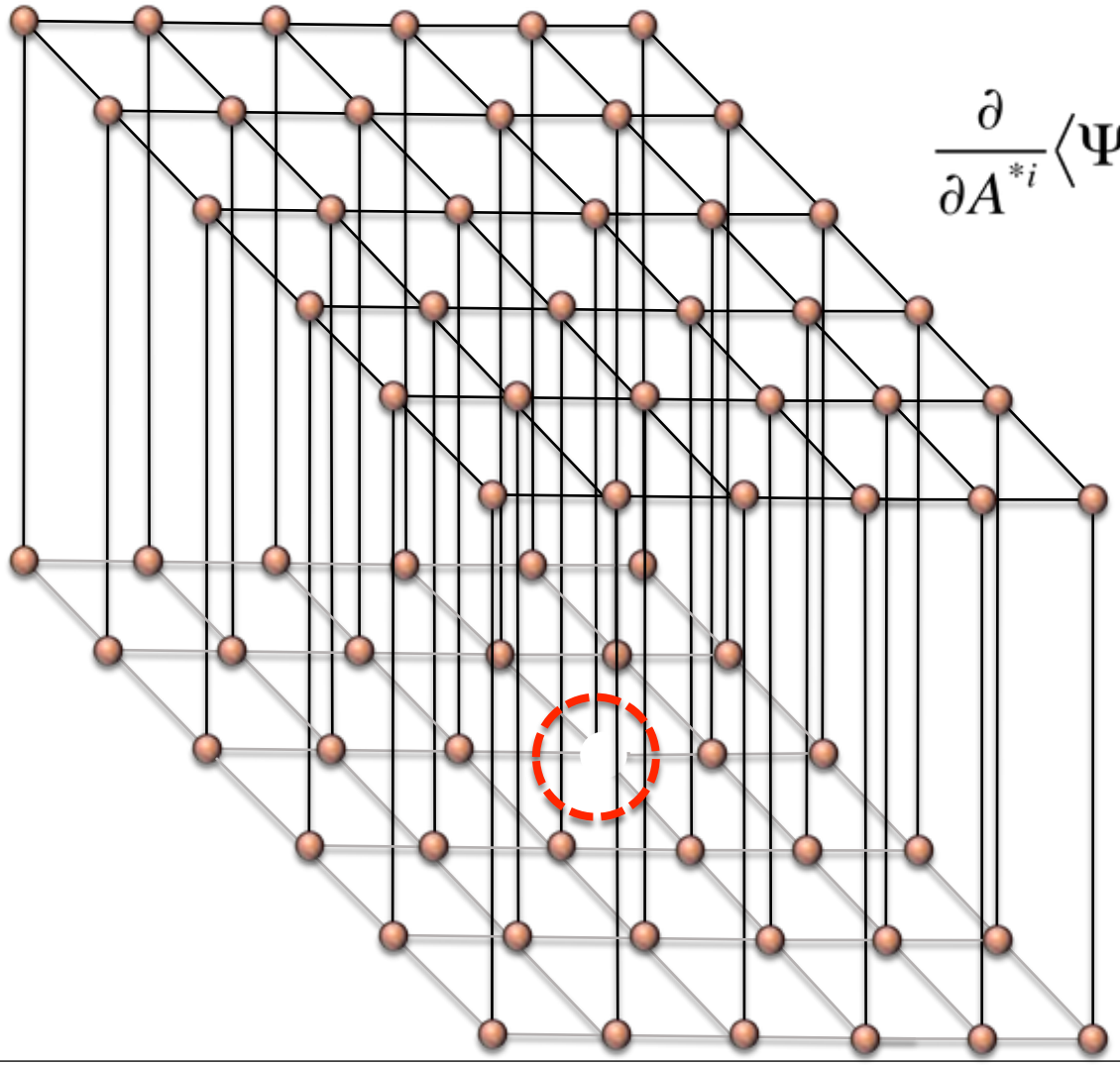

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

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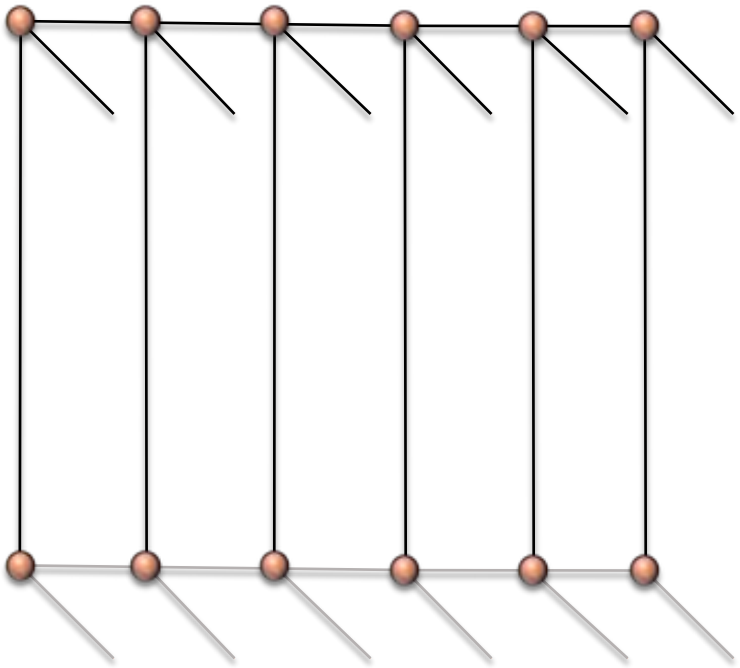

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MPS

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

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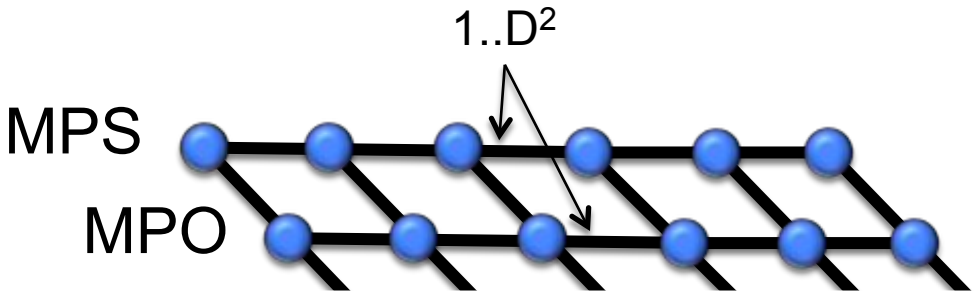
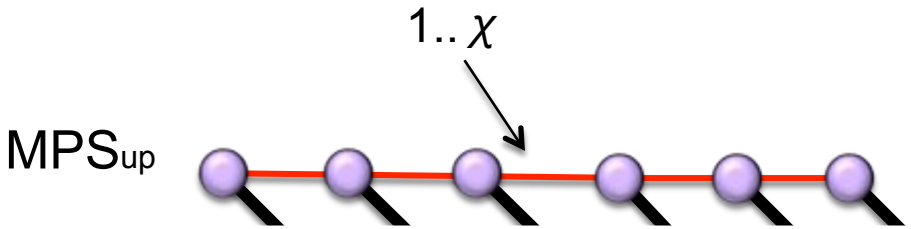


$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

1d problem: use a 1d method for MPS (e.g., DMRG or TEBD)

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



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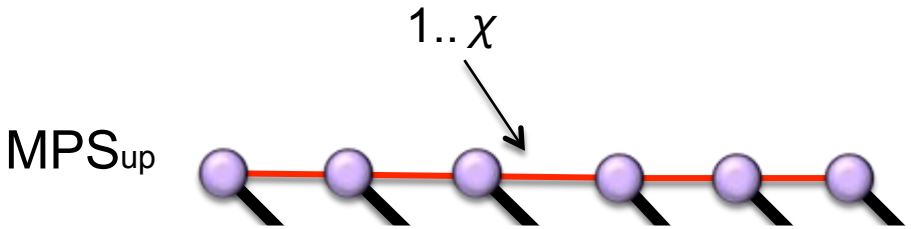

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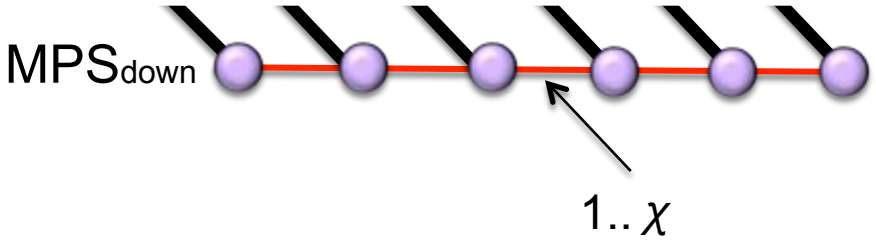

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$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

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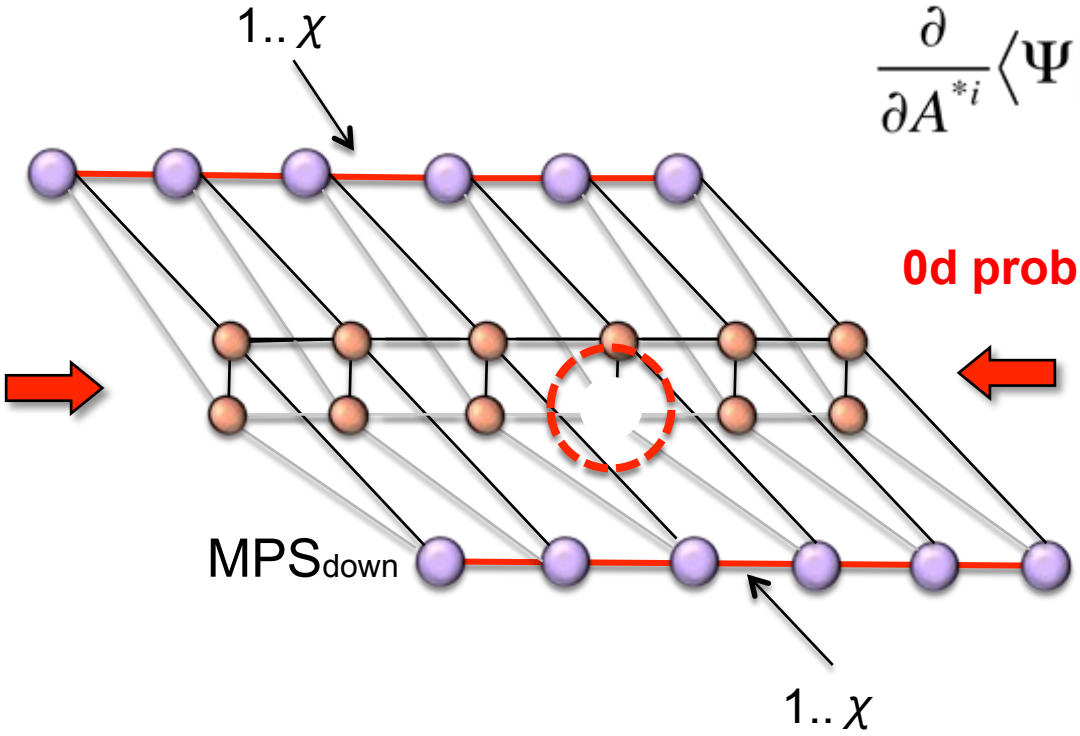


0d problem: exact!

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

Dimensional reduction

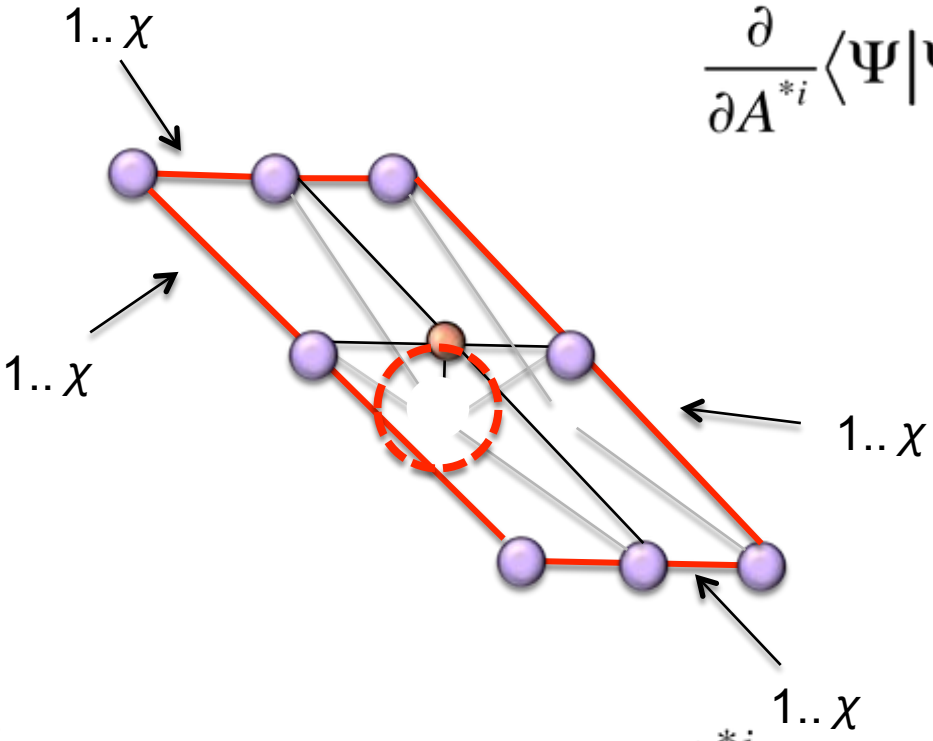
2d problem

↓

1d problem: use DMRG!

↓

0d problem: exact!



$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

$\mathbf{N}^i \vec{A}^i$ is the ***environment*** of tensor A^{*i}

$\mathbf{H}^i_{eff}\vec{A}^i$ is computed similarly, but sandwitching with the Hamiltonian

Valid also for any expectation value

Time evolution

(real, imaginary)

Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \frac{e^{-\tau H}|\Psi\rangle}{\left\| e^{-\tau H}|\Psi\rangle \right\|}$$

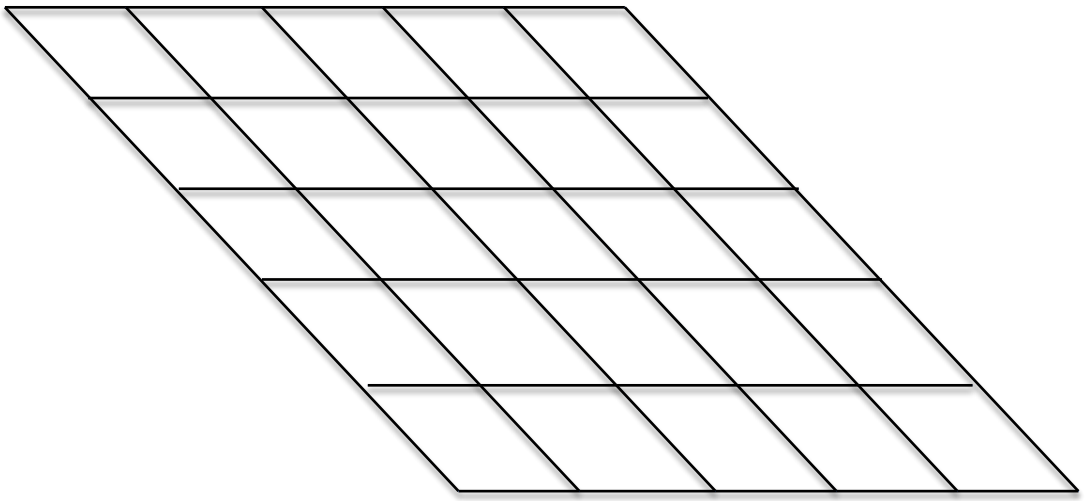
Divide into small time-steps $\delta\tau << 1$

$$|\Psi\rangle \xrightarrow{e^{-\delta\tau H}} |\Psi^1\rangle \xrightarrow{e^{-\delta\tau H}} \cdots \xrightarrow{e^{-\delta\tau H}} |\Psi^m\rangle \xrightarrow{e^{-\delta\tau H}} |\Psi^{m+1}\rangle \xrightarrow{e^{-\delta\tau H}} \cdots$$

$$E \geq E^1 \geq \cdots \geq E^m \geq E^{m+1} \geq \cdots$$

Split the Hamiltonian (e.g. 2-body n.n.)

$$H = H_{hor}^{even} + H_{hor}^{odd} + H_{ver}^{even} + H_{ver}^{odd}$$



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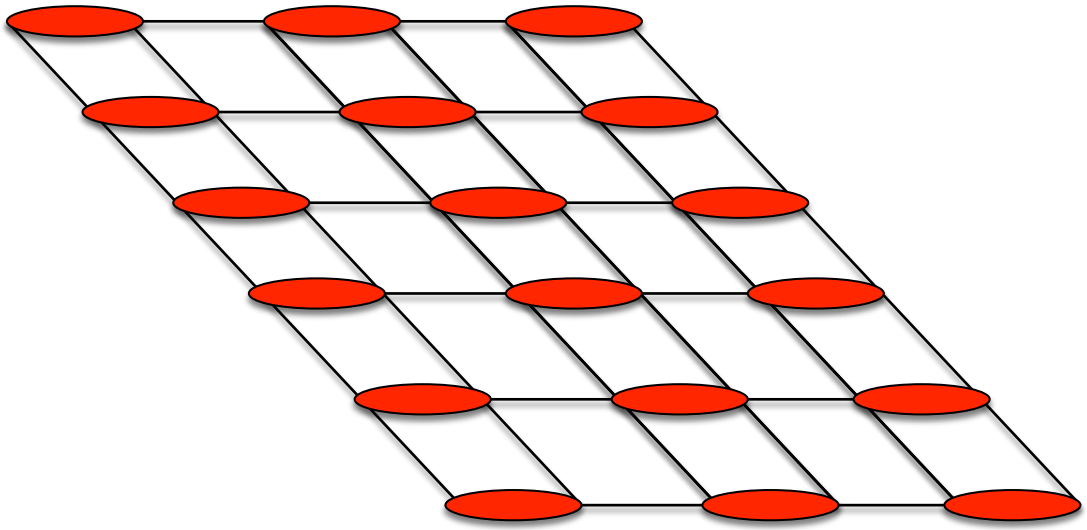

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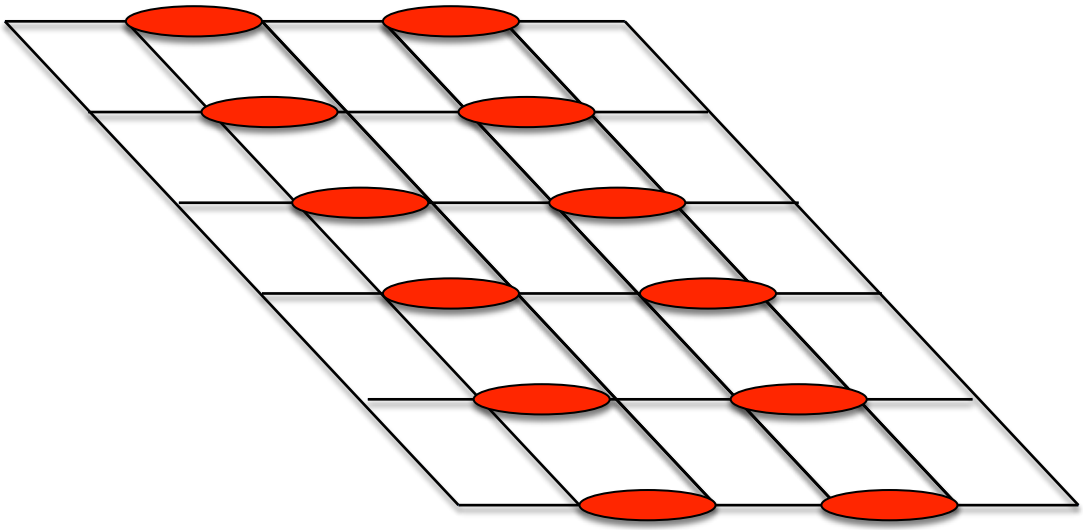

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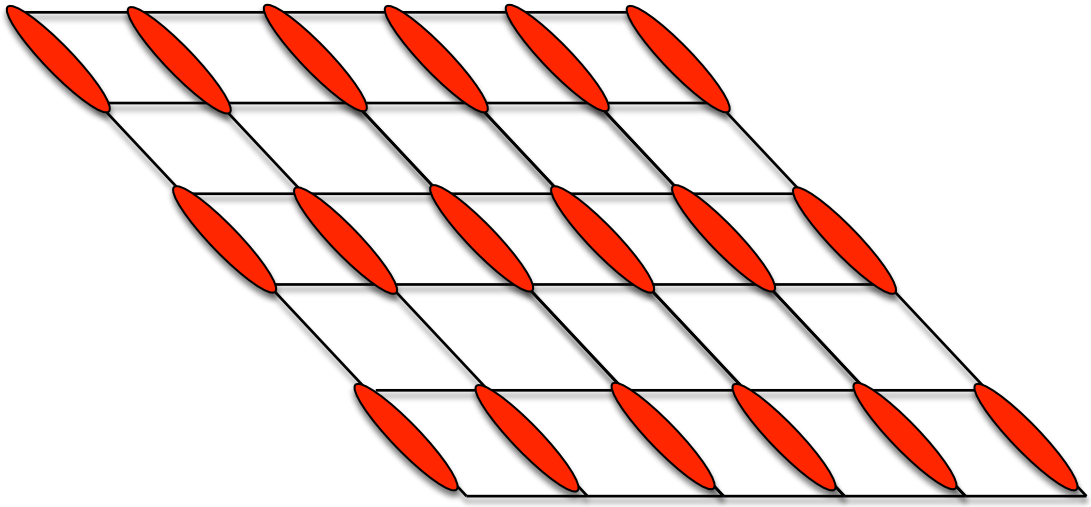

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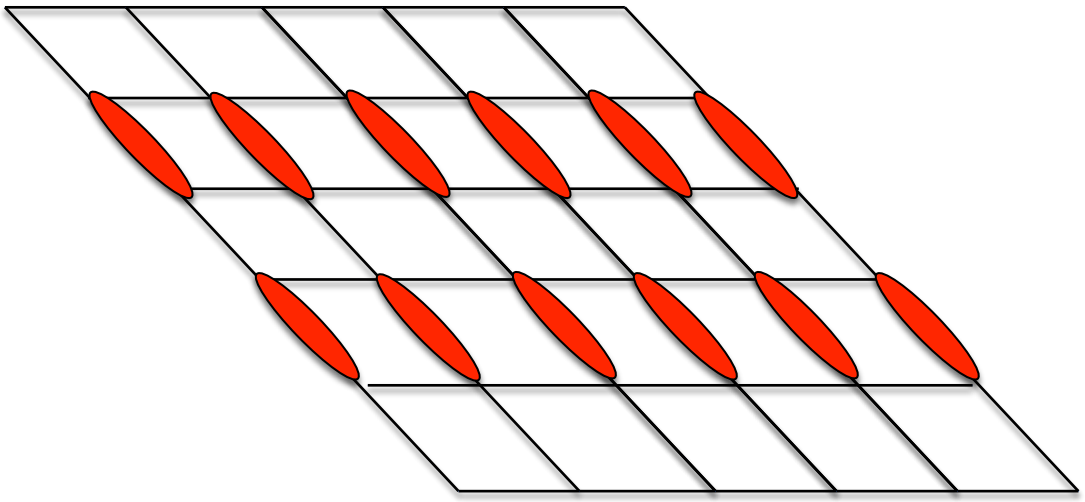

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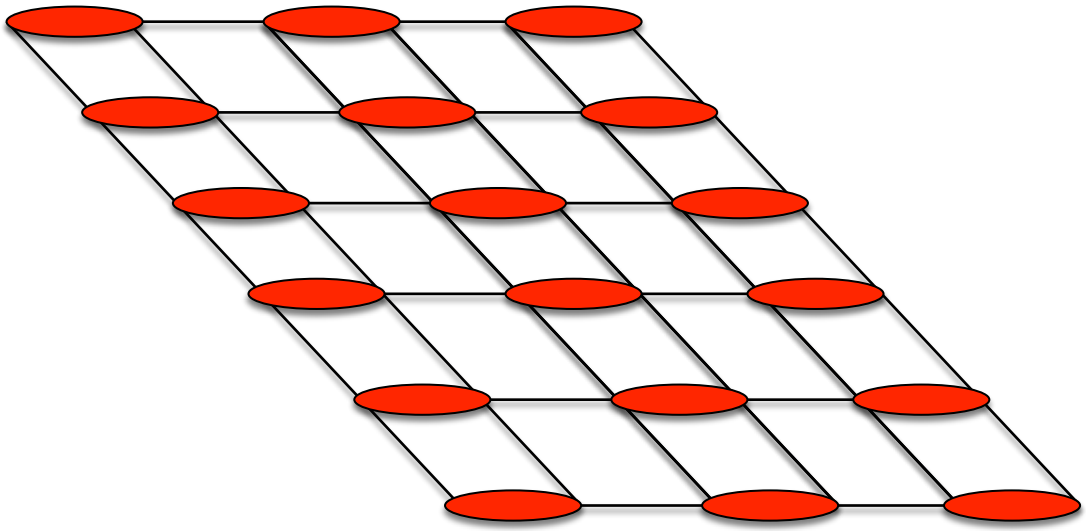
$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}}$$



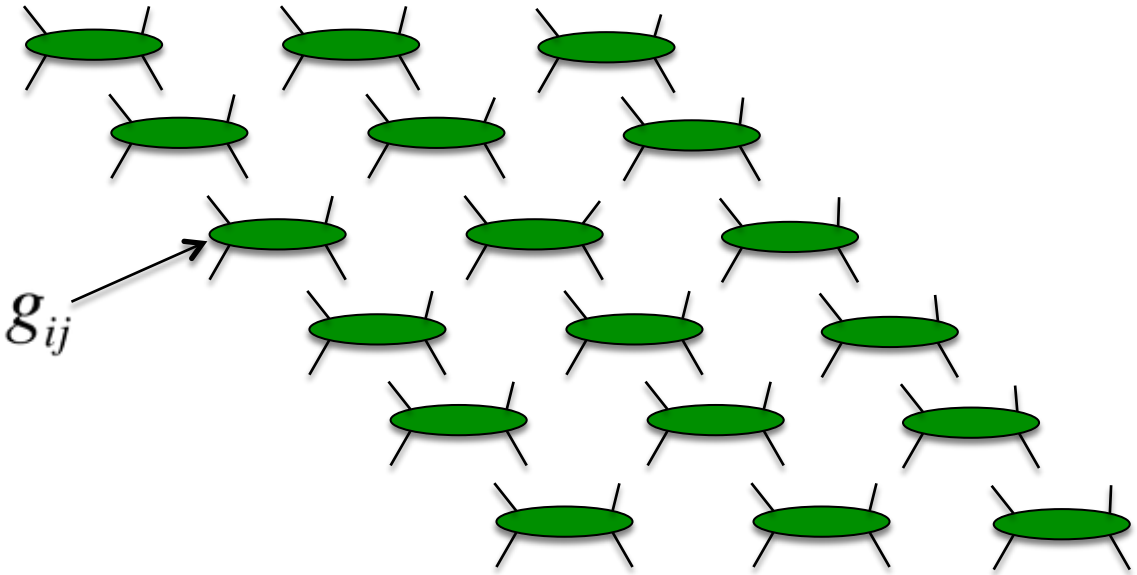
Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}} = \bigotimes_{<i,j>} e^{-\delta\tau h_{ij}} = \bigotimes_{<i,j>} g_{ij}$$

2-body gates



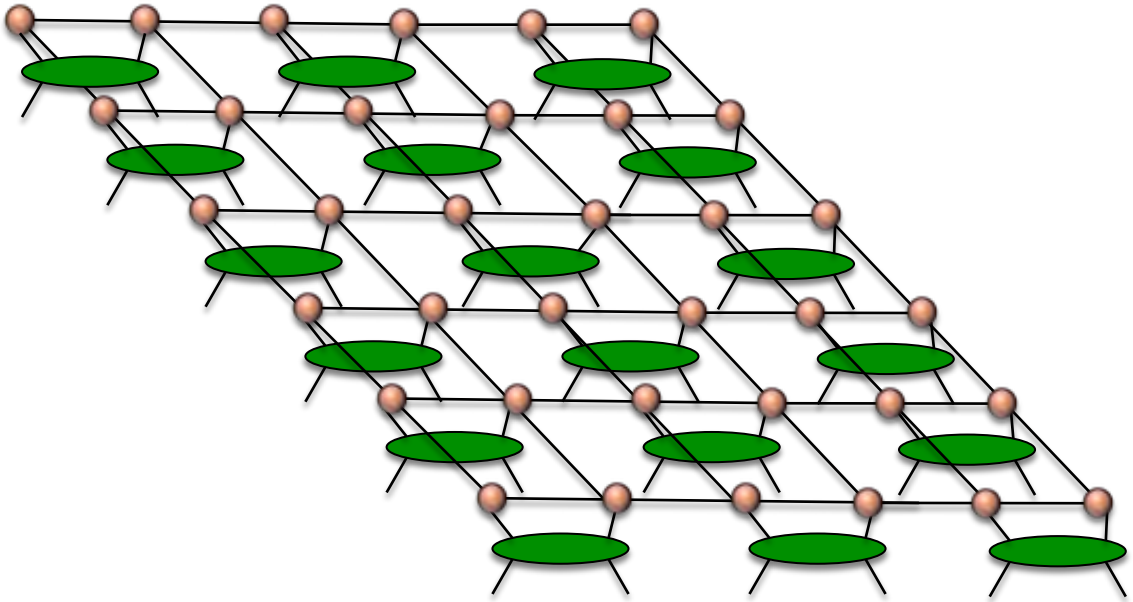
Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}} \left|\Psi\right\rangle = \left|\tilde{\Psi}\right\rangle$$

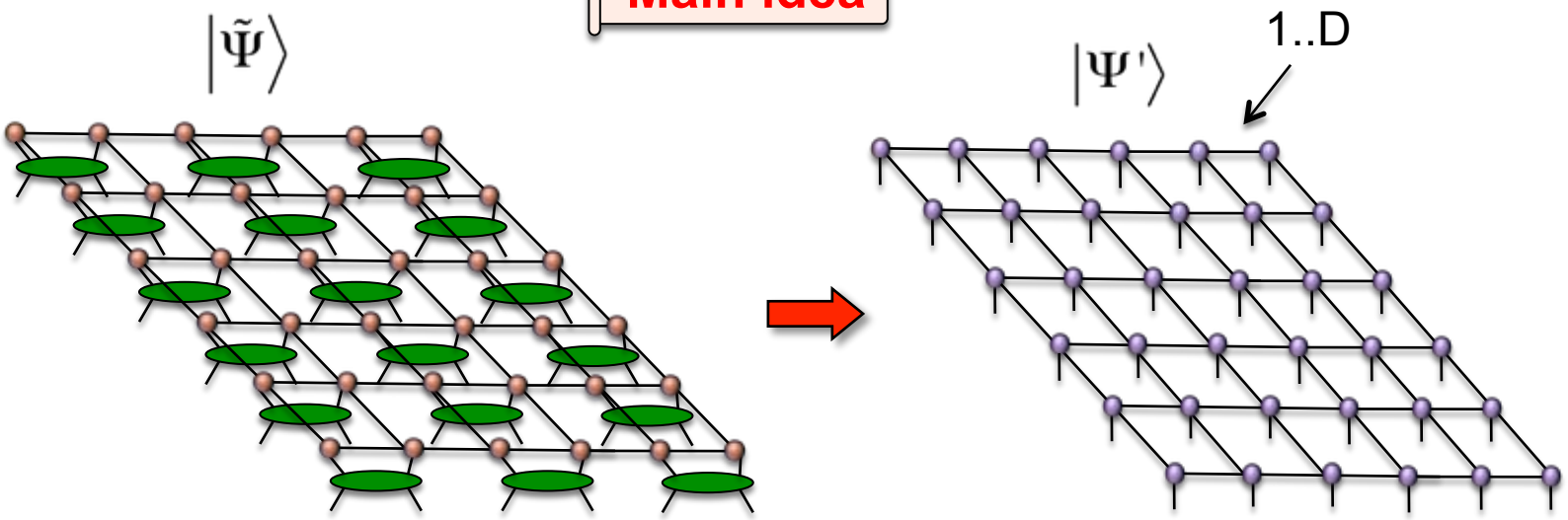
evolved state



Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

Main idea



Different approaches to this problem: full update, simplified update, TPVA...

Full update: $\min \left\| |\tilde{\Psi}\rangle - |\Psi'\rangle \right\|^2$

Finite systems: optimize over all tensors in the PEPS (as before)

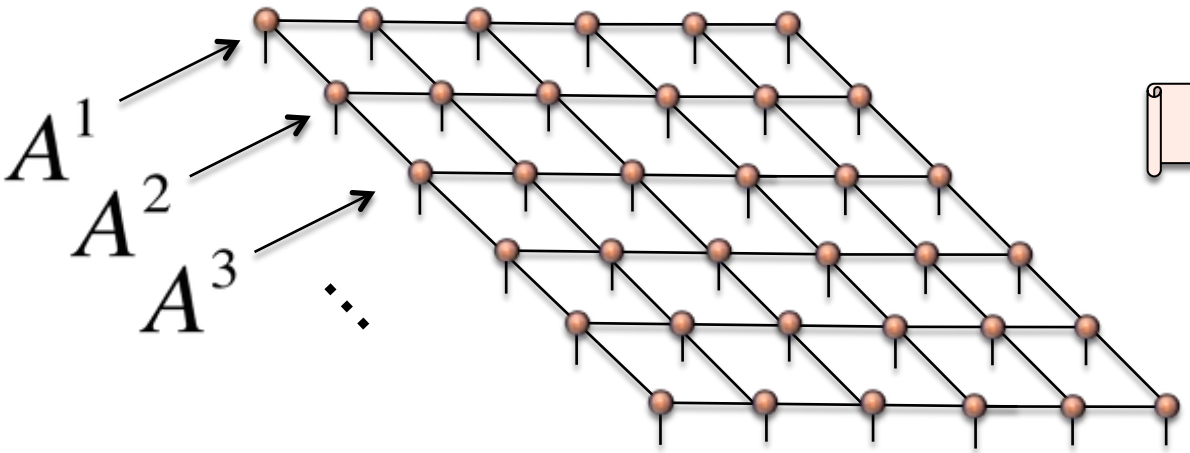
Infinite systems: optimize over tensors in the PEPS unit cell (iPEPS)

Require ***calculations of environments***, like the one shown before.

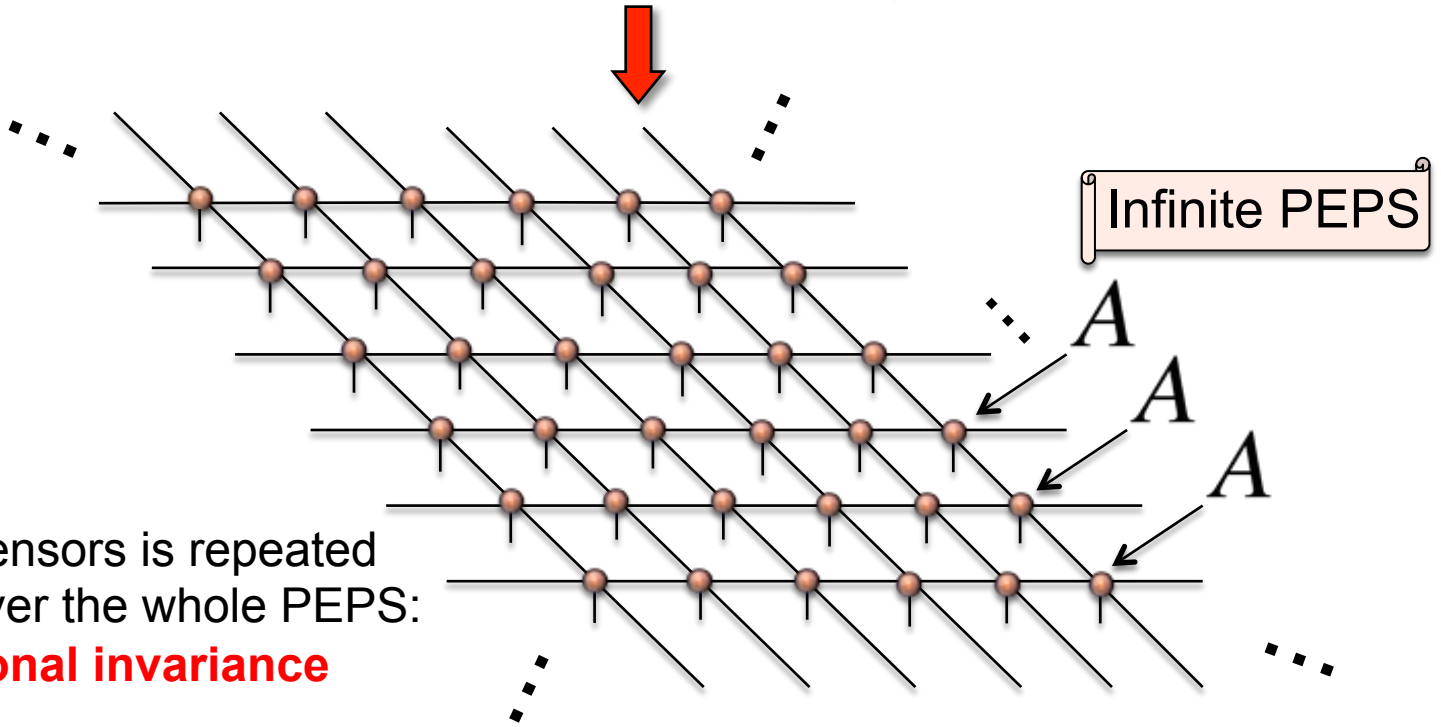
Environments with infinite PEPS

Infinite systems

e.g. J. Jordan et al, PRL 101, 250602 (2008)
R. Orús, G. Vidal, PRB 80 094403 (2009)



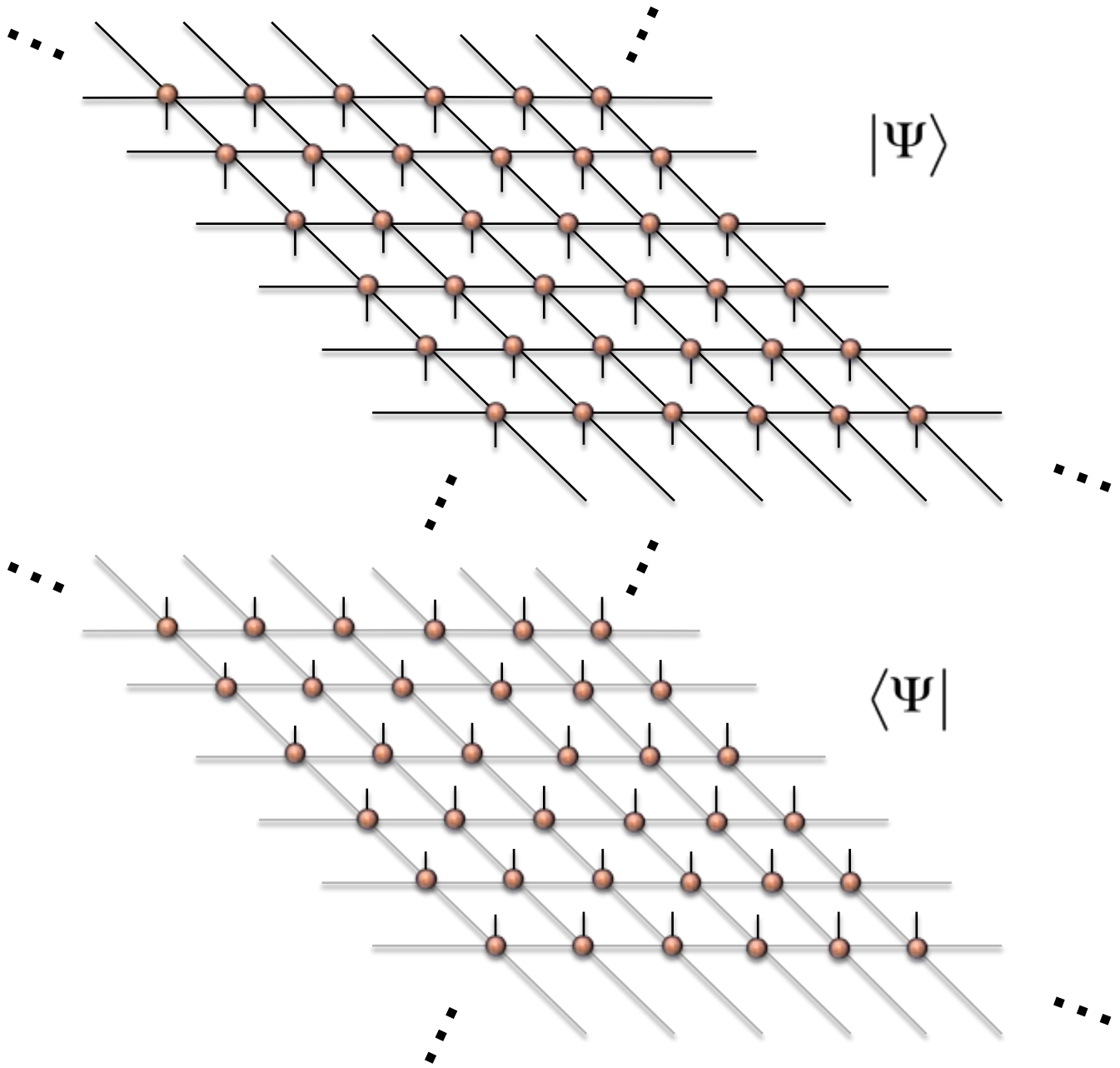
Finite PEPS

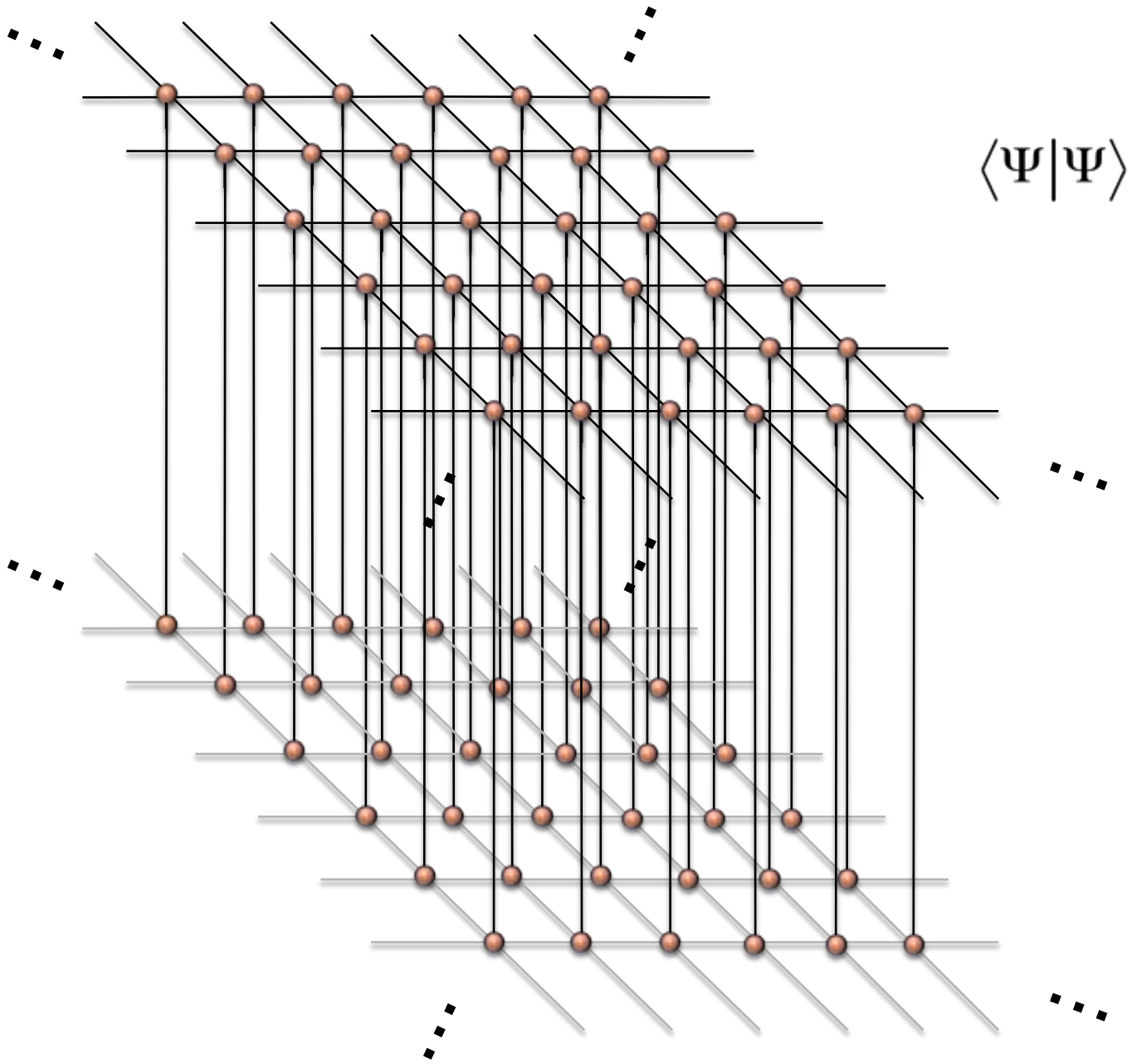


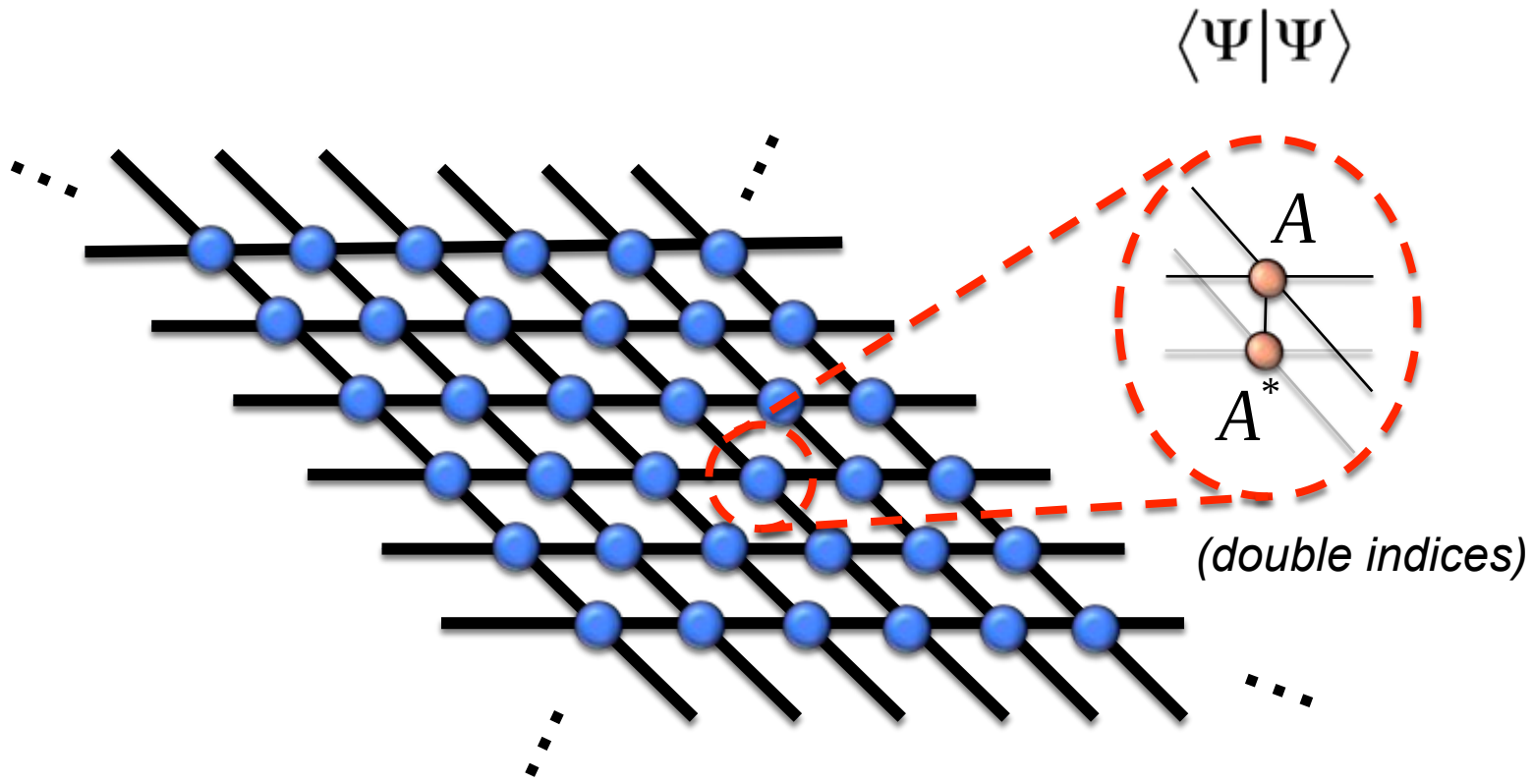
Infinite PEPS

Unit cell of tensors is repeated periodically over the whole PEPS:

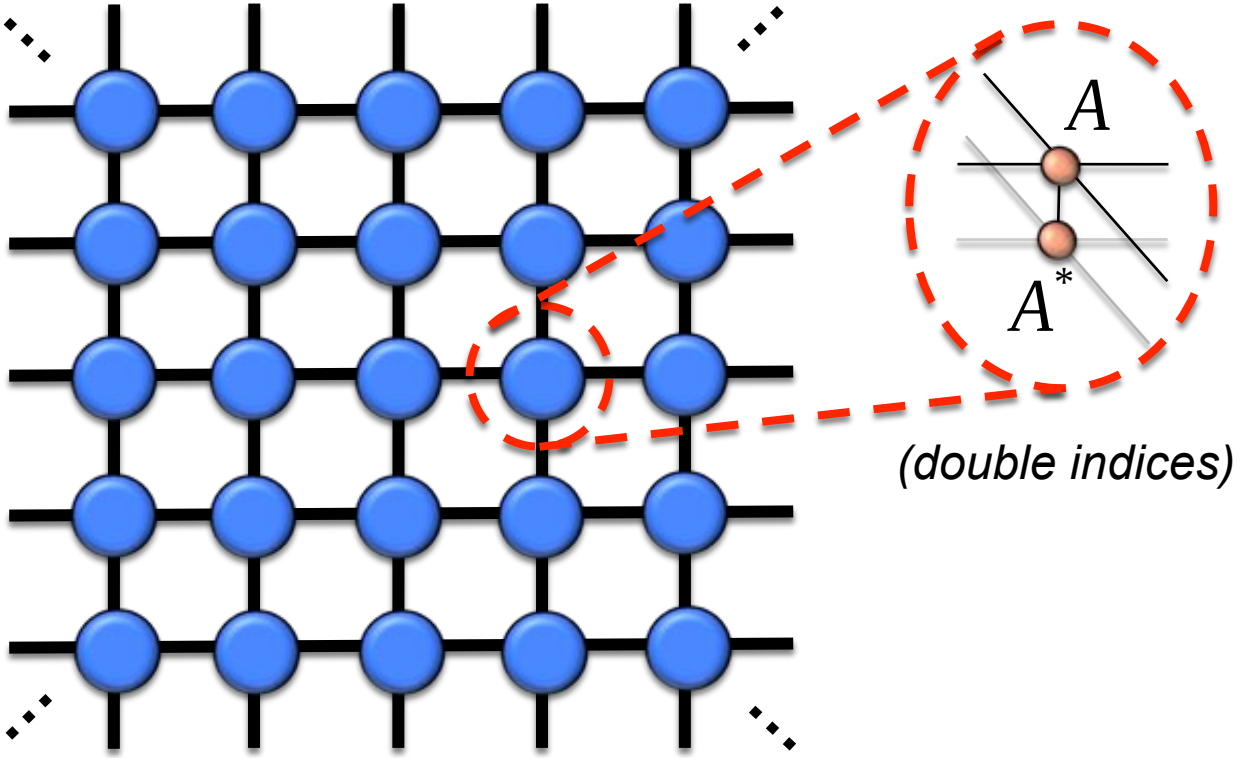
translational invariance


$|\Psi\rangle$
$\langle\Psi|$


$\langle \Psi | \Psi \rangle$

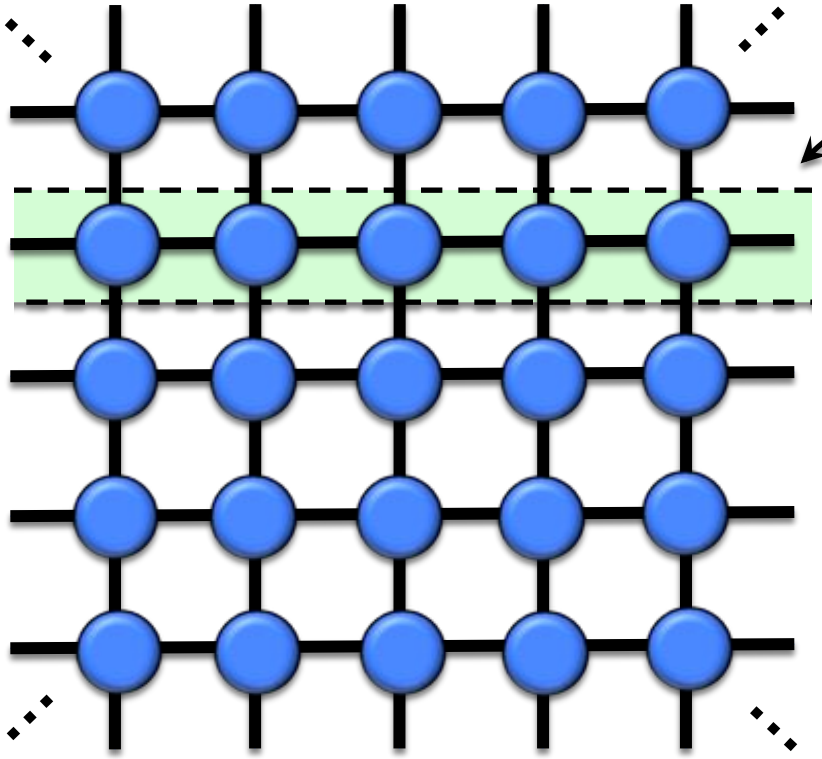


Let's put it on the plane of the screen!



Environment calculations ⟺ ***Contraction of this infinite lattice***

Contracting the infinite 2d lattice



1-dim transfer matrix:

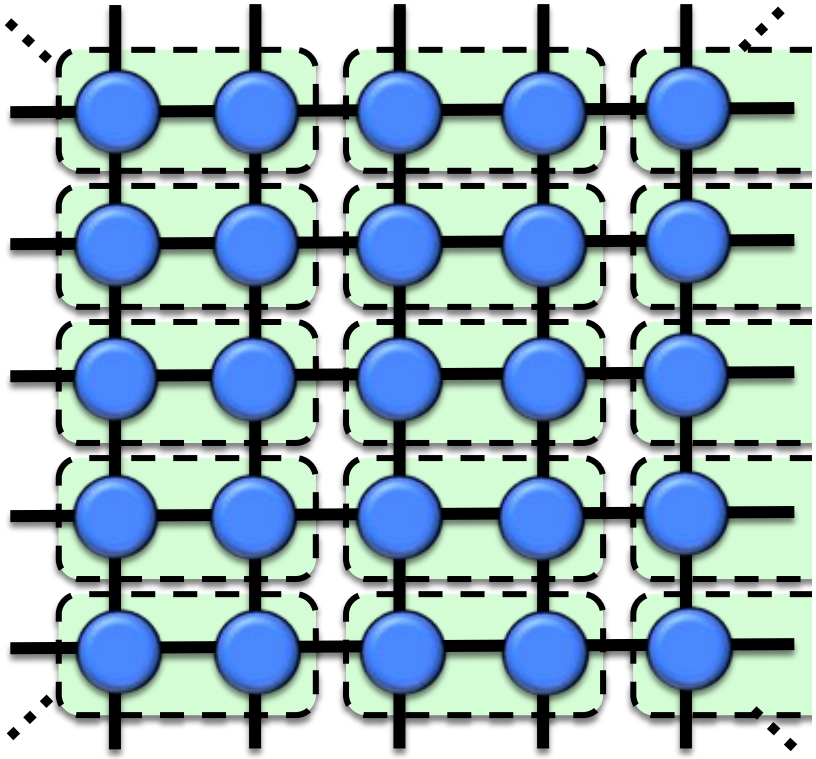
dominant eigenvector?

Can be approximated using infinite MPS

1.. χ

iTEBD, iDMRG, PWFRG, etc

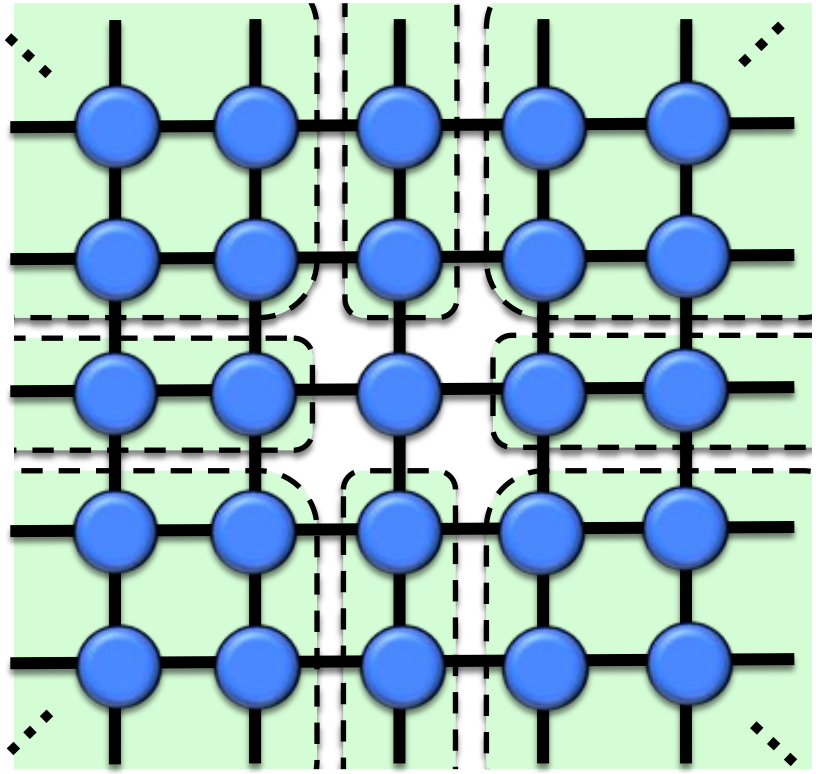
Contracting the infinite 2d lattice



Coarse-graining approaches:

TRG/SRG, HOSVD

Contracting the infinite 2d lattice



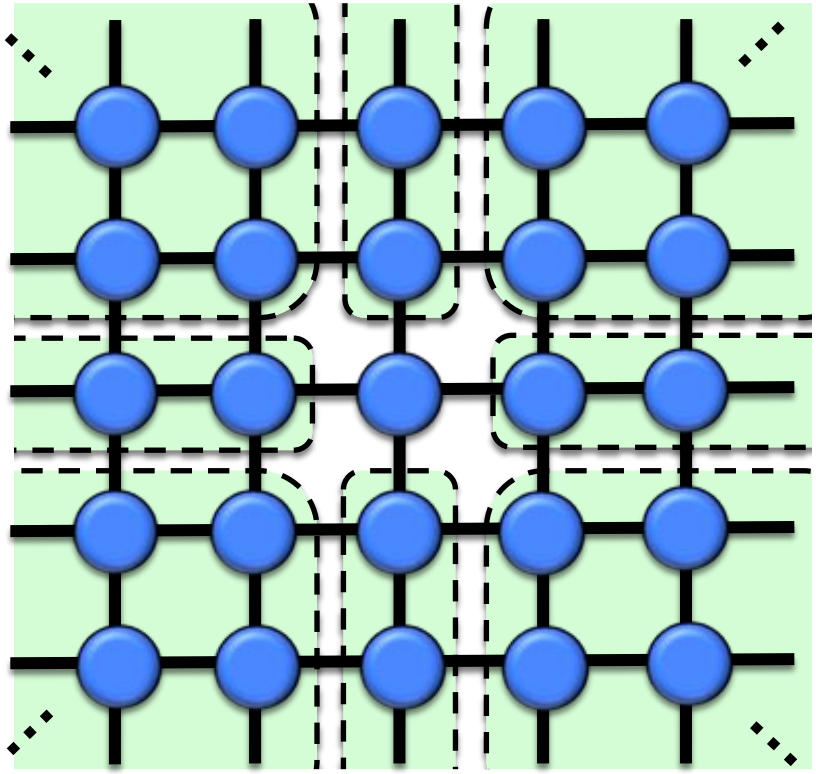
corner transfer matrix *(Baxter, 1968)*

(nice spectral properties)

half-row transfer matrix

half-column transfer matrix

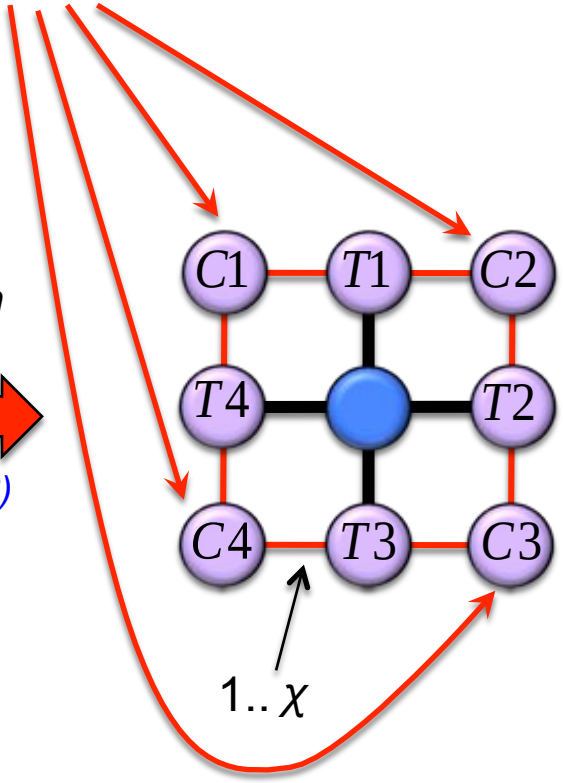
Contracting the infinite 2d lattice



Renormalization
(numerical)

(Baxter, 1968,1978)

Renormalized Corner Transfer Matrices



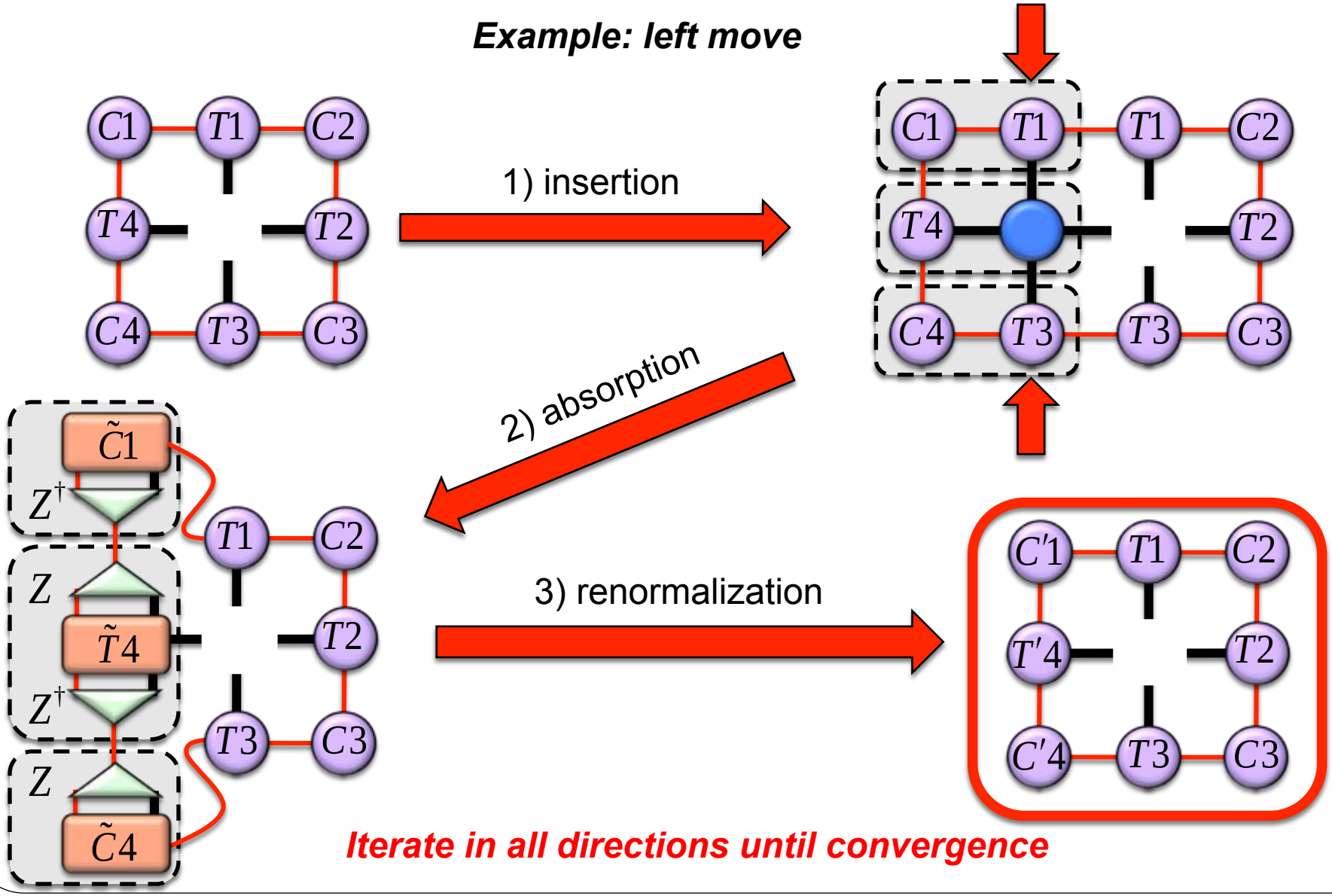
Directional version *of the* ***corner transfer matrix renormalization group***

(faster than 1d transfer matrix methods) *T. Nishino, K. Okunishi, JPS Jpn. 65, 891 (1996)*

Corner transfer matrix and iPEPS

R. Orús, G. Vidal, PRB 80 094403 (2009), R. Orús, PRB 85, 205117 (2012)

Example: left move

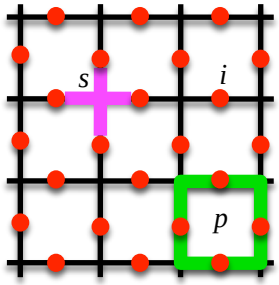


Iterate in all directions until convergence

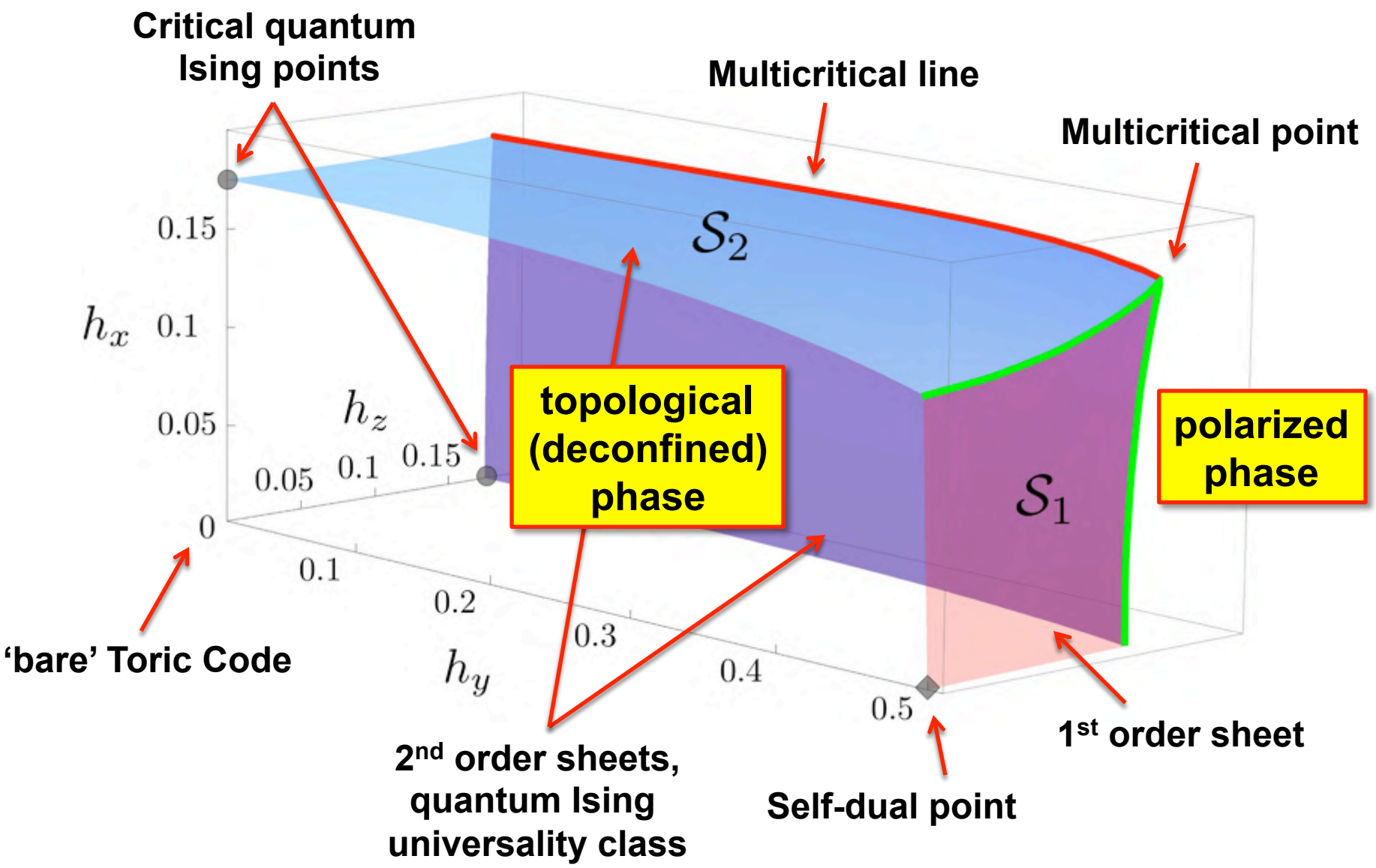
An example:
Toric Code in arbitrary field

S. Dusuel et al., PRL 106, 107203 (2011)

$$H = -J\sum_s A_s - J\sum_p B_p - h_x\sum_i \sigma_i^x - h_y\sum_i \sigma_i^y - h_z\sum_i \sigma_i^z$$



Phase diagram



Outline

1) Generalities

2) 2d PEPS

- 2.1) Basics
- 2.2) Computing
- 2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

Outline

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Fermionic 2d systems

Fermionic systems are extremely interesting physical systems, e.g. the **2d fermionic Hubbard model** may be the key to understand the emergence of **high-T_c superconductivity**

Unfortunately, fermionic systems are also **amongst the most difficult** to simulate, because of the **sign problem in Quantum MonteCarlo (sampling of negative probabilities)**

Fermions are a NUMERICAL MONSTER

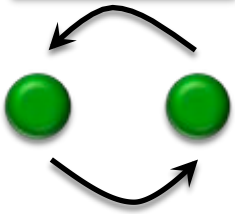
for Quantum MonteCarlo because of the sign problem!

but there is hope...

Tensor Networks + Fermions

e.g., P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)

Bosons



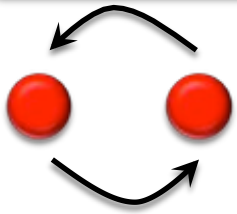
$$\Psi_B(x_1, x_2) = \Psi_B(x_2, x_1)$$

Symmetric wavefunction

$$b_i b_j = b_j b_i$$

Operators commute

Fermions



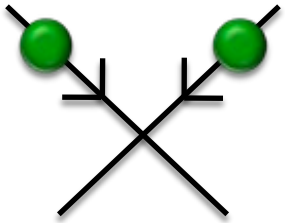
$$\Psi_F(x_1, x_2) = -\Psi_F(x_2, x_1)$$

Antisymmetric wavefunction

$$c_i c_j = -c_j c_i$$

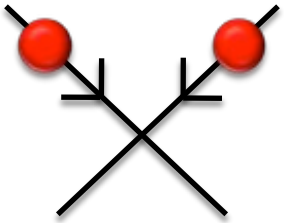
Operators anticommute

Crossings in a TN



+

Ignore crossings



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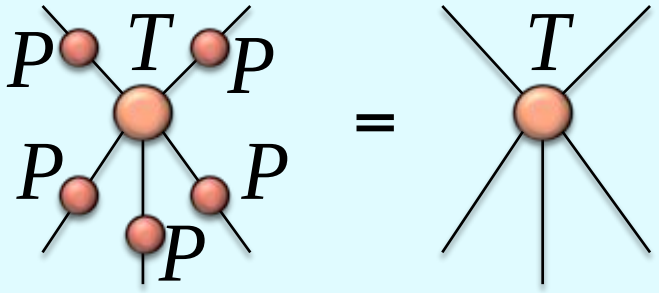
Careful!!!!

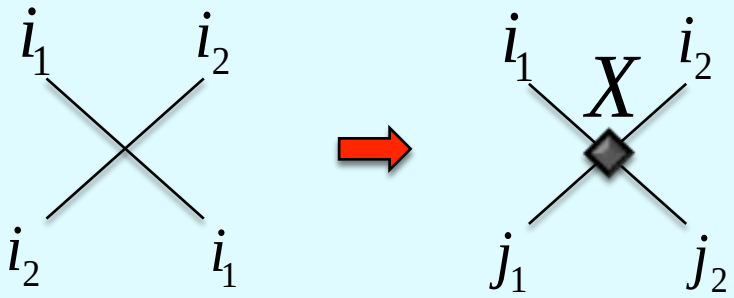
Tensor Network "fermionization" rules



*Use **parity-preserving** tensors*

$$T_{i_1 i_2 \ldots i_M} = 0 \quad \text{if} \quad P(i_1)P(i_2)\ldots P(i_M) \neq 1$$

Symmetry of the Hamiltonian



Replace crossings by

fermionic swap gates

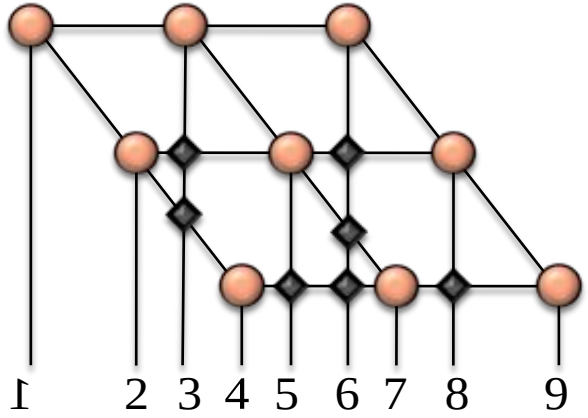
$$X_{i_2 i_1 j_1 j_2} = \delta_{i_1 j_1}\delta_{i_2 j_2} S(P(i_1), P(i_2))$$

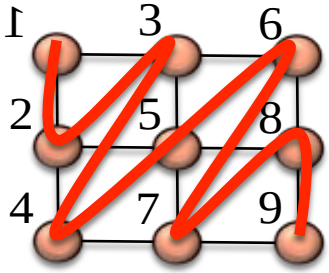
$$S(P(i_1), P(i_2)) = \begin{cases} -1 & \text{if} \quad P(i_1) = P(i_2) = -1 \\ +1 & \text{otherwise} \end{cases}$$

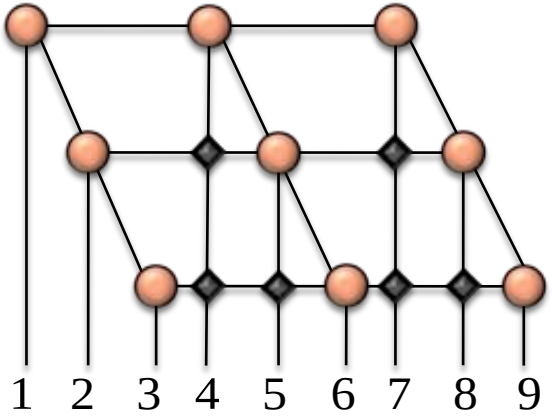
*Fermionic operators **anti**commute*

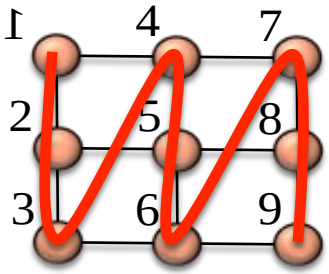
The leading order of the computational cost is the same as in the bosonic case

fermionic order ~ graphical projection of a PEPS







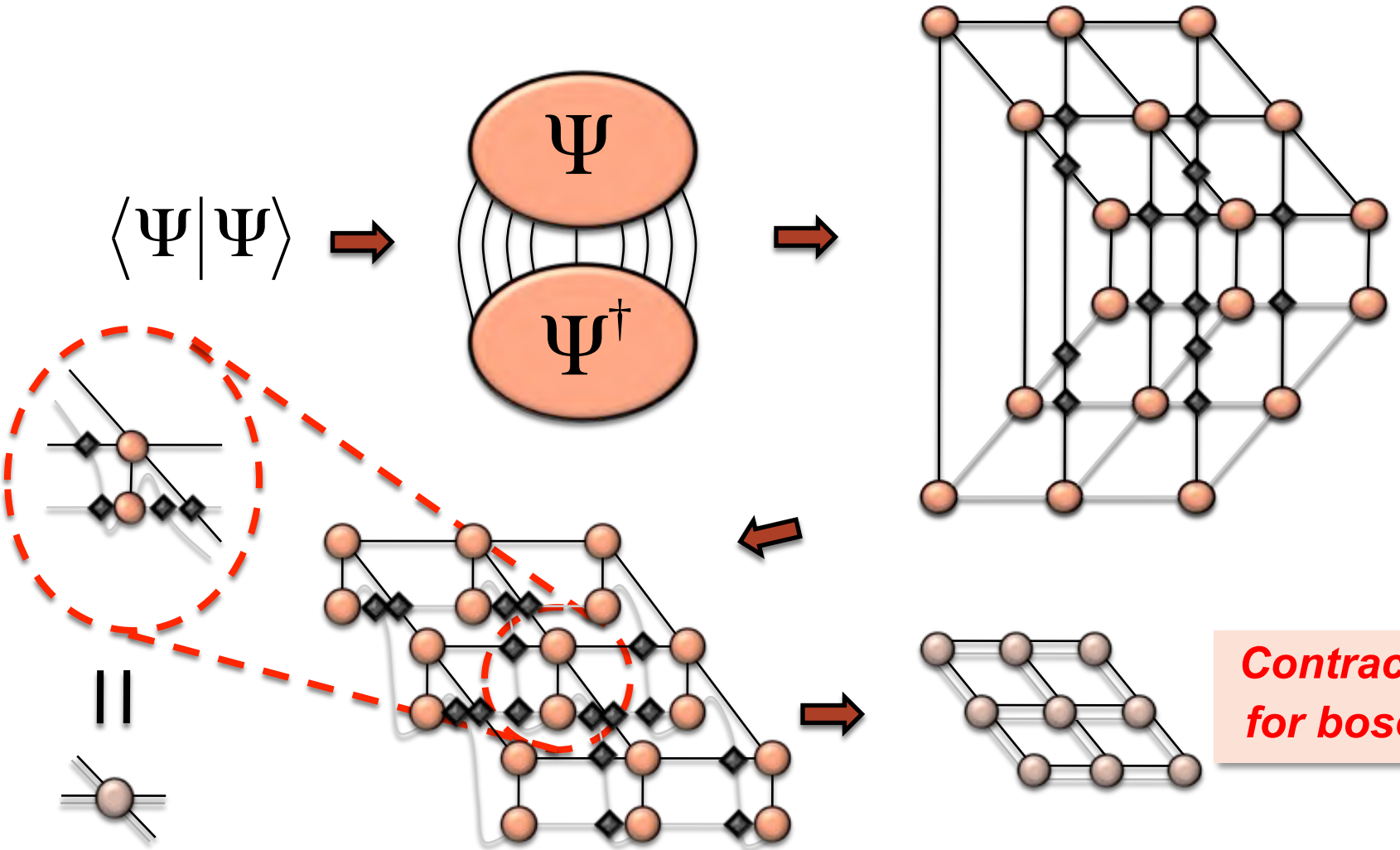


physics is independent of the order

physics is independent of graphical projection

(different choices of Jordan-Wigner transformation, if mapping to a spin system)

Example: *scalar product of 3x3 PEPS*



Contract as for bosons!

An example:
2d t-J model

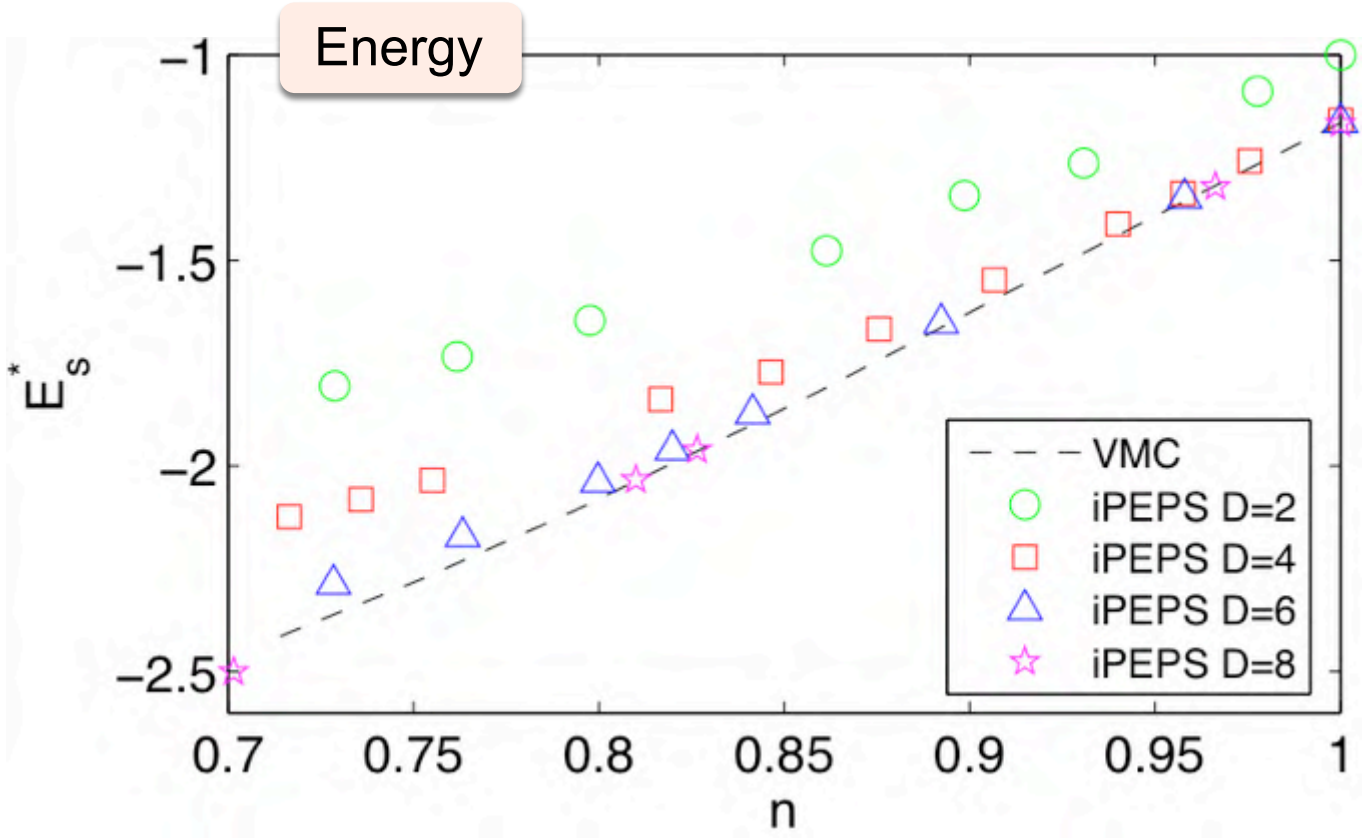
$$H = -t \sum_{\langle i,j \rangle \sigma} \left(\tilde{c}^{\dagger}_{i\sigma} \tilde{c}_{j\sigma} + hc \right) + J \sum_{\langle i,j \rangle} (S_i S_j - \frac{1}{4} n_i n_j) - \mu \sum_i n_i$$

P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)

P. Corboz, S. White, G. Vidal, M. Troyer, PRB 84 041108 (2011)

First simulations

P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)



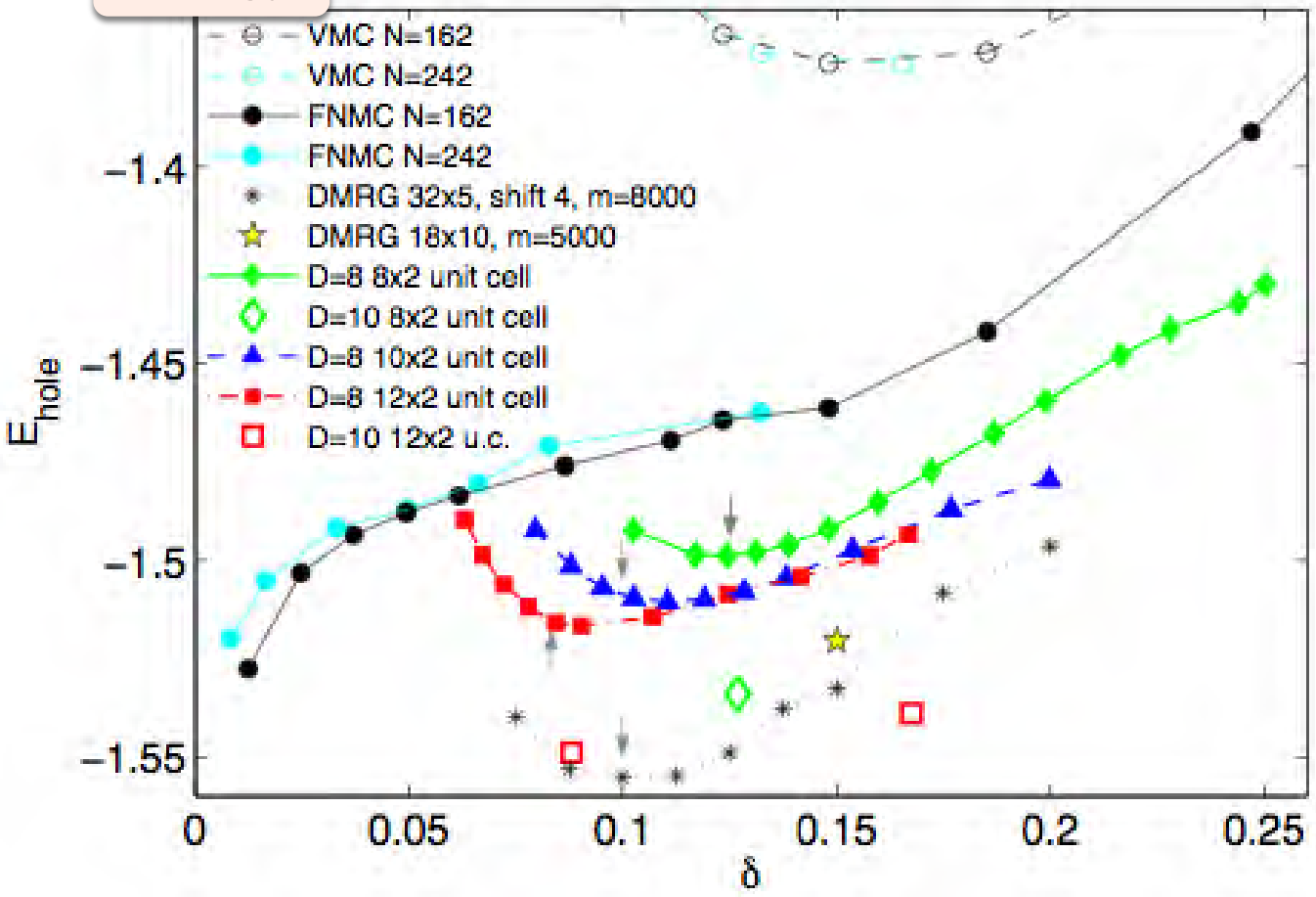
$(t/J = 3)$

Compatible results

Some improved simulations

P. Corboz, S. White, G. Vidal, M. Troyer, PRB 84 041108 (2011)

Energy



$(t / J = 2.5)$

Promising!

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1) Generalities

2) 2d PEPS

2.1) Basics

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3) 1d MERA: Basics

4) Some further topics

Outline

JG|U

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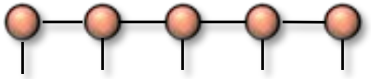
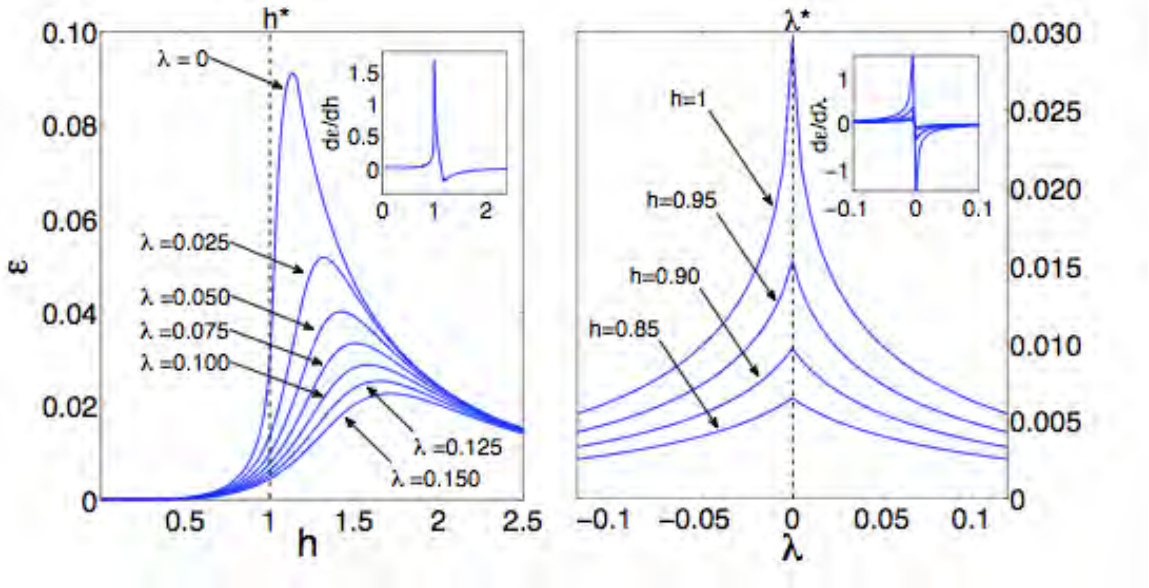
From MPS to MERA

Matrix Product States (MPS)



1d systems

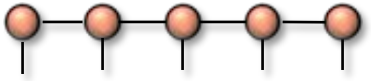
But we want to do critical systems!!!



Also very painful for DMRG...

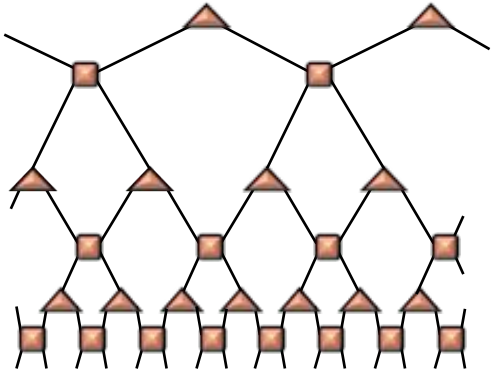
From MPS to MERA

Matrix Product States (MPS)

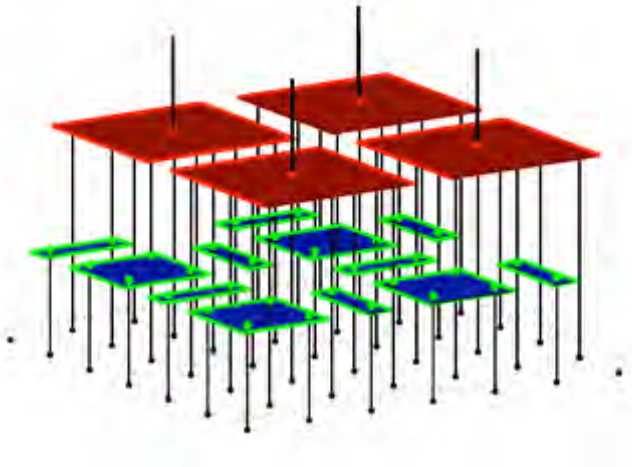


1d systems

Multiscale Entanglement Renormalization Ansatz (MERA)



1d systems

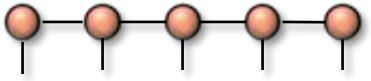


2d systems

and so on…

From MPS to MERA

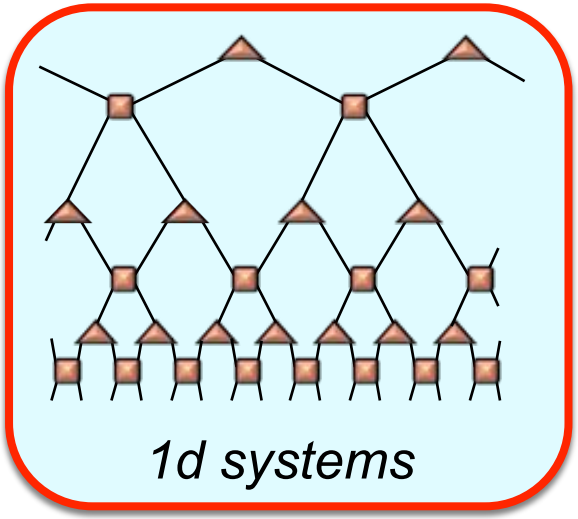
Matrix Product States (MPS)



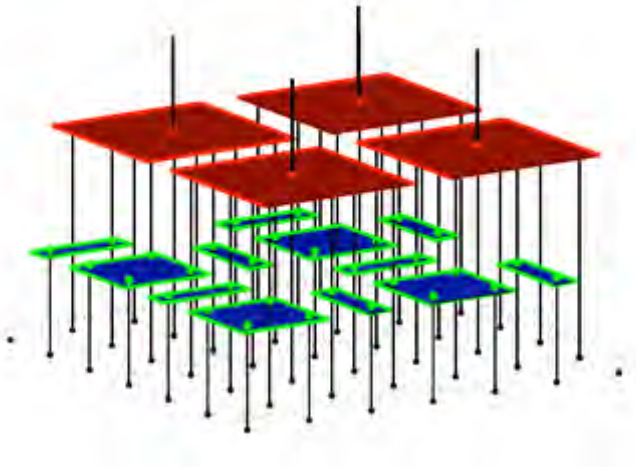
1d systems

Multiscale Entanglement Renormalization Ansatz (MERA)

This lecture



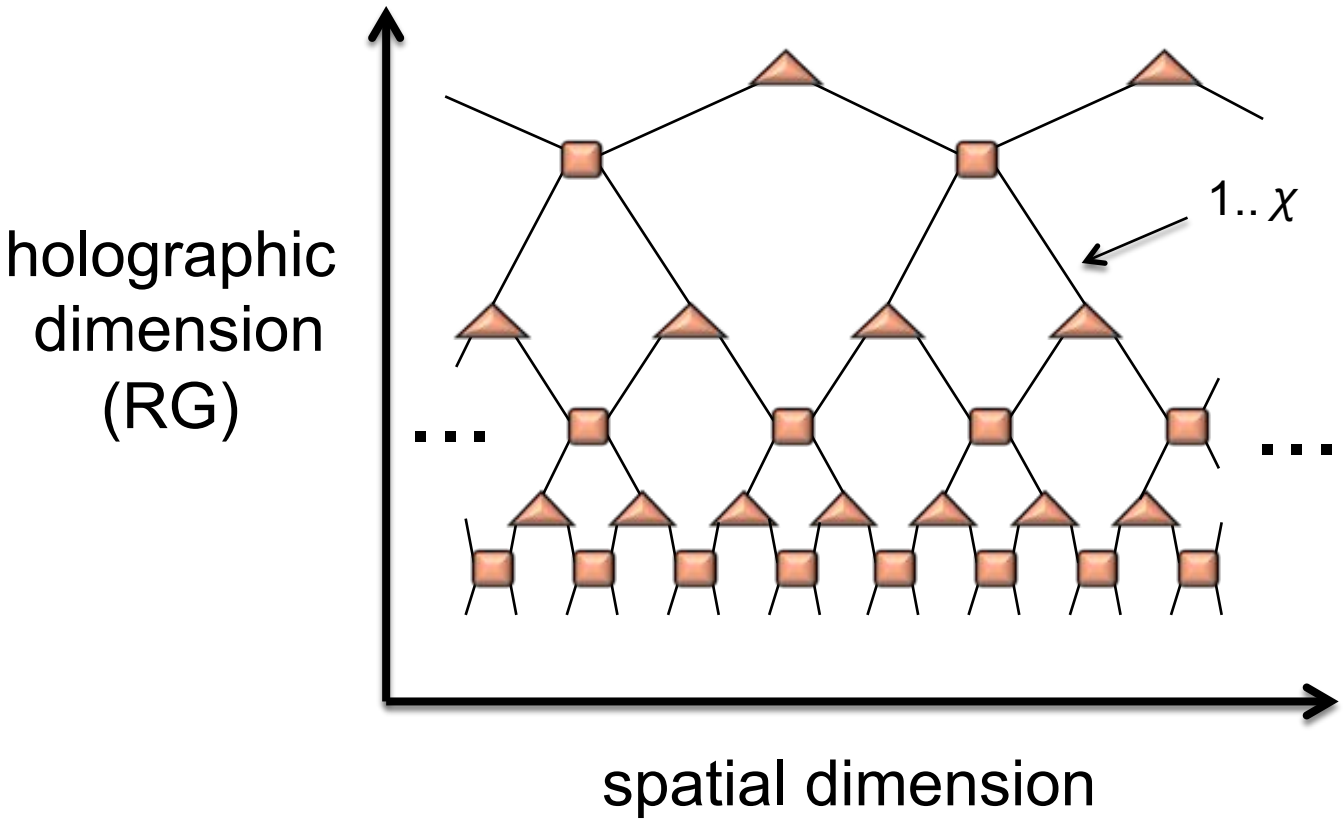
1d systems



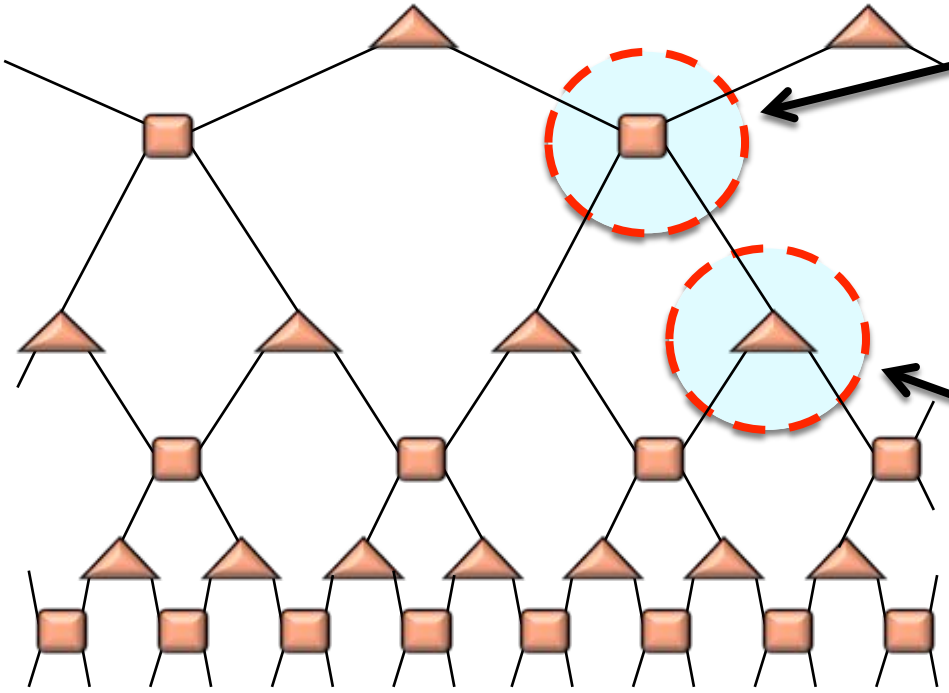
2d systems

and so on…

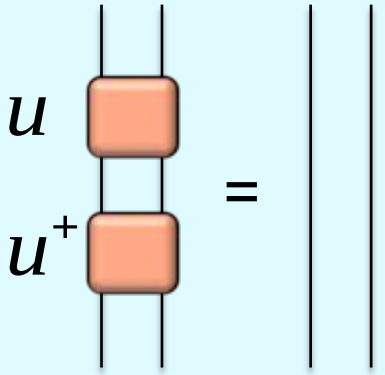
1d MERA



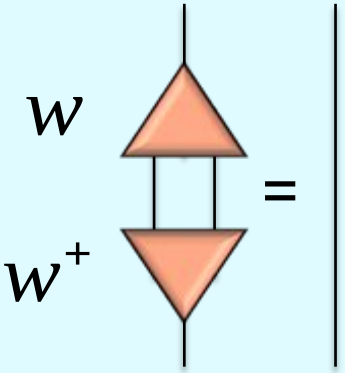
Tensors obey constraints



Unitaries (disentanglers)

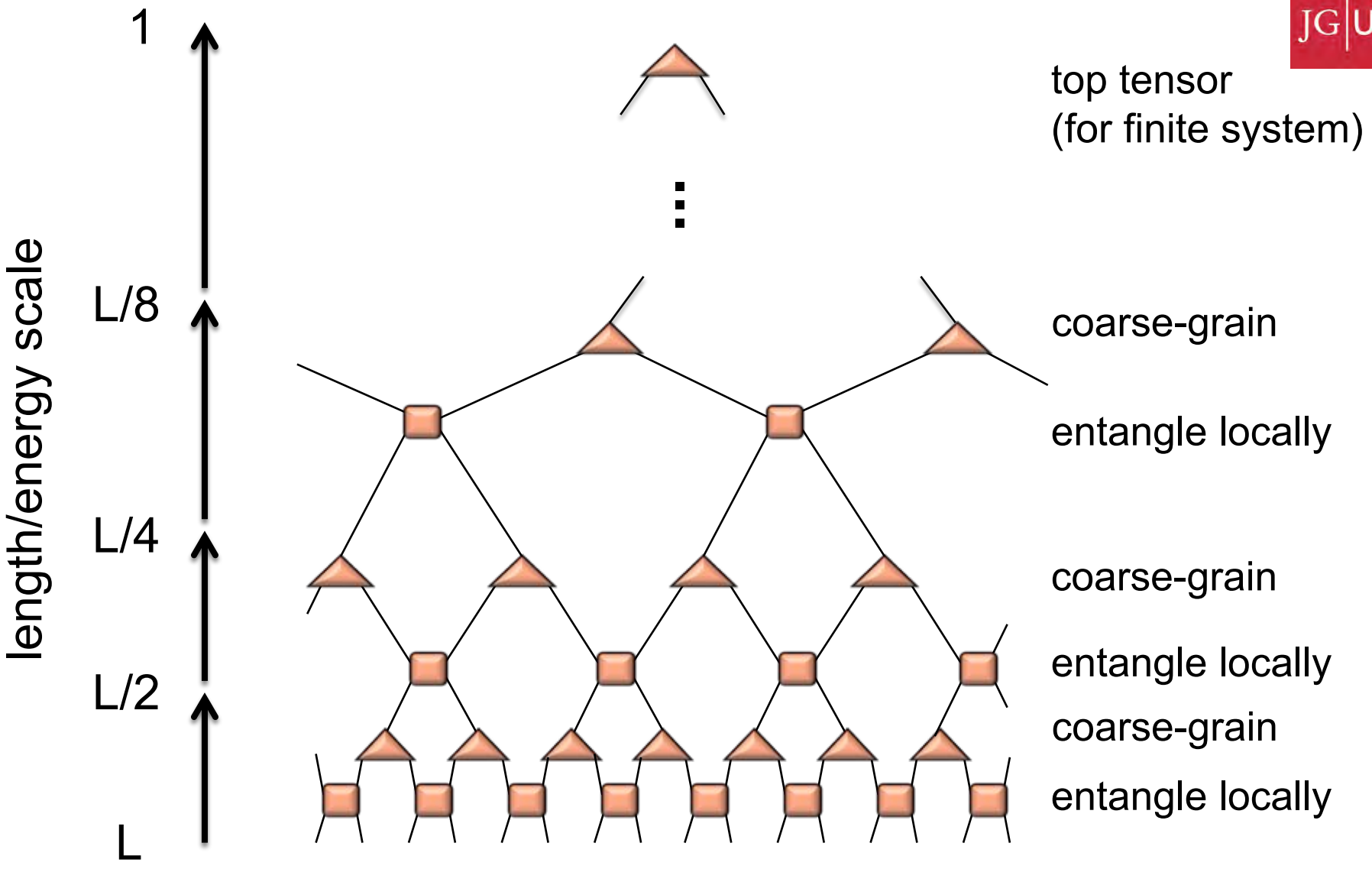


Isommetries (coarse-grainings)



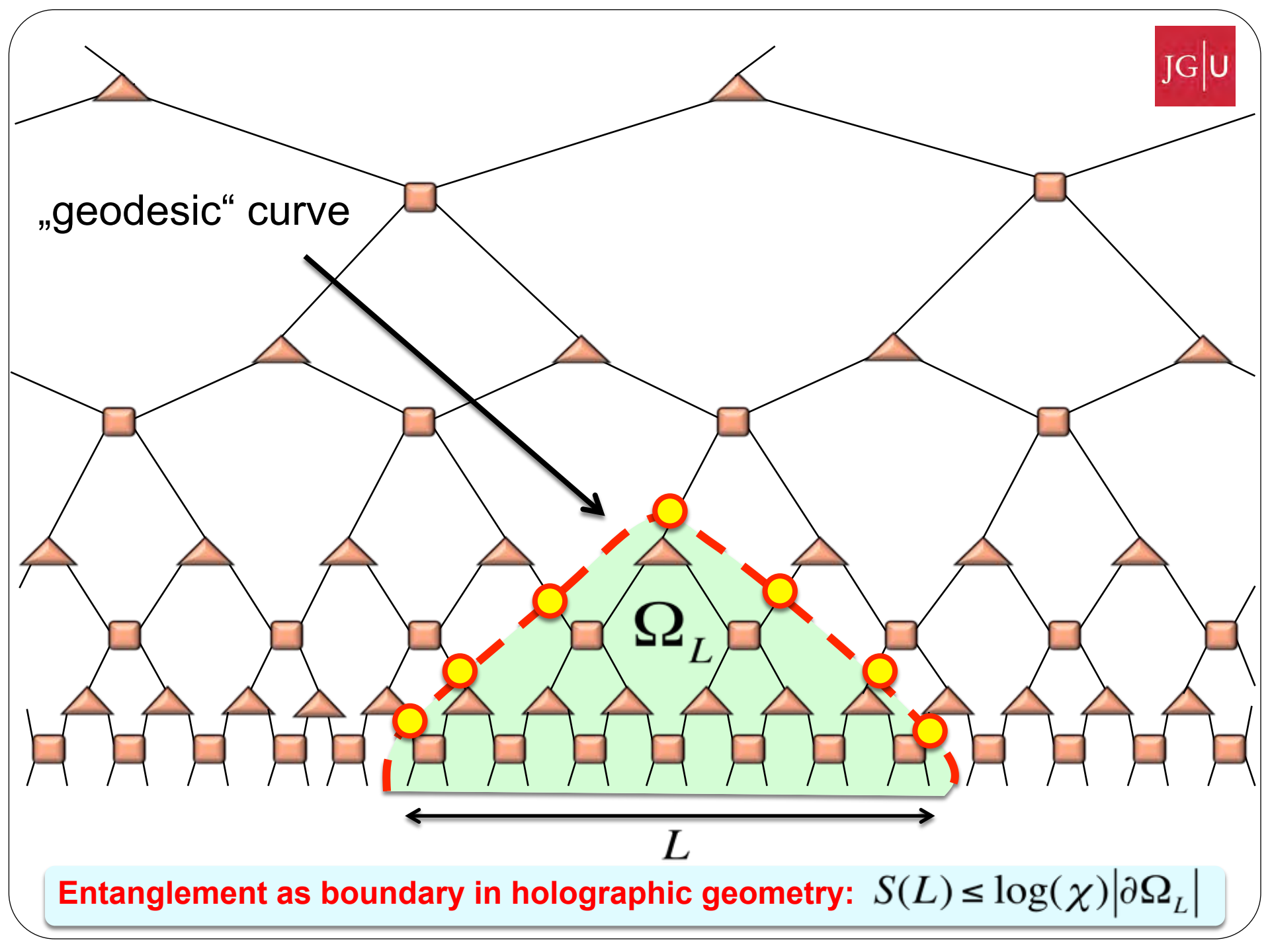
Reason:

entanglement is built locally
at all length scales

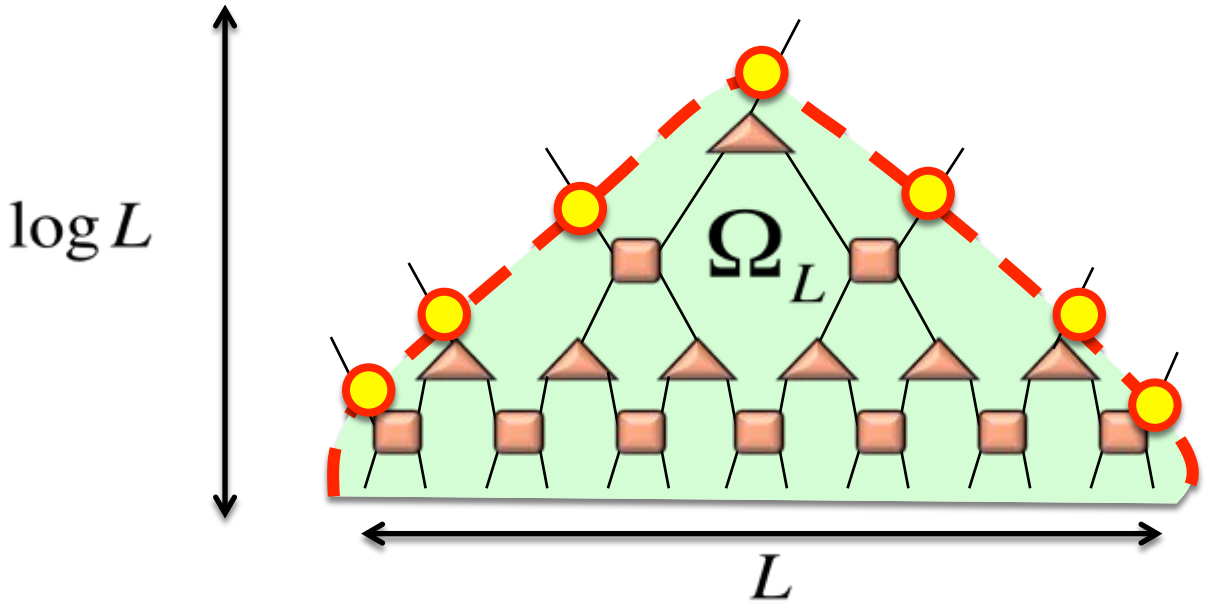


Extra dimension defines an RG flow: Entanglement Renormalization

Entropy of 1d MERA



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



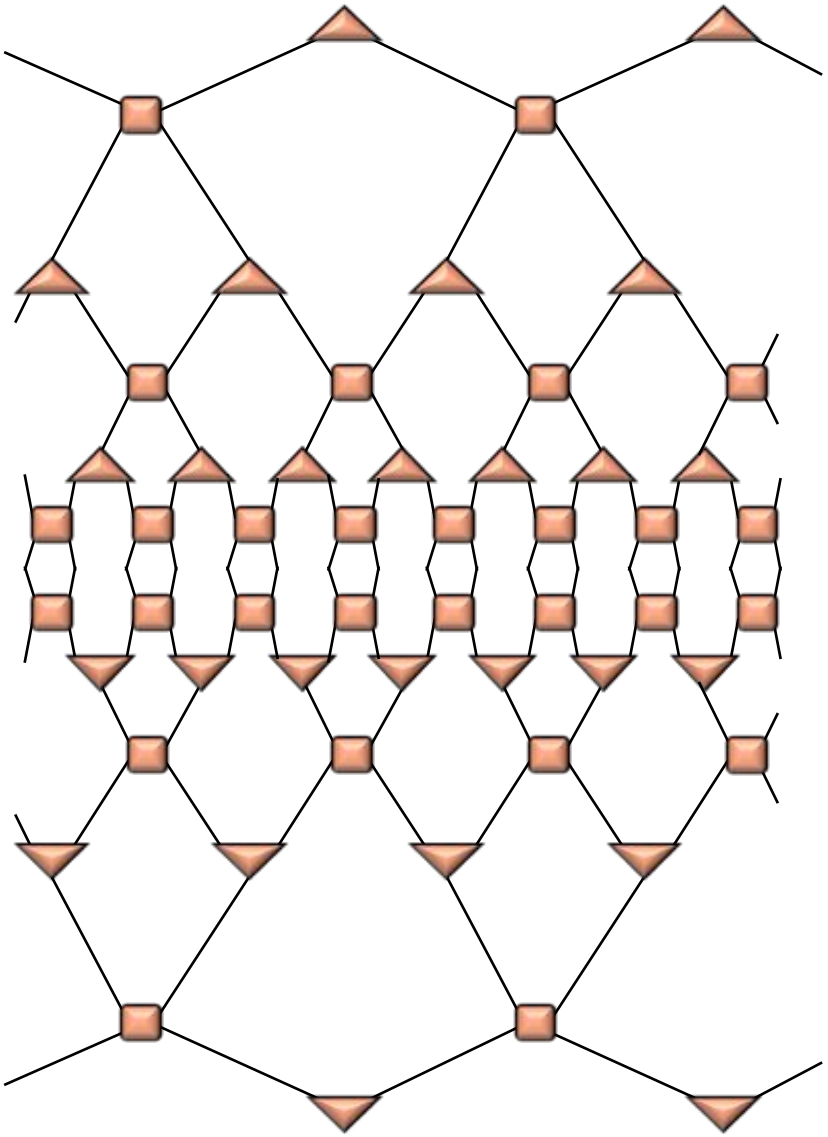
Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi) \left| \partial \Omega_L \right|$

Constant contribution at every layer

1d MERA can produce logarithmic violations to the area-law: $S(L) \approx \log L$

(like 1d critical systems!)

Norm of MERA



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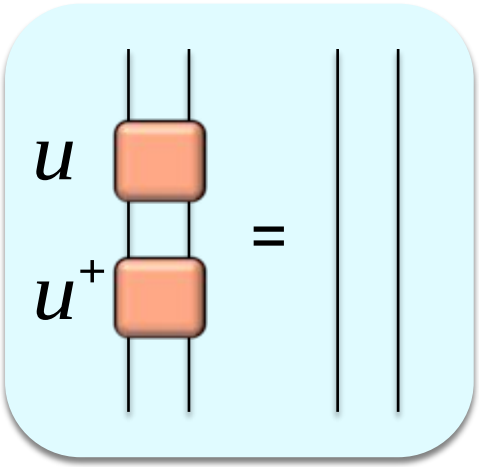
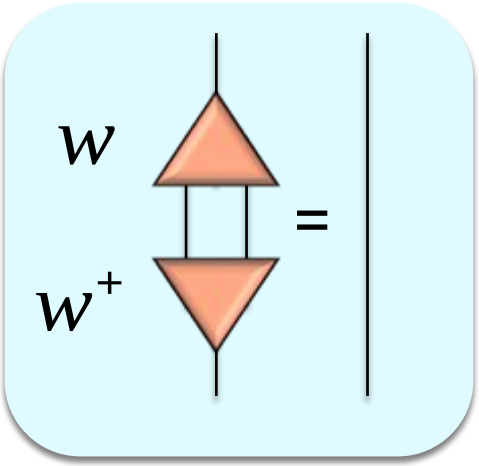
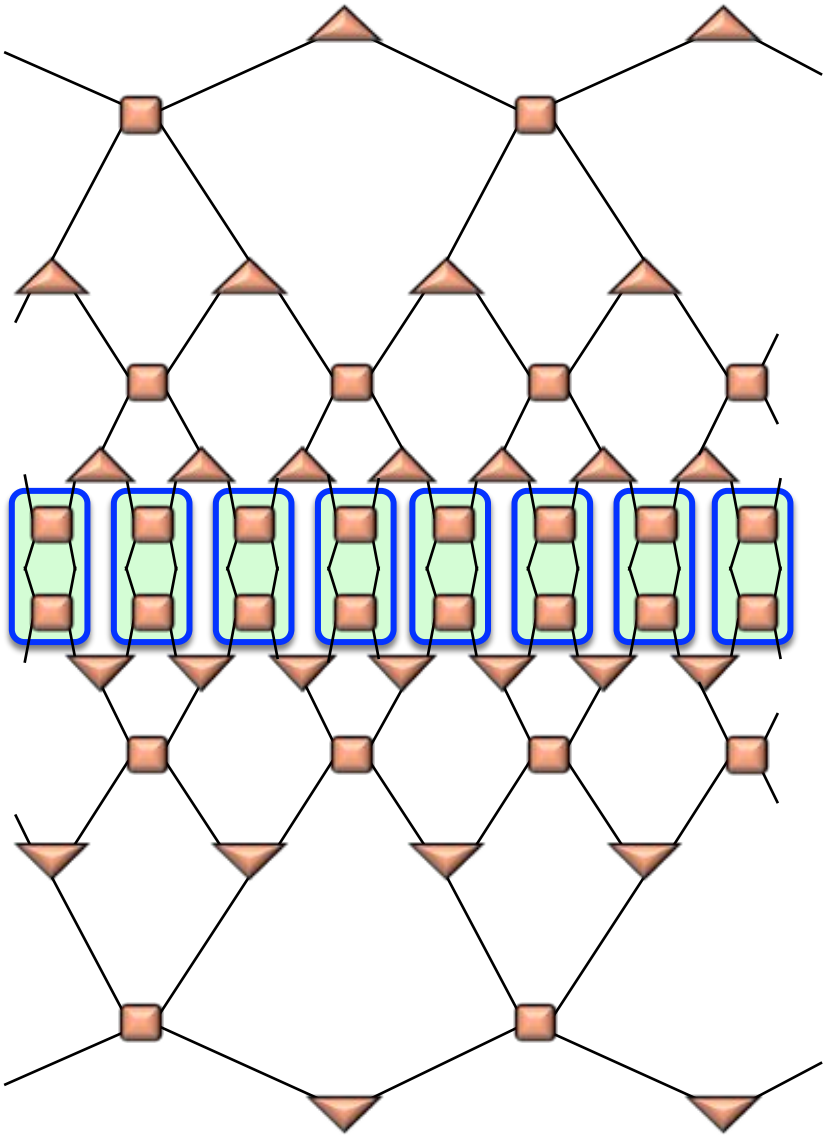




Norm of MERA



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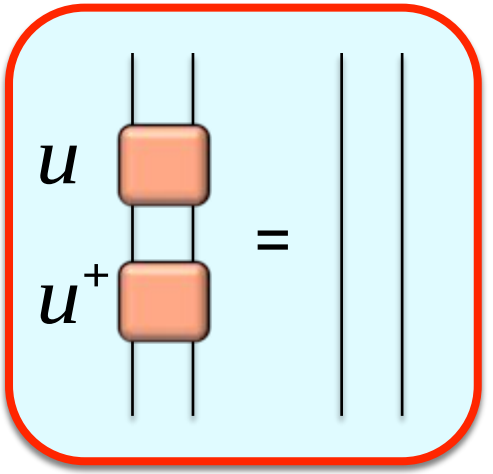
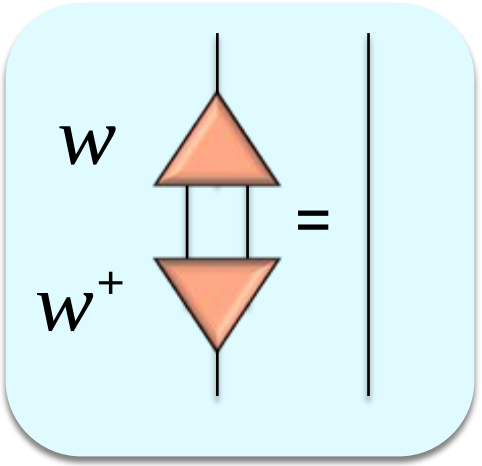
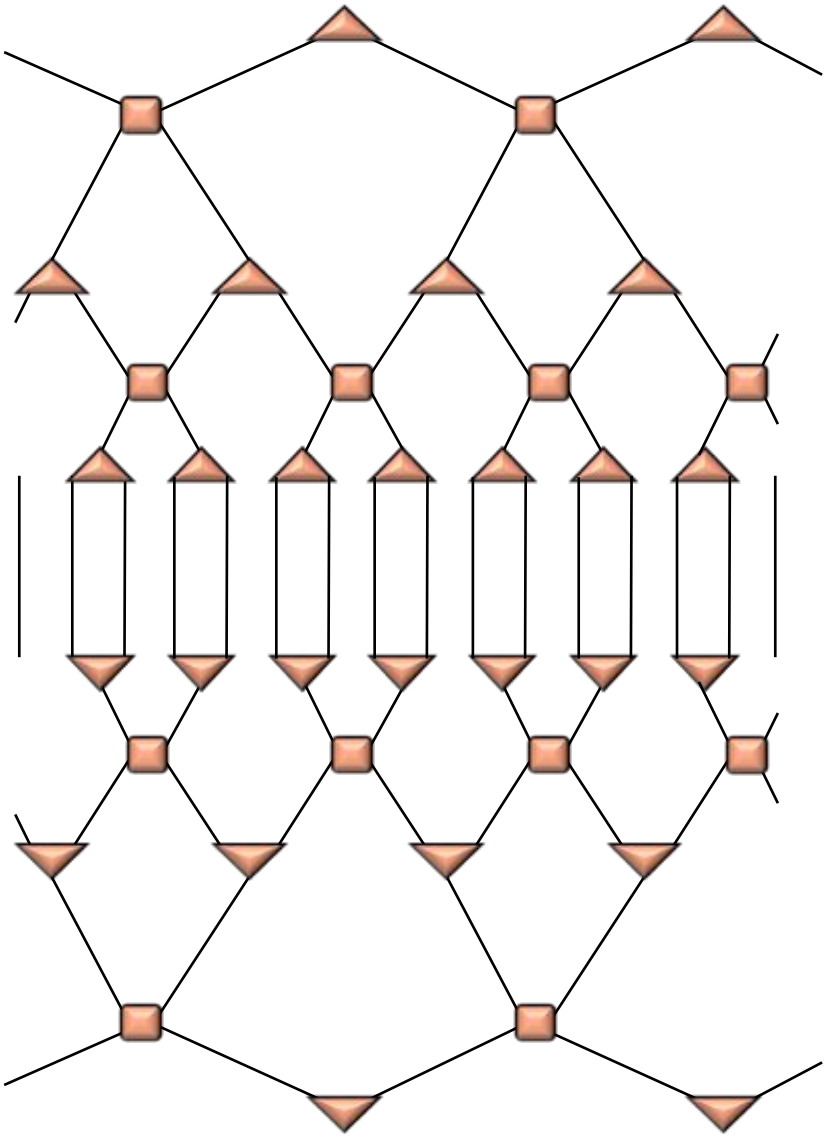




Norm of MERA



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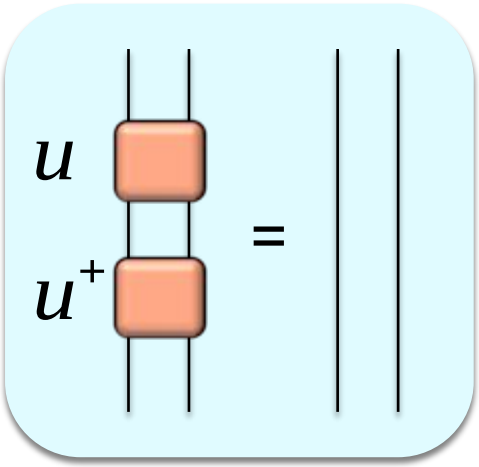

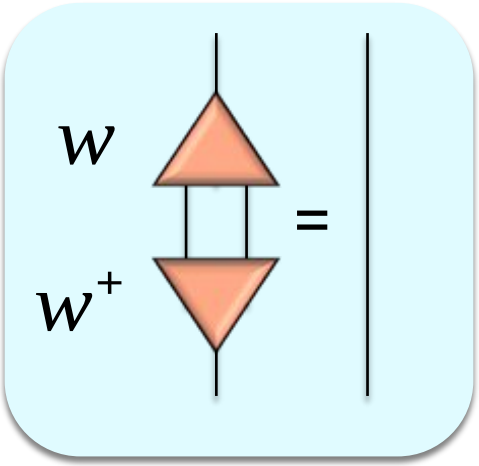

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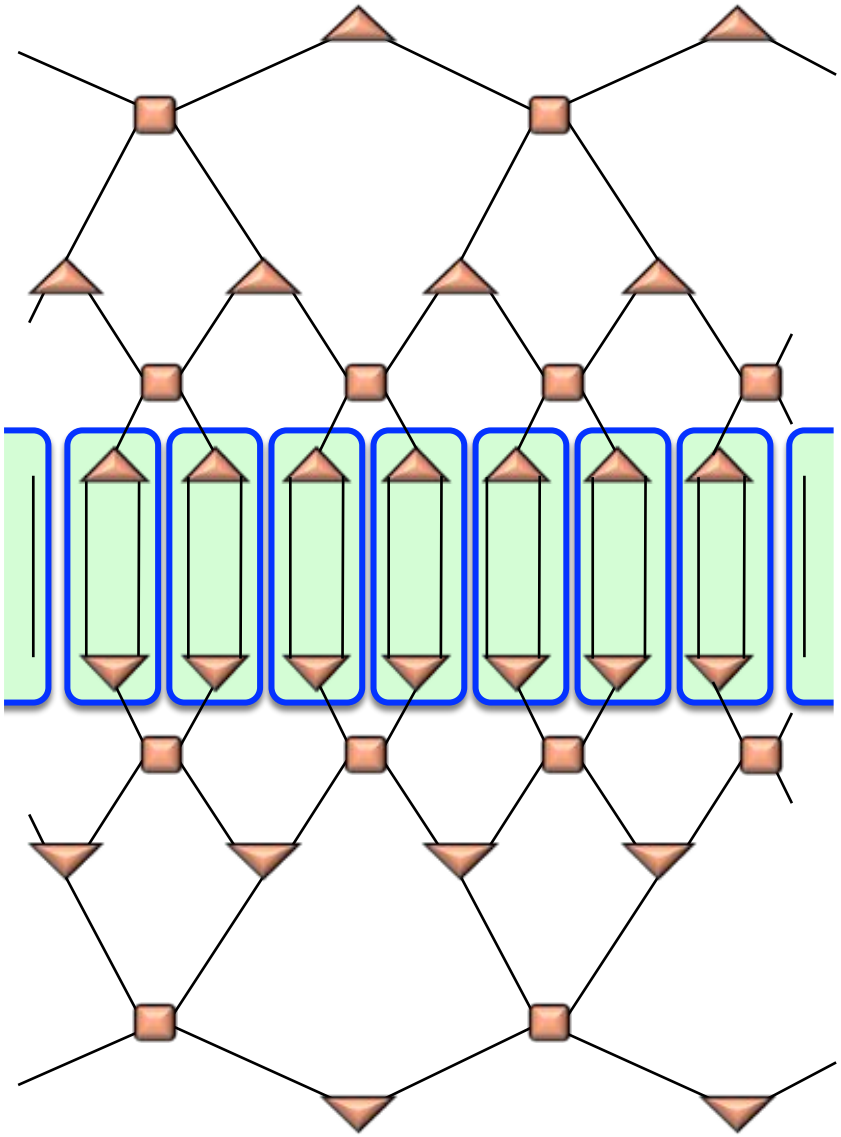

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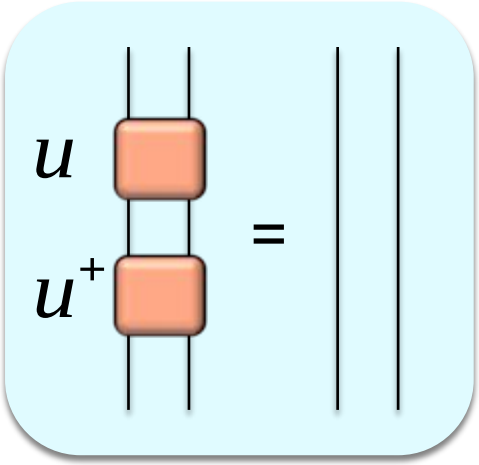
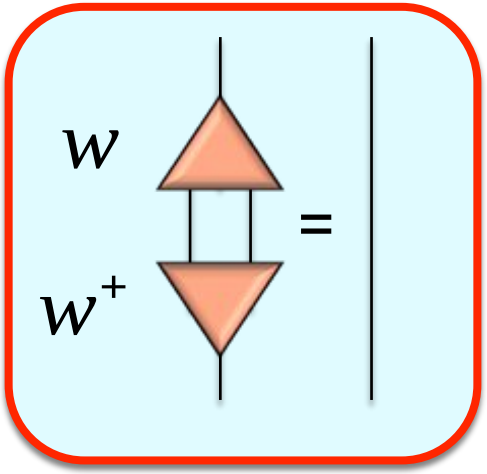
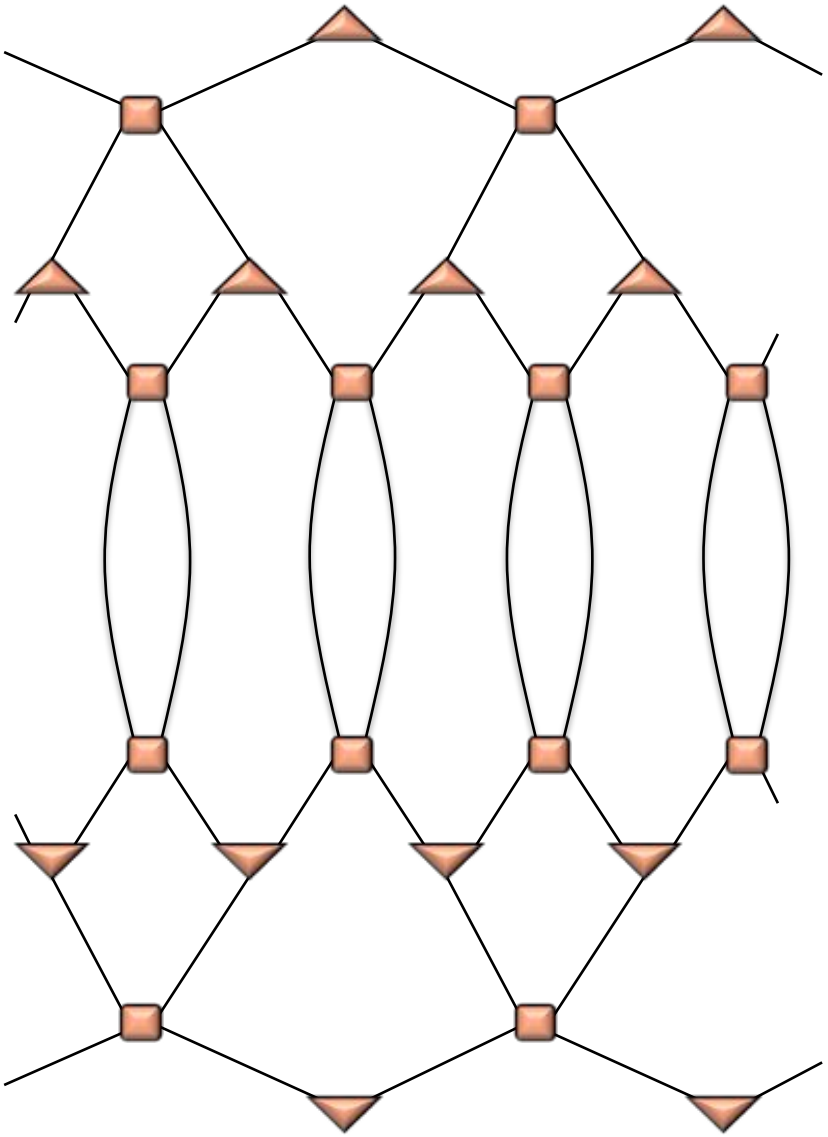




Norm of MERA



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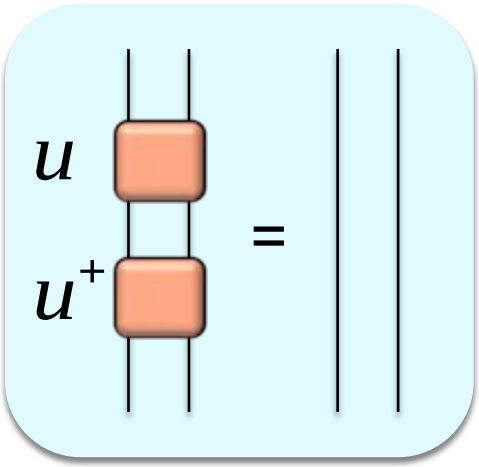

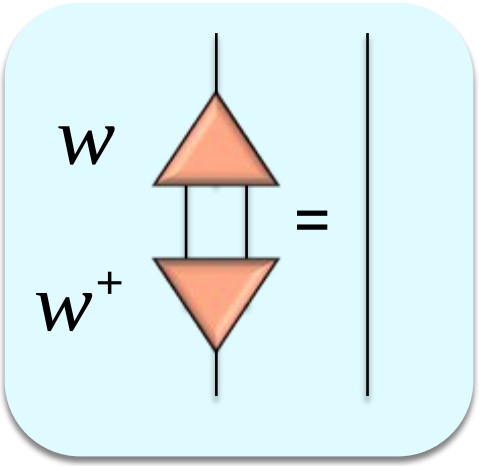

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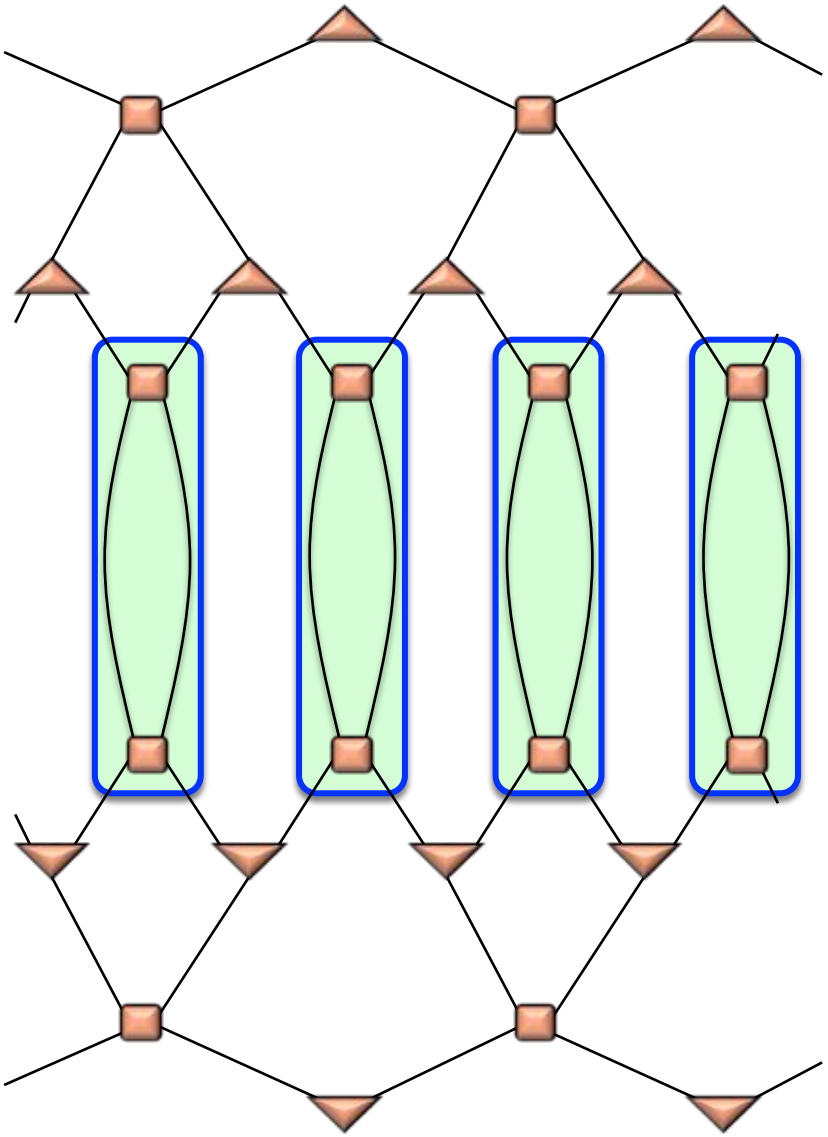

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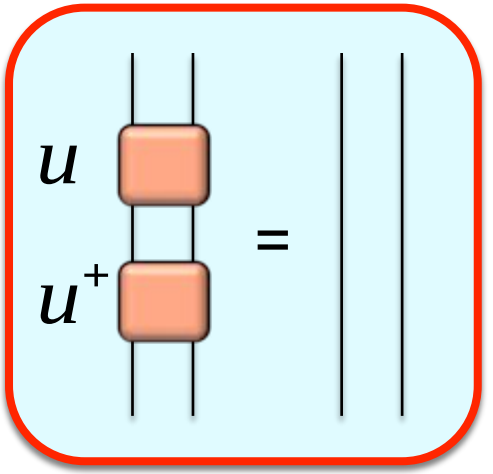
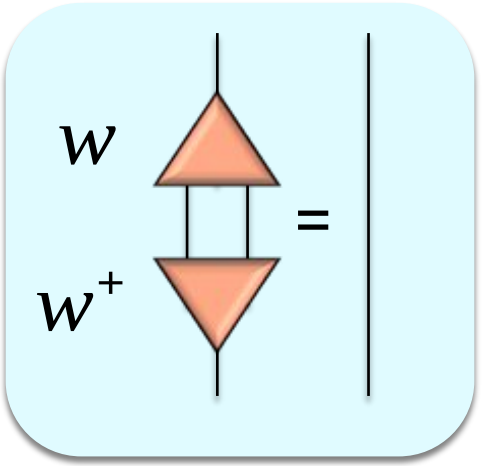
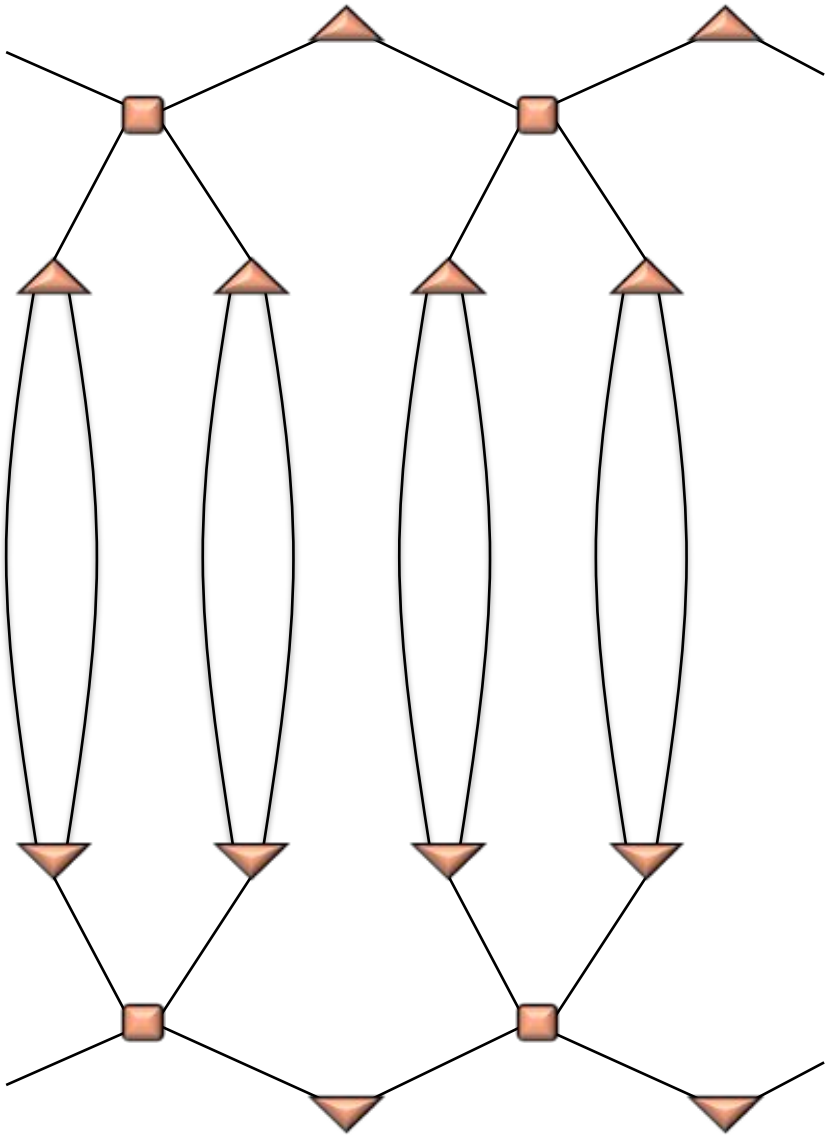




Norm of MERA



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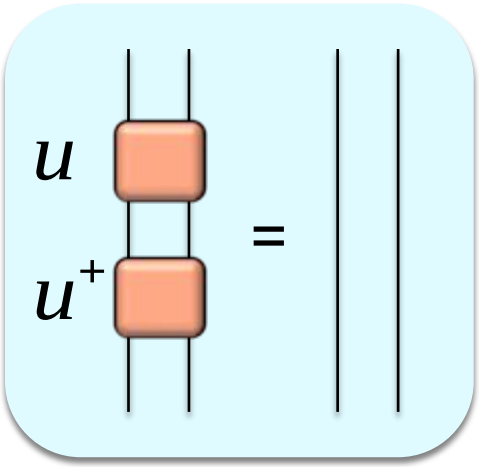

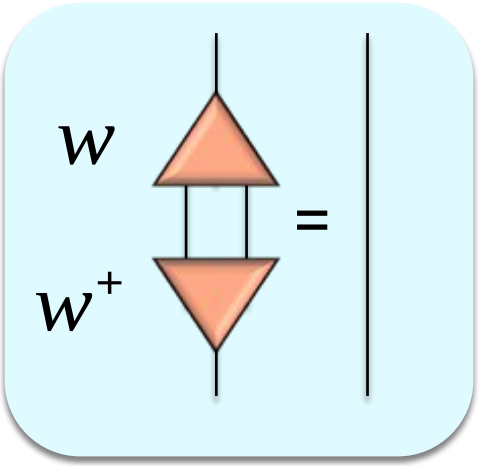

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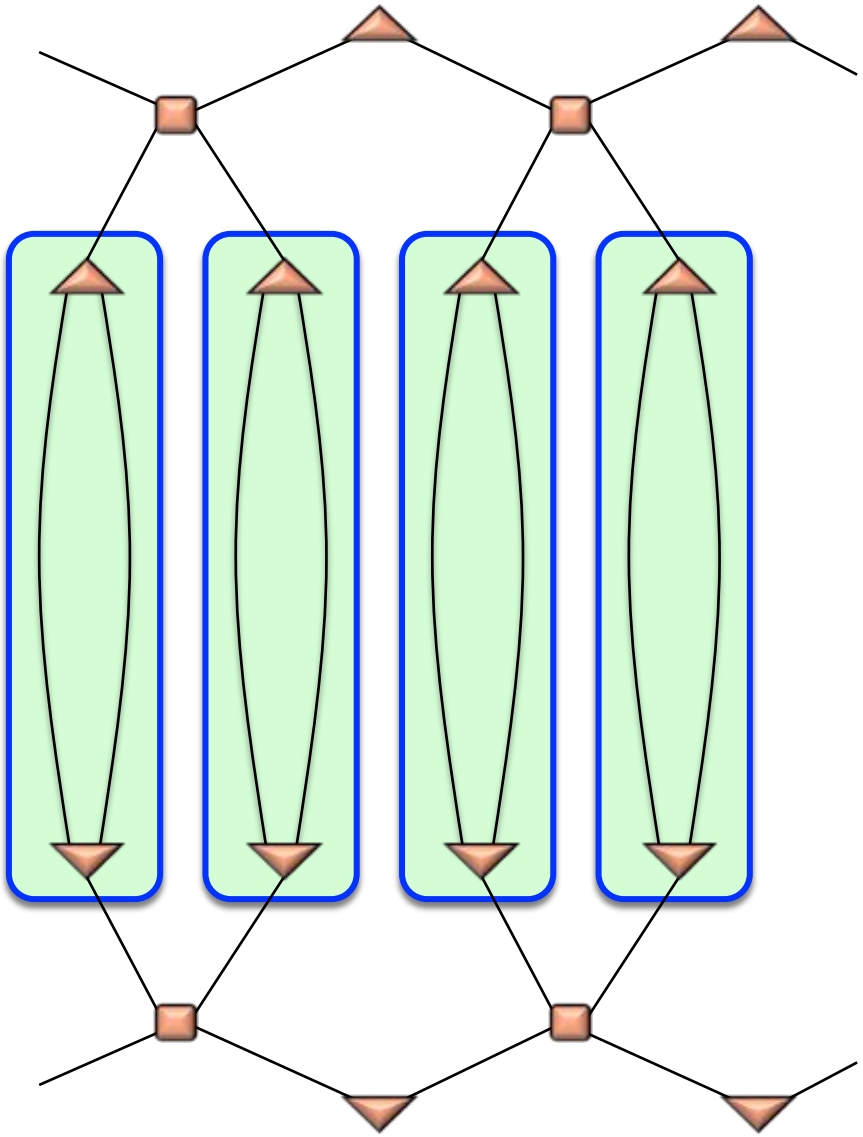

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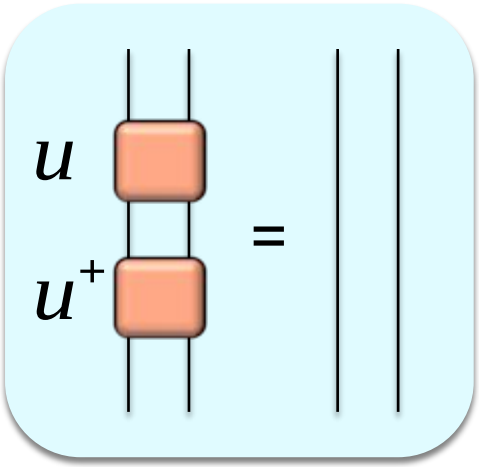
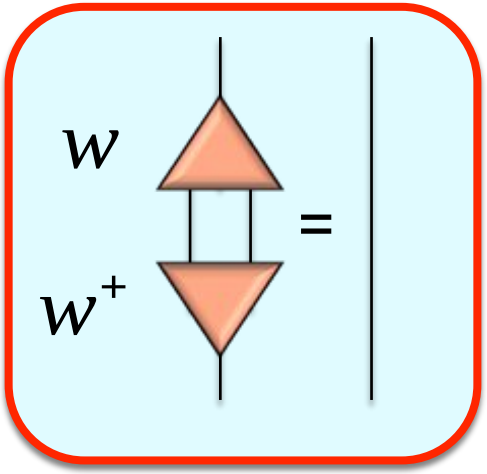
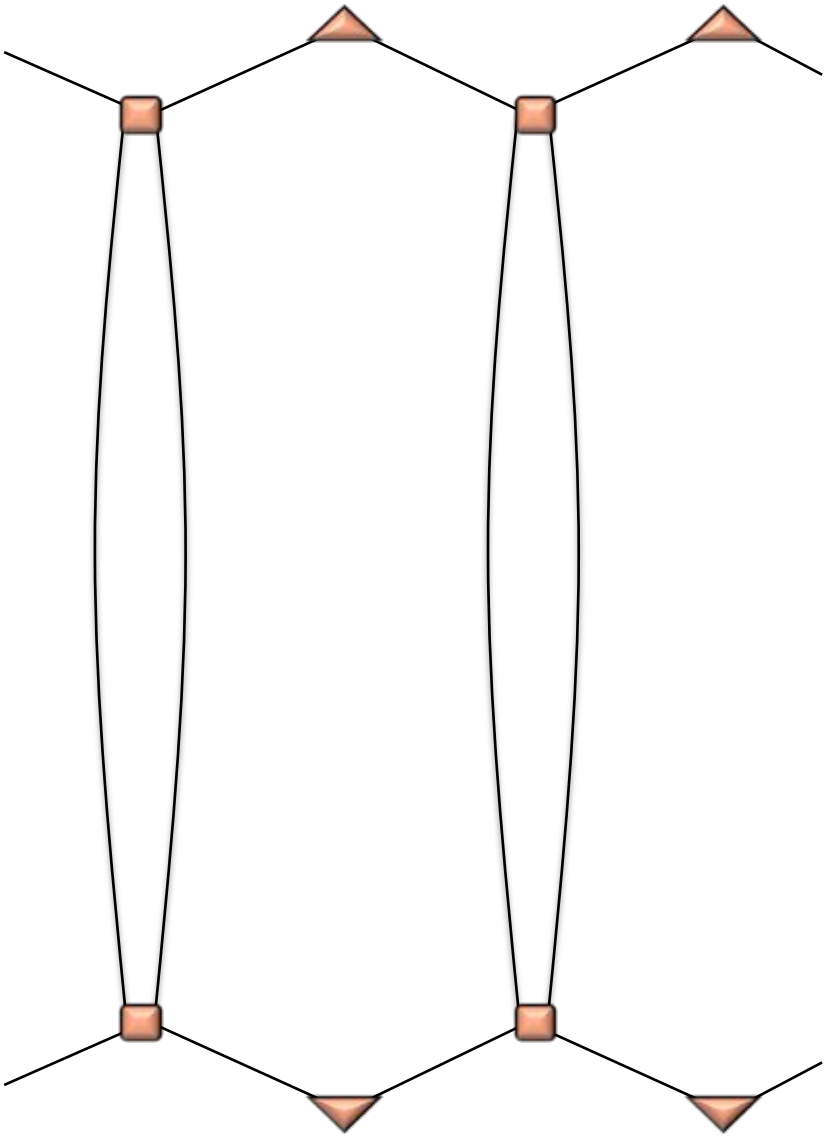




Norm of MERA



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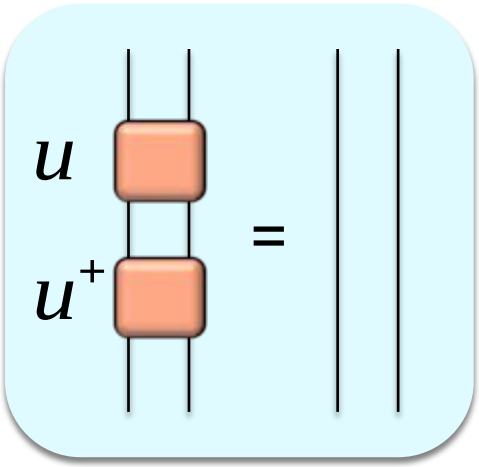

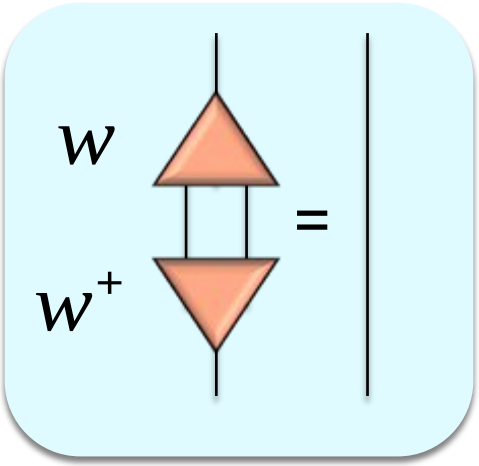

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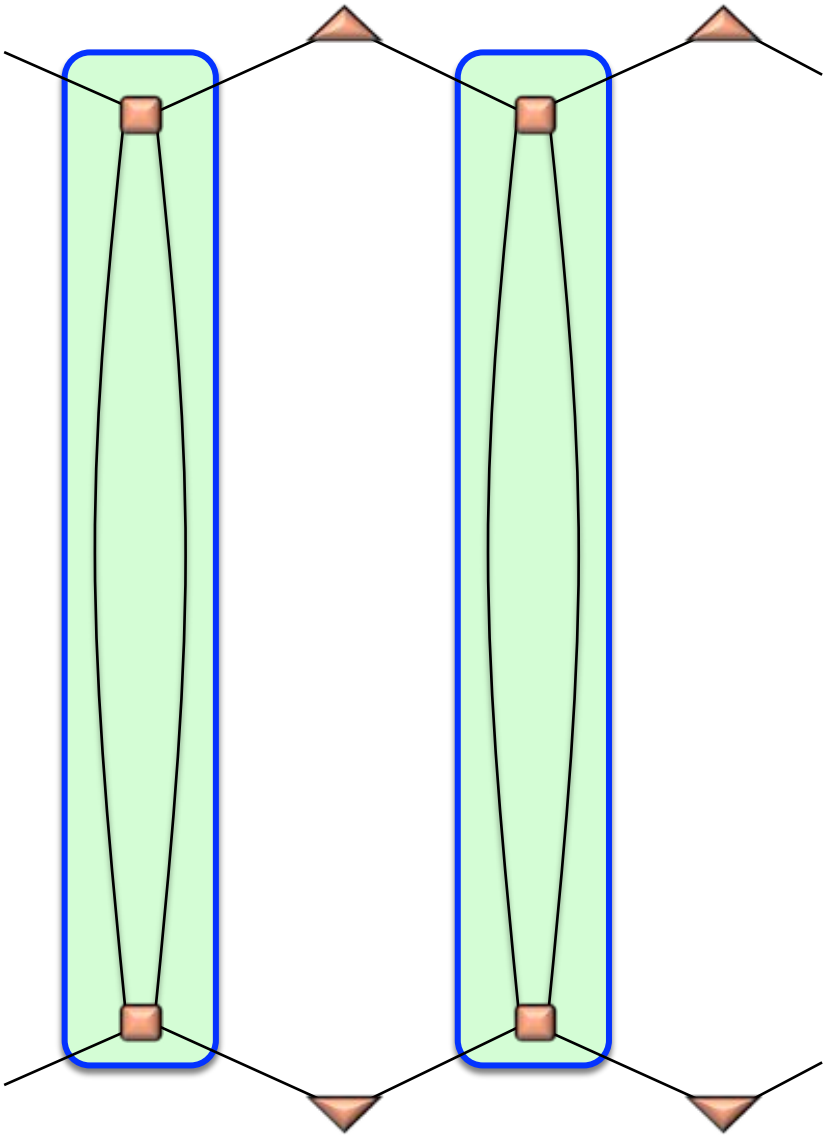

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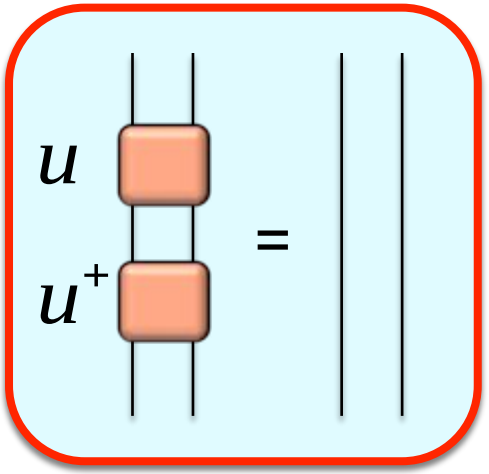
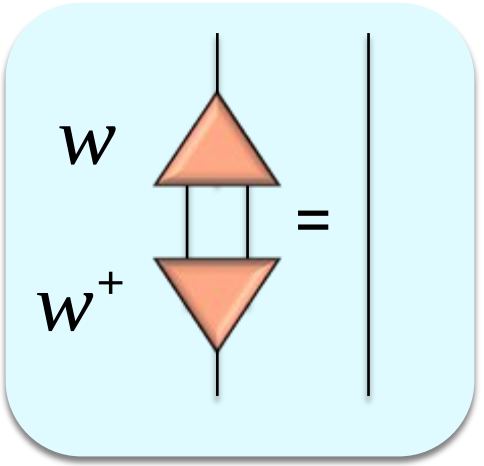
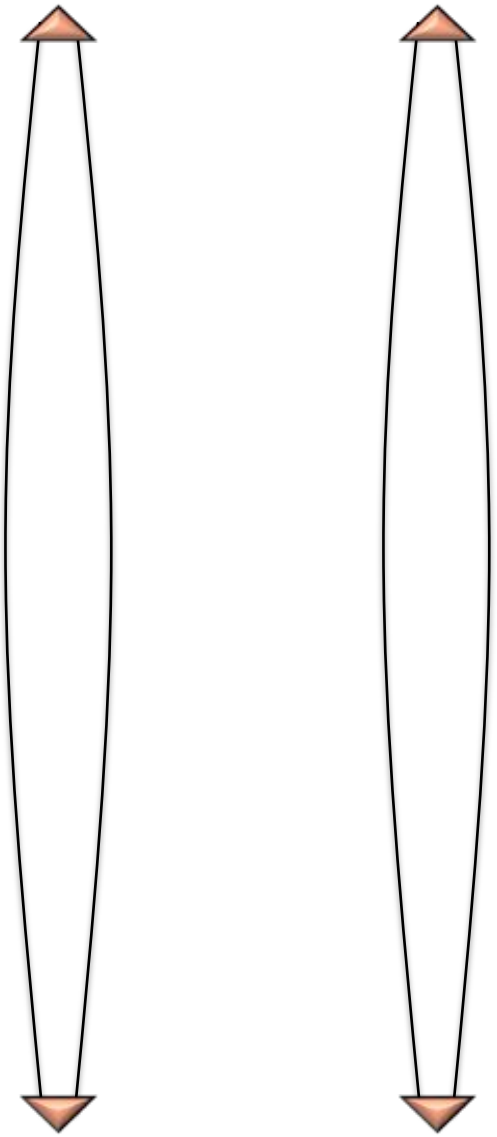




Norm of MERA



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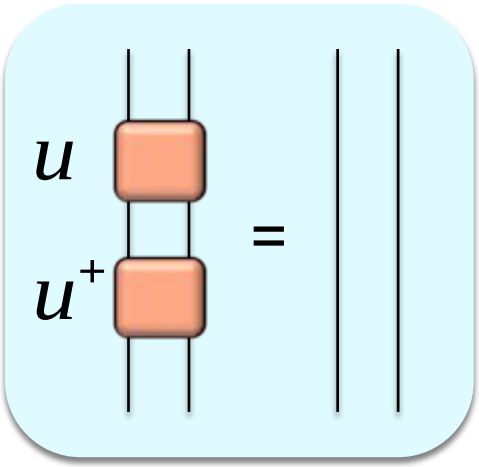

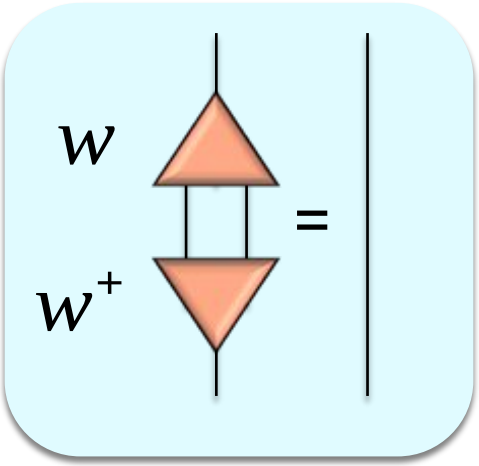

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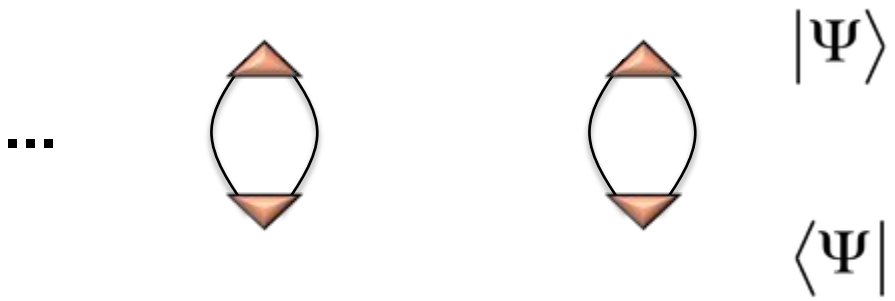
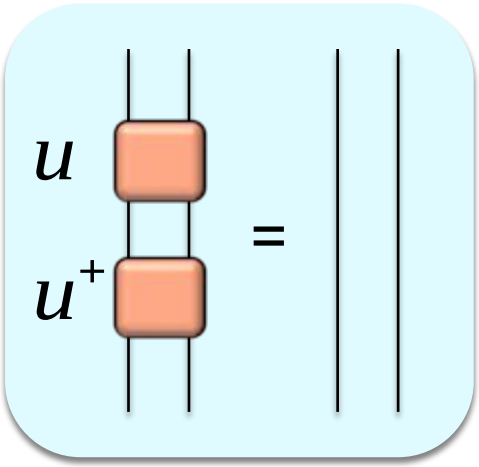
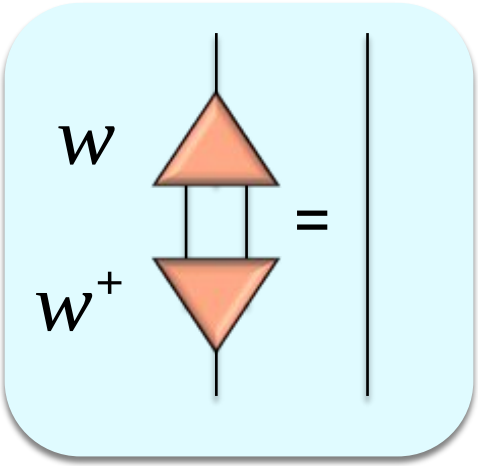
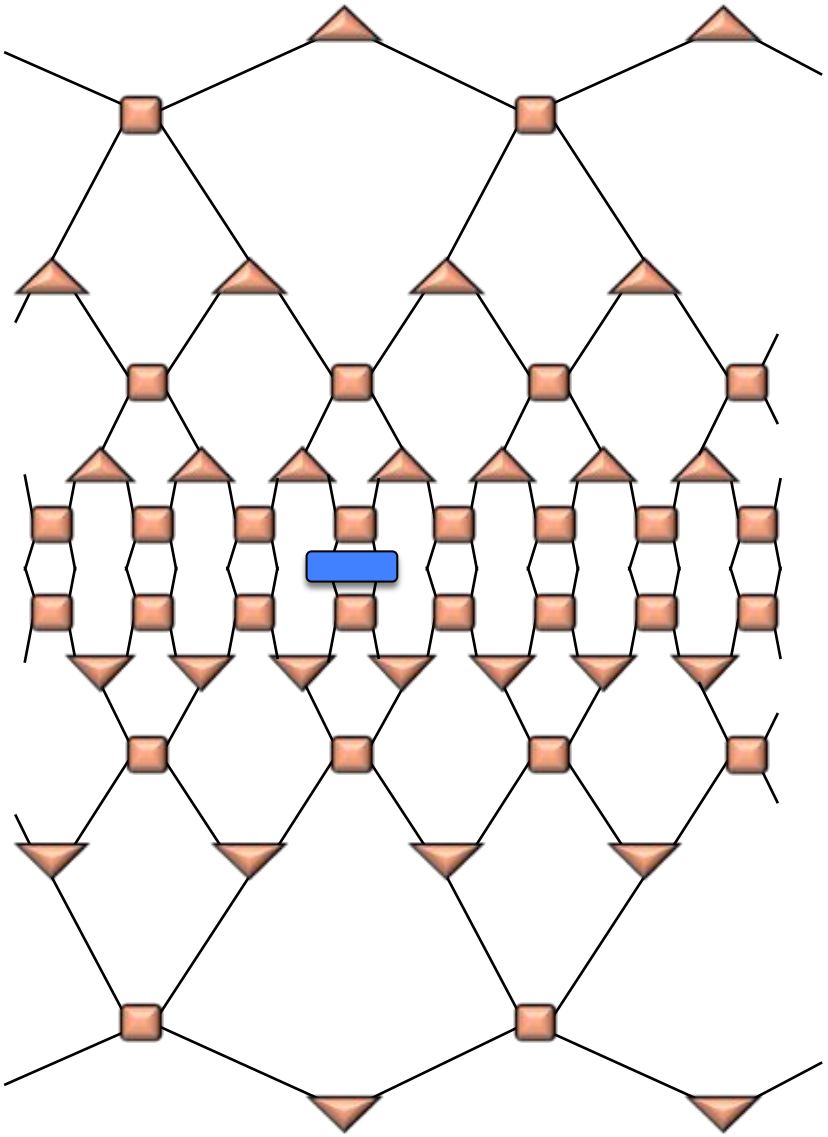






The norm is just the contraction of the top tensors

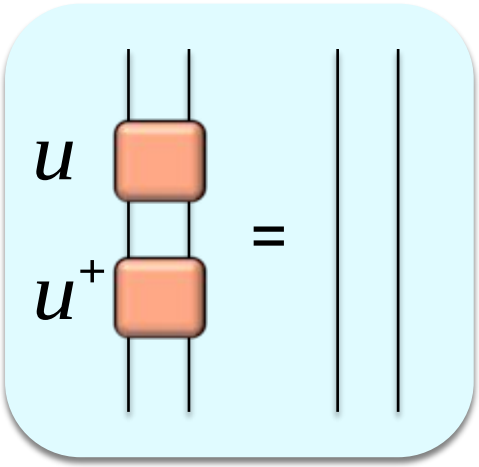
Expectation values

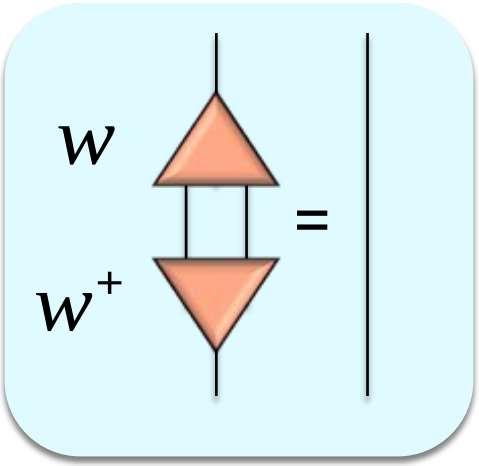


$|\Psi\rangle$

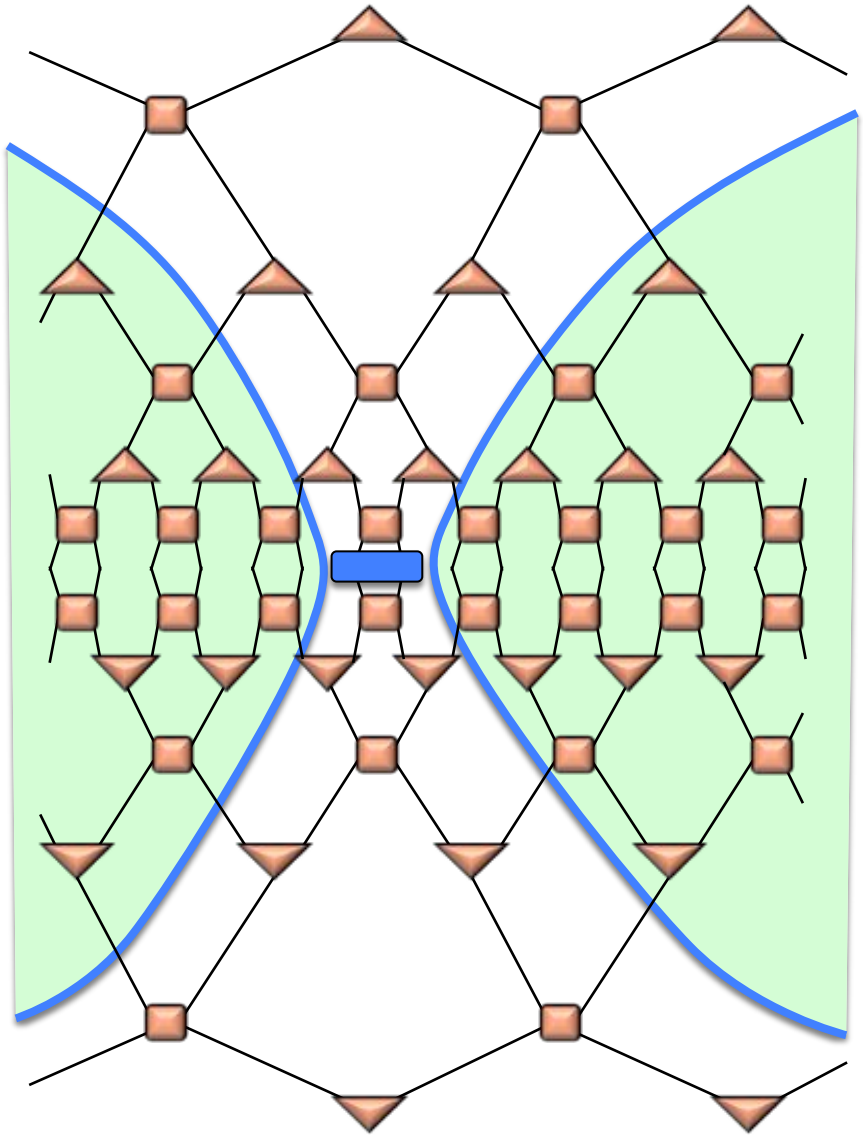
o_{12}

$\langle\Psi|$





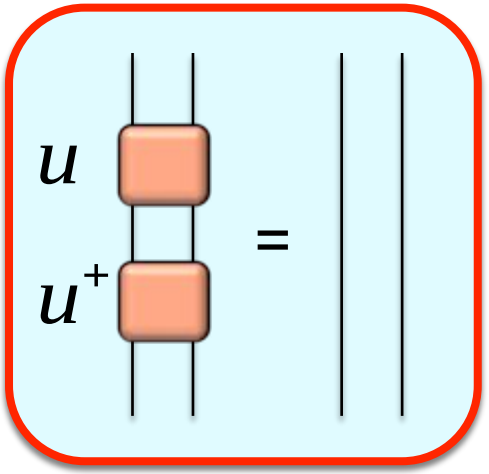
Expectation values

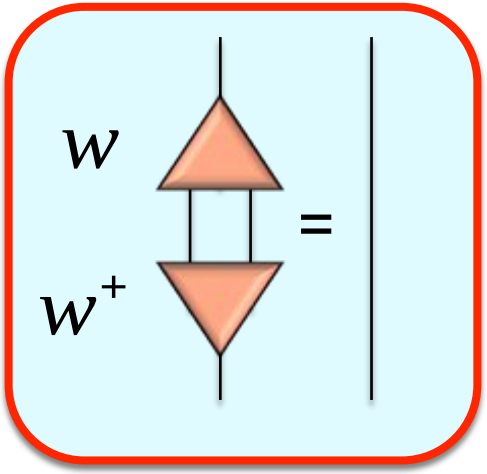


$|\Psi\rangle$

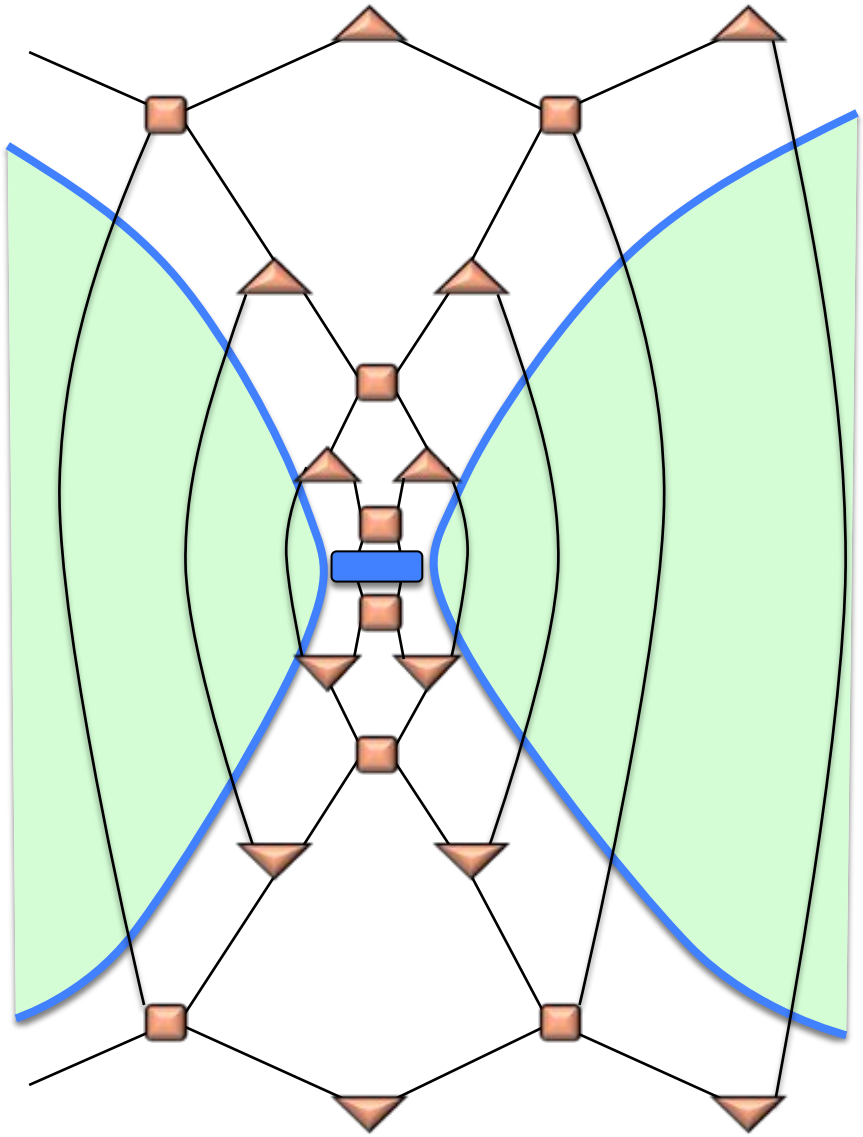
o_{12}

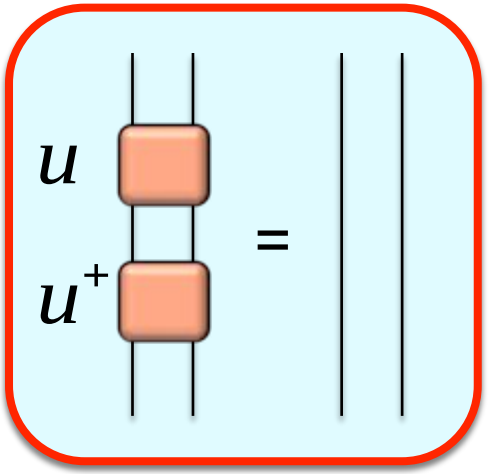
$\langle\Psi|$

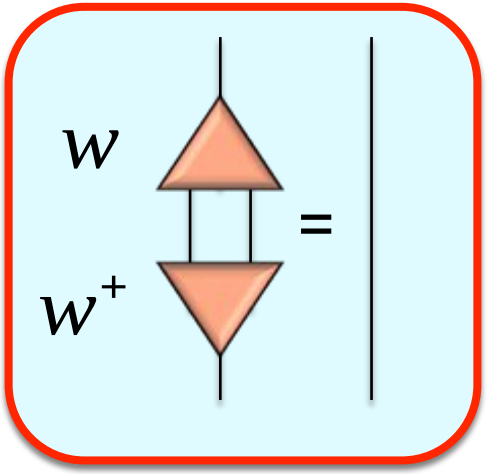




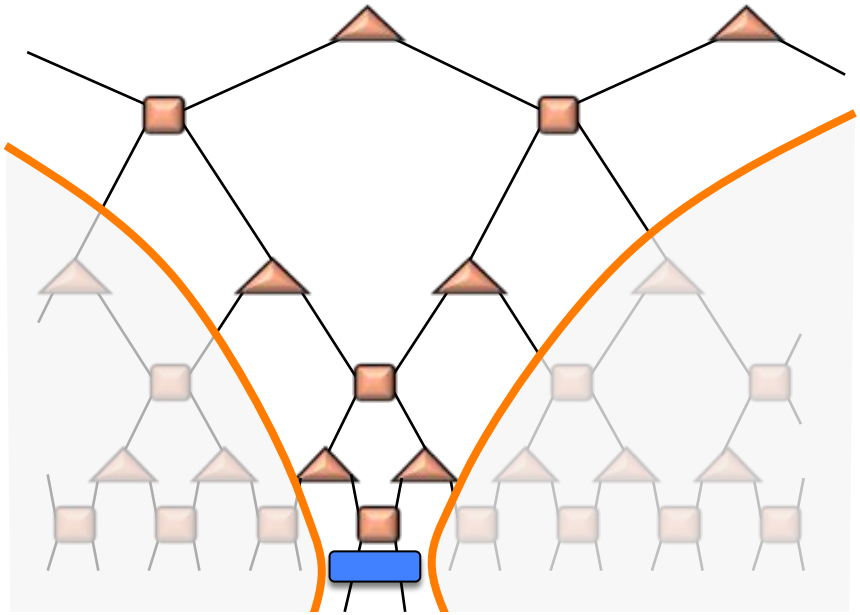
Expectation values







Expectation values



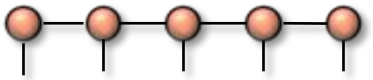
„causal cone“ with bounded width

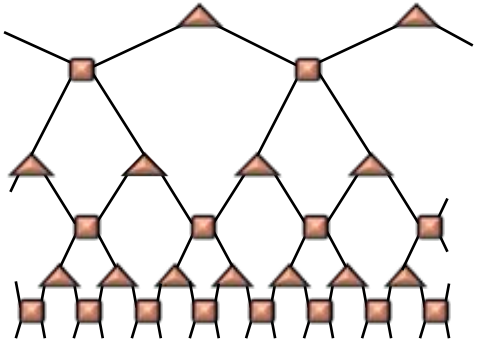
$|\Psi\rangle$

o_{12}

Only tensors inside of the causal cone contribute to the expectation value

MPS vs 1d MERA





MPS in 1d

1d area law $S(L) = O(1)$

Exact contraction is efficient

Finite correlation length

to/from 1d Hamiltonians

Arbitrary tensors

MERA in 1d

Beyond 1d area law $S(L) = O(\log L)$

Exact contraction is efficient

Finite and infinite correlation lengths

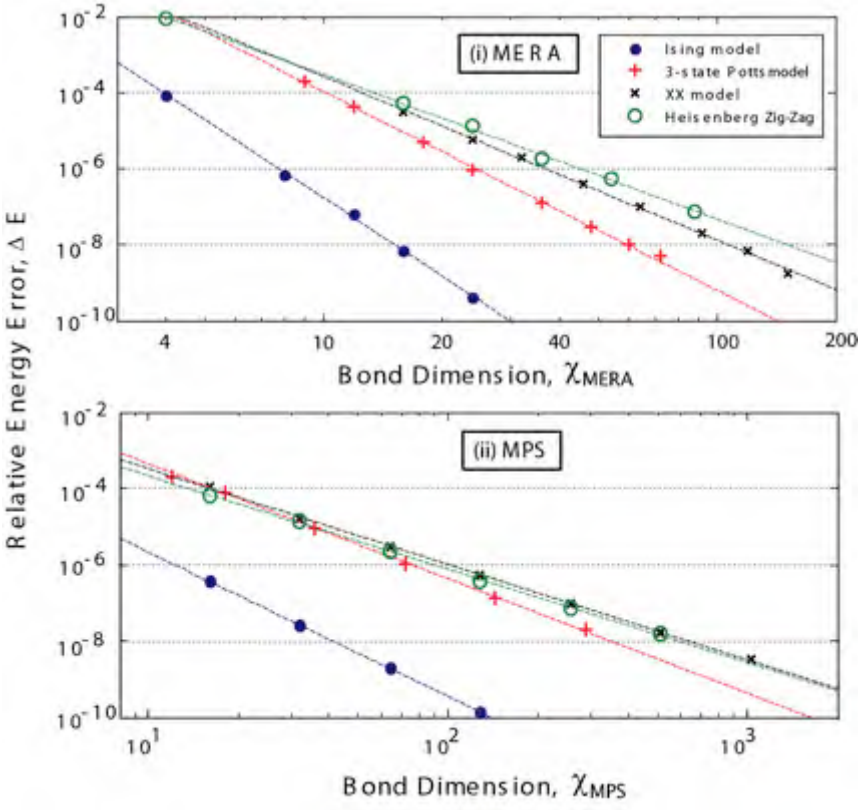
to/from 1d Hamiltonians

Constrained tensors

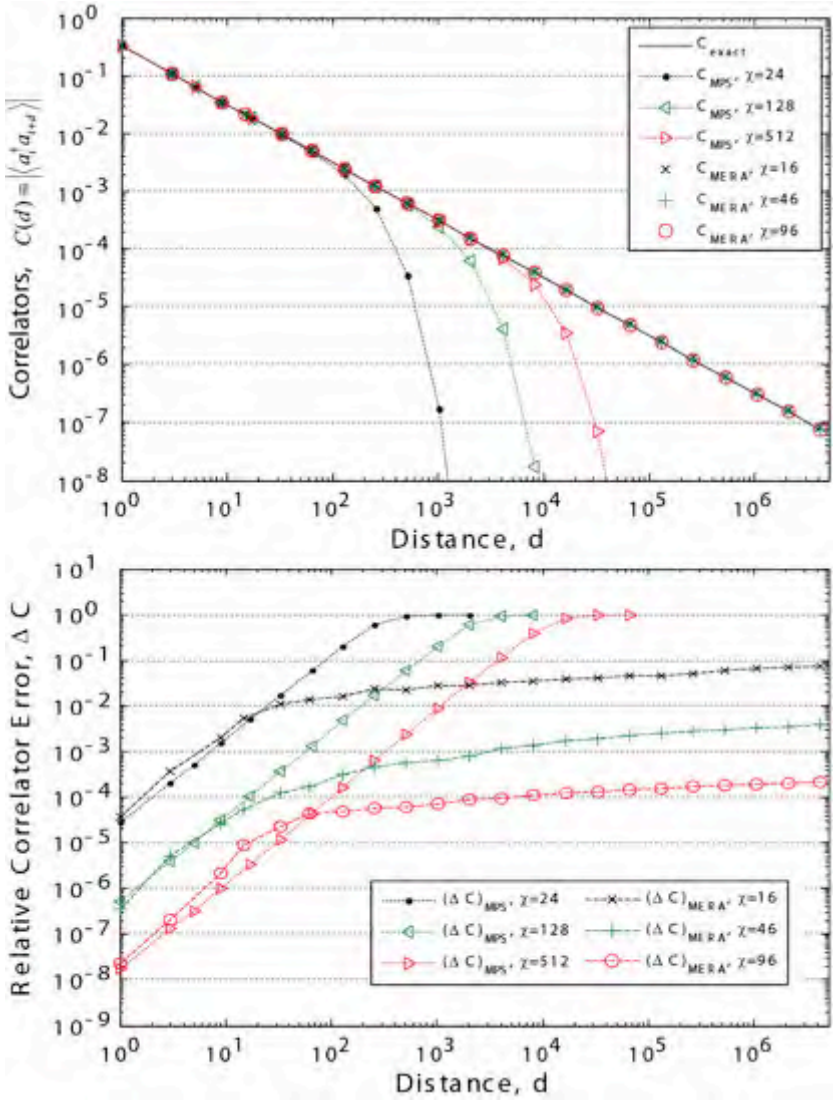
An example:
1d critical systems

G. Evenbly, G. Vidal, in "Strongly Correlated Systems. Numerical Methods", Springer, Vol. 176 (2013)

Critical Energies



Critical XX Correlators



Outline

JG|U

1) Generalities

2) 2d PEPS

2.1) Basics

2.2) Computing

2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

Outline

1) Generalities

2) 2d PEPS

2.1) Basics

2.2) Computing

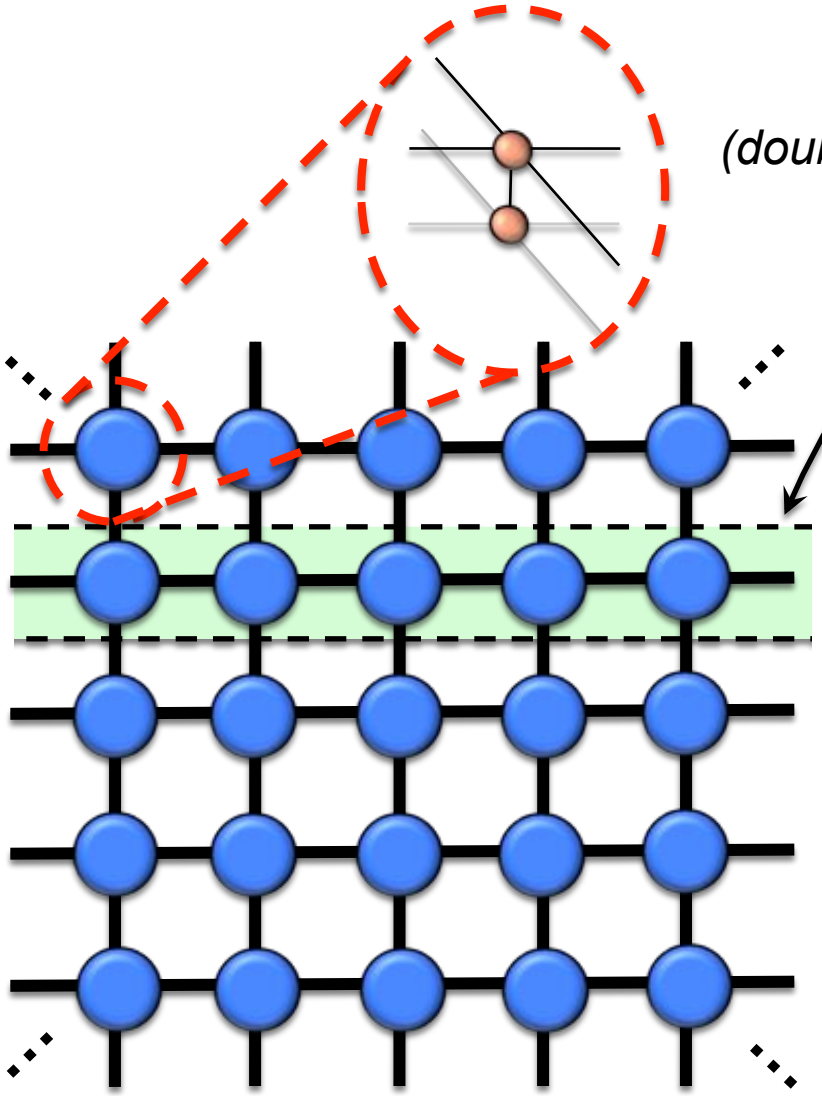
2.3) Fermions

3) 1d MERA: Basics

4) Some further topics

PEPS & Entanglement Hamiltonians

e.g. I. Cirac et al, PRB 83, 245134 (2011), N. Schuch et al, PRL 111, 090501 (2013)

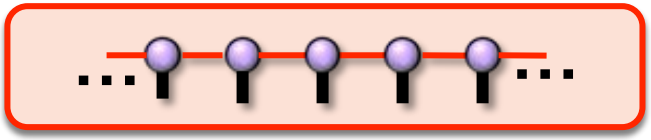


(double indices)

1-dim transfer matrix:

dominant eigenvector?

Can be approximated using infinite MPS

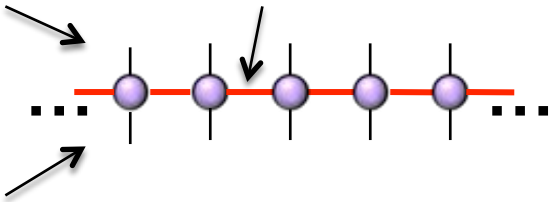


iTEBD, iDMRG, PWFRG, etc

Virtual indices of bra
1…D

Virtual indices of ket
1…D

Boundary virtual index 1... χ



It is also hermitian and positive by construction (up to finite- χ effects)

||

1d Entanglement Hamiltonian

$$\rho = \exp(-H_E)$$

Who is H_E ???

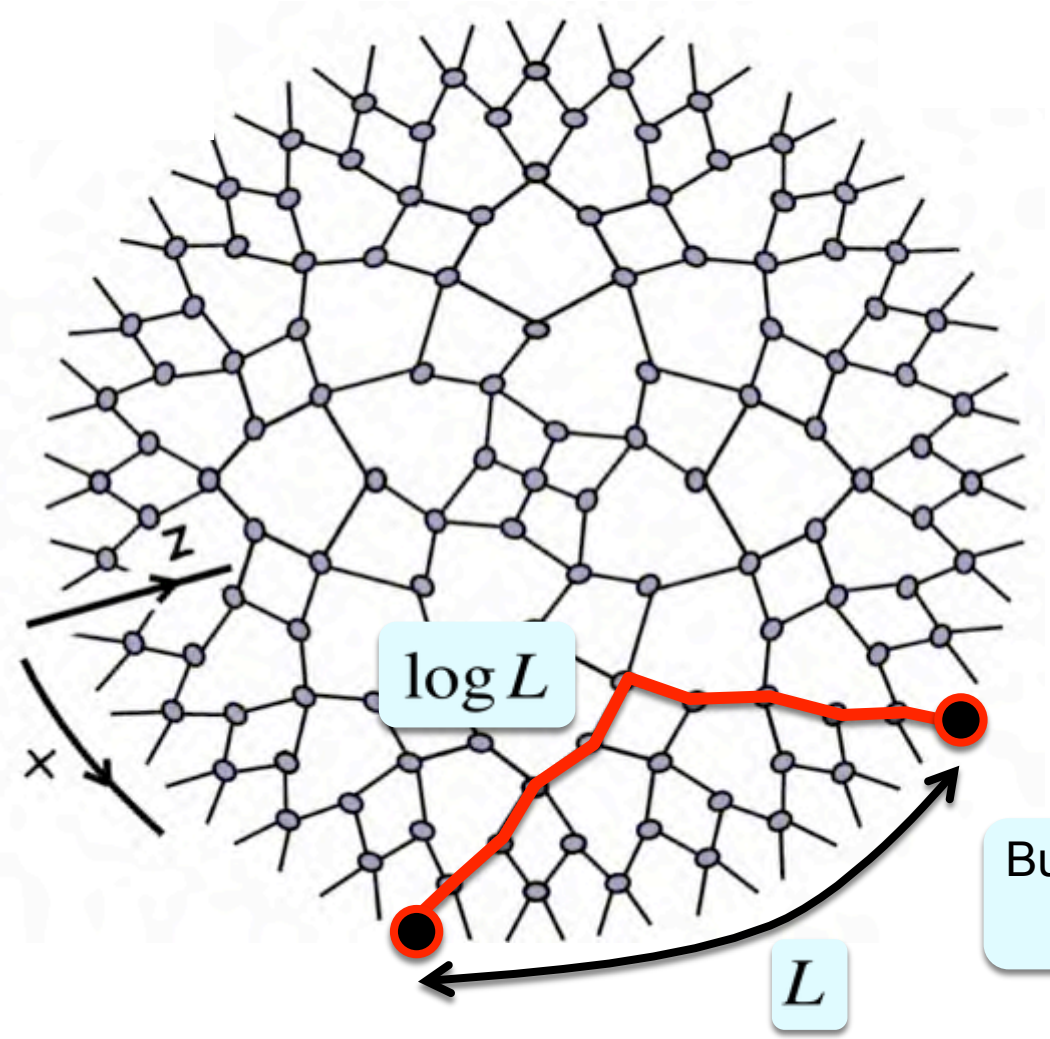
Bulk	**Holographic principle**	**Boundary**
Gapped 2d systems, trivial phase	→	**1d Hamiltonian, short-range**
Critical 2d systems	→	**1d Hamiltonian, long-range**
Gapped 2d systems, topological order	→	**Completely non-local (projector)**

Based on examples

Contraction of 2d PEPS for gapped trivial phases could be approximated efficiently!

MERA & AdS/CFT

e.g. B. Swingle, PRD 86, 065007 (2012), G. Evenbly,G. Vidal, JSTAT 145:891-918 (2011)

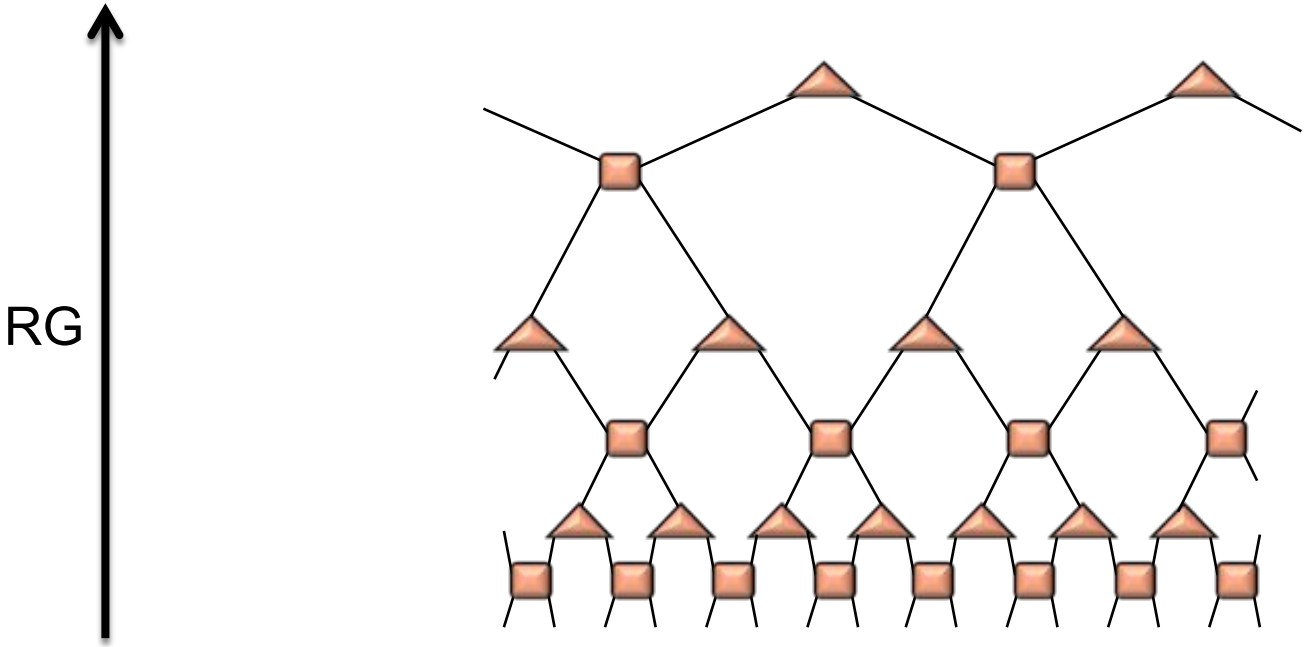


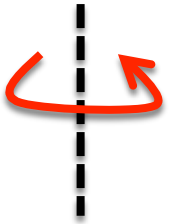
Bulk is a discretized AdS space

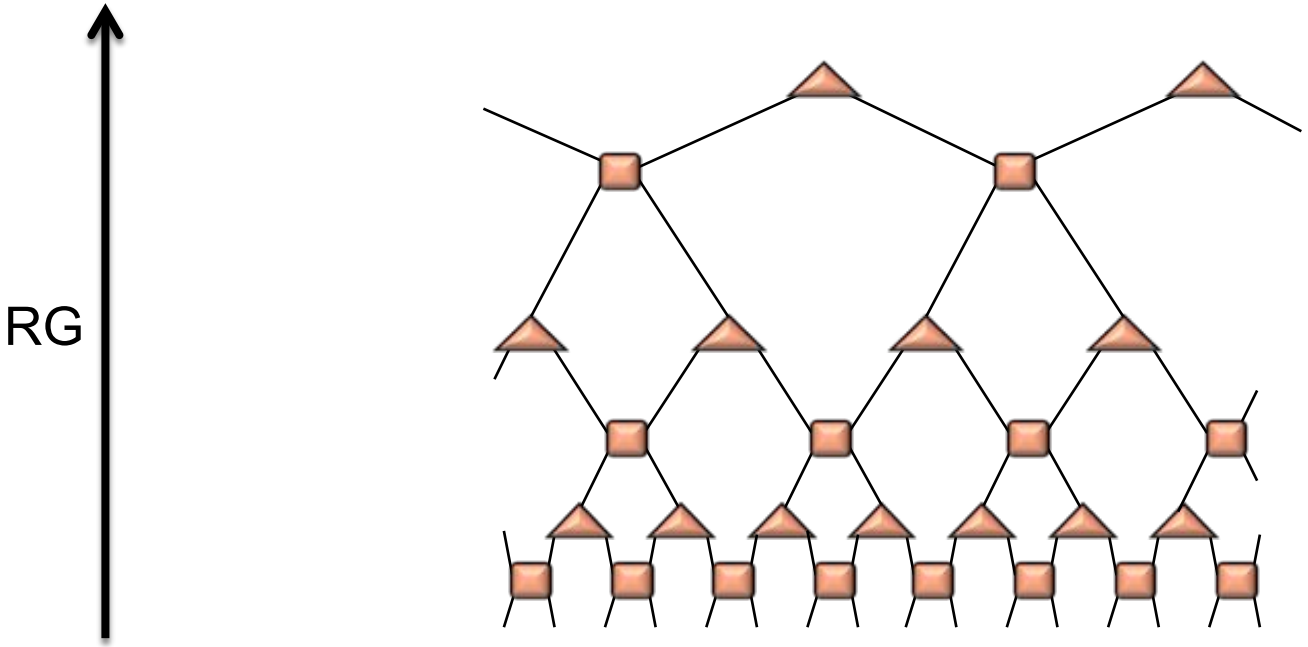
For a scale-invariant MERA, the tensors of a critical model with a CFT limit correspond to a „gravitational“ description in a discretized AdS space: „lattice“ realization of AdS/CFT correspondance

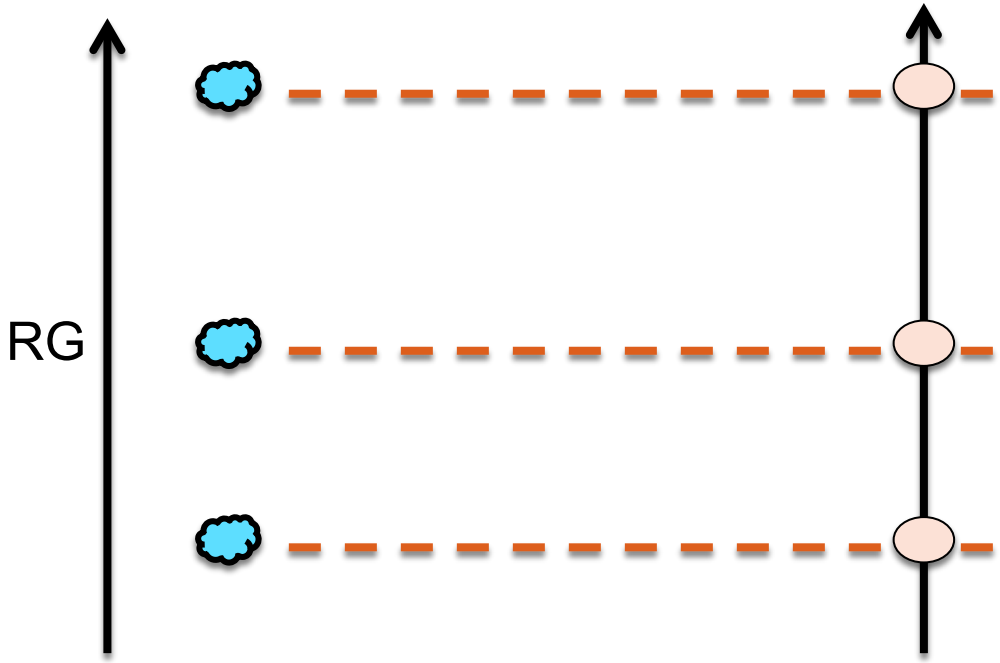
“branching” MERA

G. Evenbly, G. Vidal, arXiv:1210.1895

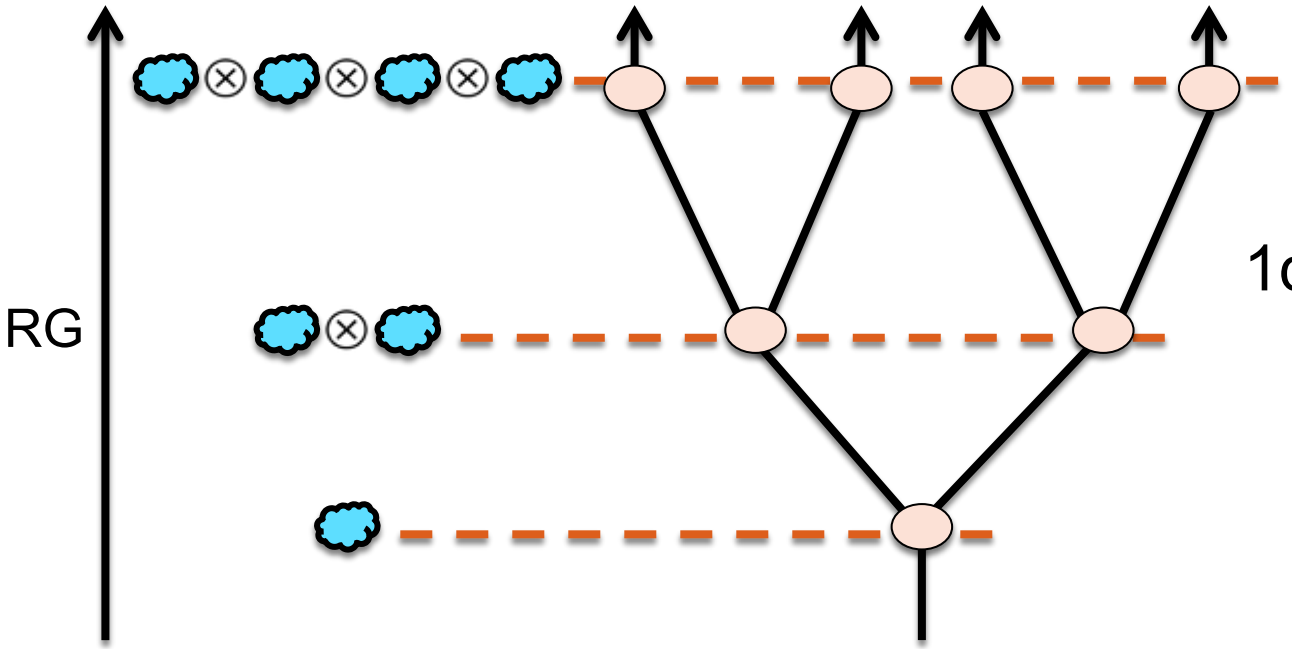

RG




RG



1d MERA



1d branching MERA

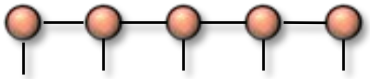
Decoupling of degrees of freedom along RG (e.g. spin-charge separation), and allows arbitrary scalings of the entanglement entropy

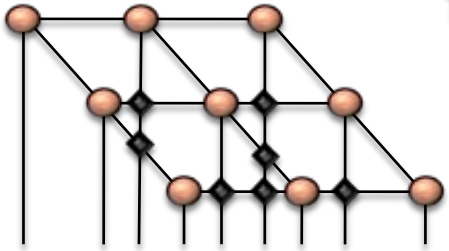
In 2d, ansatz for e.g., Fermi & Bose liquids, $S(L) \approx L \log L$

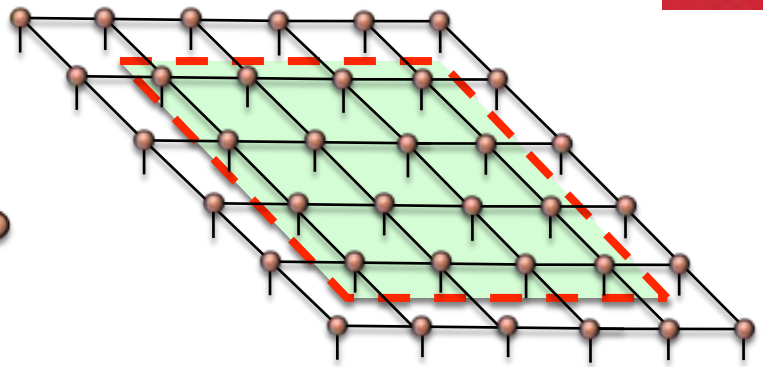
Lots of other things:

Chiral PEPS
Entanglement measures
Time evolution
Infinite systems
Classical systems
Corner Transfer Matrices
Tensor Renormalization Group
Symmetries
2d MERA
3d PEPS
MERA optimization
Topological systems
Excited states
Thermal states & dissipation
Continuous tensor networks
Tensor Networks & Montecarlo
Practical implementations
Blablabla...

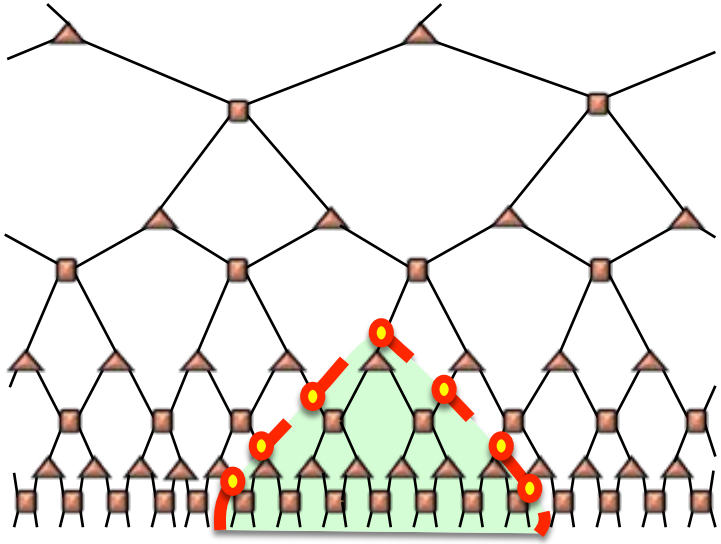
Ask me
if interested!

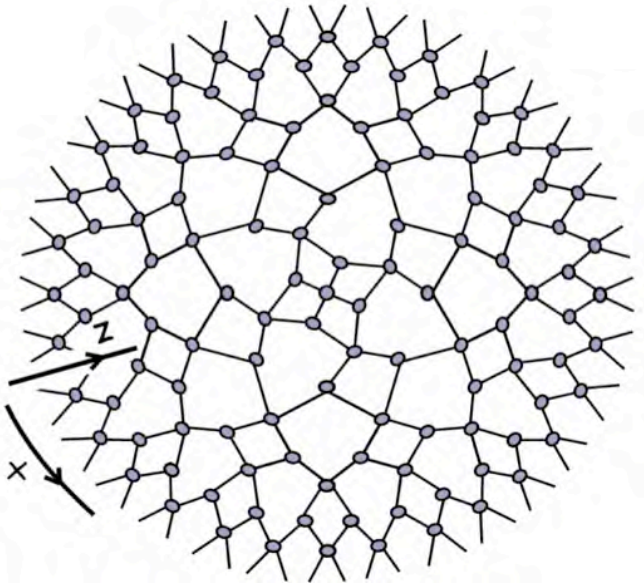






Thank you!





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