On measurement-based quantum computation with the toric code states

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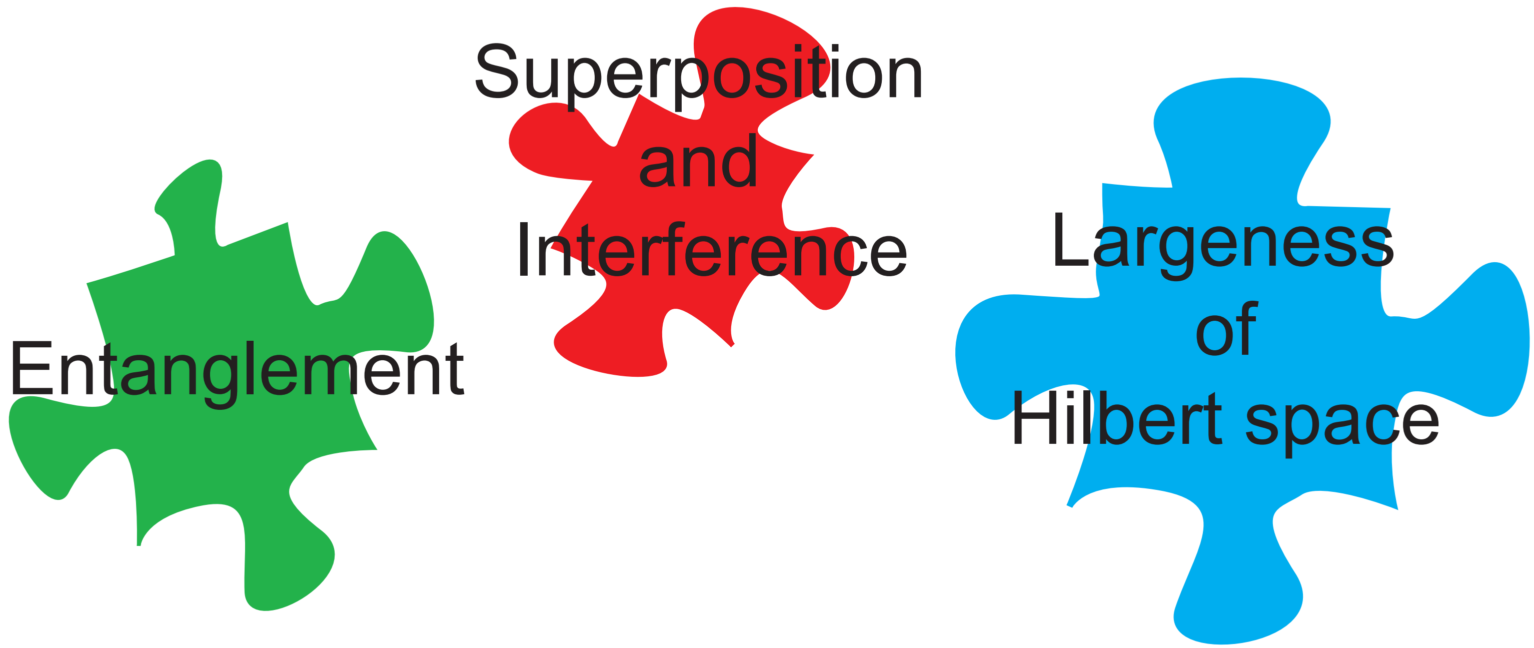
Perimeter Institute, moving to UBC in Jan 08

PITP Vancouver, Dec 2, 2007

Joint work with Sergey Bravyi, IBM; PRA 76, 022304 (2007).

Where does the power in quantum computation come from?



- A prerequisite for a speed-up in quantum computation is the hardness of its classical simulation.

Quantum compuation and statistical mechanics

- Characteristic *state overlaps* in measurement-based quantum computation can be related to the *partition function* of the Ising model.

Ising model		**One-way QC**
Z_{Ising}	$\sim$	$\langle$local state$\vert$quantum resource$\rangle$
planar + magn. field simulation >= NP hard[1]		$\vert$cluster state$\rangle$ universal
planar, no magn. field simulation efficient		$\vert$planar code state$\rangle$ not universal

1: F. Barahona (1982).

Talk outline

Part I: One-way quantum computer ($QC_{\mathcal{C}}$) and cluster states

What is the one-way quantum computer?

Part II: Efficient classical simulation of MQC based on the tree-ness of graphs

The $QC_{\mathcal{C}}$ on graph states of tree graphs can be efficiently simulated classically.

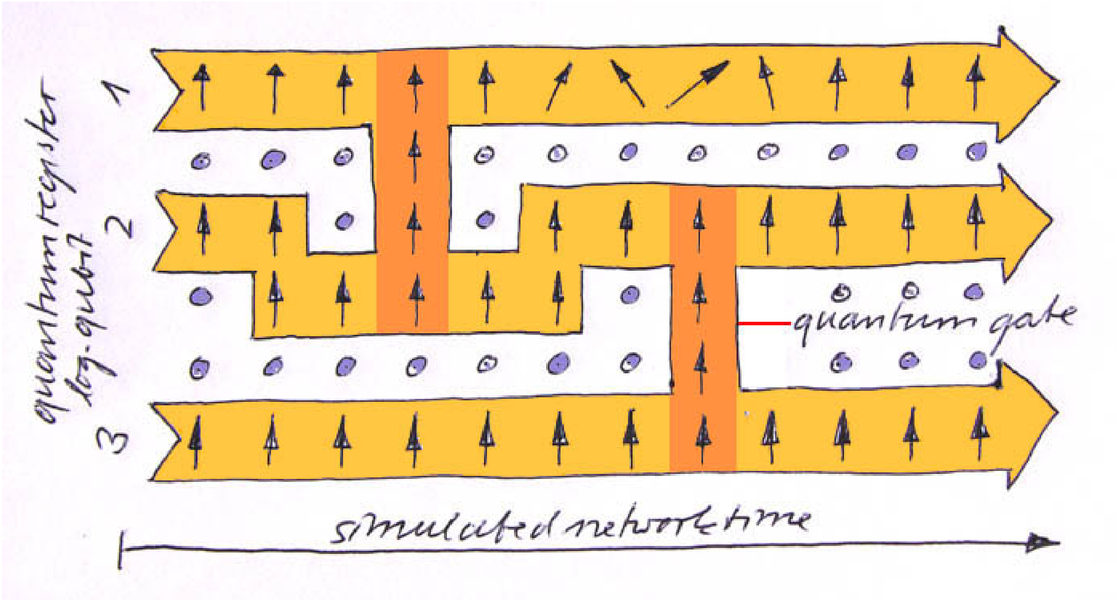
Part III: Efficient classical simulation of MQC based on planarity of graphs

The $QC_{\mathcal{C}}$ on the planar code state can be efficiently simulated classically.

Part I:

The one-way quantum computer and cluster states

The one-way quantum computer



measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

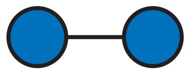
R. Raussendorf and H.-J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).

Cluster states - creation

1. Prepare product state $\bigotimes_{a\in\mathcal{C}} \frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}$ on d-dimensional qubit lattice $\mathcal{C}$.

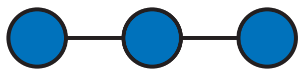
2. Apply the Ising interaction for a fixed time T (conditional phase of π accumulated).

Cluster states - simple examples



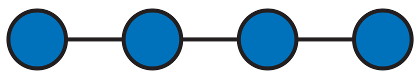
$$|\psi\rangle_2 = |0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2$$

Bell state



$$|\psi\rangle_3 = |+\rangle_1|0\rangle_2|+\rangle_3 + |-\rangle_1|1\rangle_2|-\rangle_3$$

GHZ-state



$$|\psi\rangle_4 = |0\rangle_1|+\rangle_2|0\rangle_3|+\rangle_4 + |0\rangle_1|-\rangle_2|1\rangle_3|-\rangle_4 + \\ + |1\rangle_1|-\rangle_2|0\rangle_3|+\rangle_4 + |1\rangle_1|+\rangle_2|1\rangle_3|-\rangle_4$$

Number of terms exponential in number of qubits!

Cluster states - definition

A cluster state $|\phi\rangle_{\mathcal{C}}$ on a cluster $\mathcal{C}$ is the single common eigenstate of the stabilizer operators $\{K_a\}$,

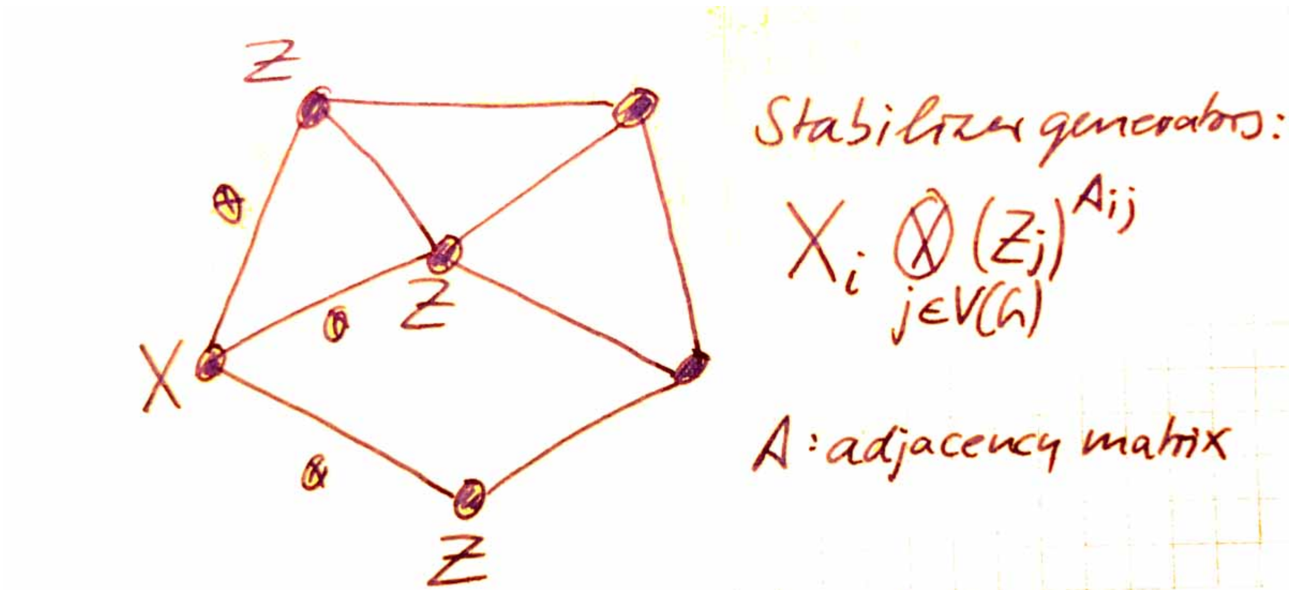
$$K_a|\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \ \ \forall\, a, \tag{1}$$

where

$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in \mathcal{C}, \tag{2}$$

and $b \in N(a)$ if a,b are spatial next neighbors in $\mathcal{C}$.

Graph states and local complementation



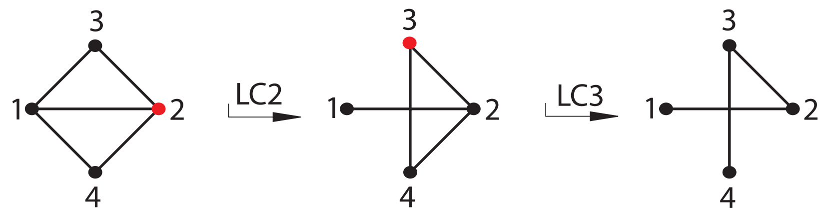
- Graph states are a straightforward generalization of cluster states.
- Cluster states are graph states corresponding to lattice graphs.

Graph states and local complementation

- For a given graph state, there exist local unitary equivalent graph states corresponding to different graphs.
- The equivalent graph states can be reached by a graph transformation, namely *local complementation*.

How local complementation works:

- Pick a vertex v in the graph.
- Find all its neighbors $\{u_i\}$.
- Invert all edges (u_i, u_j).



Part II:

Classical simulation of the $QC_{\mathcal{C}}$ on tree-like graph states via tensor networks

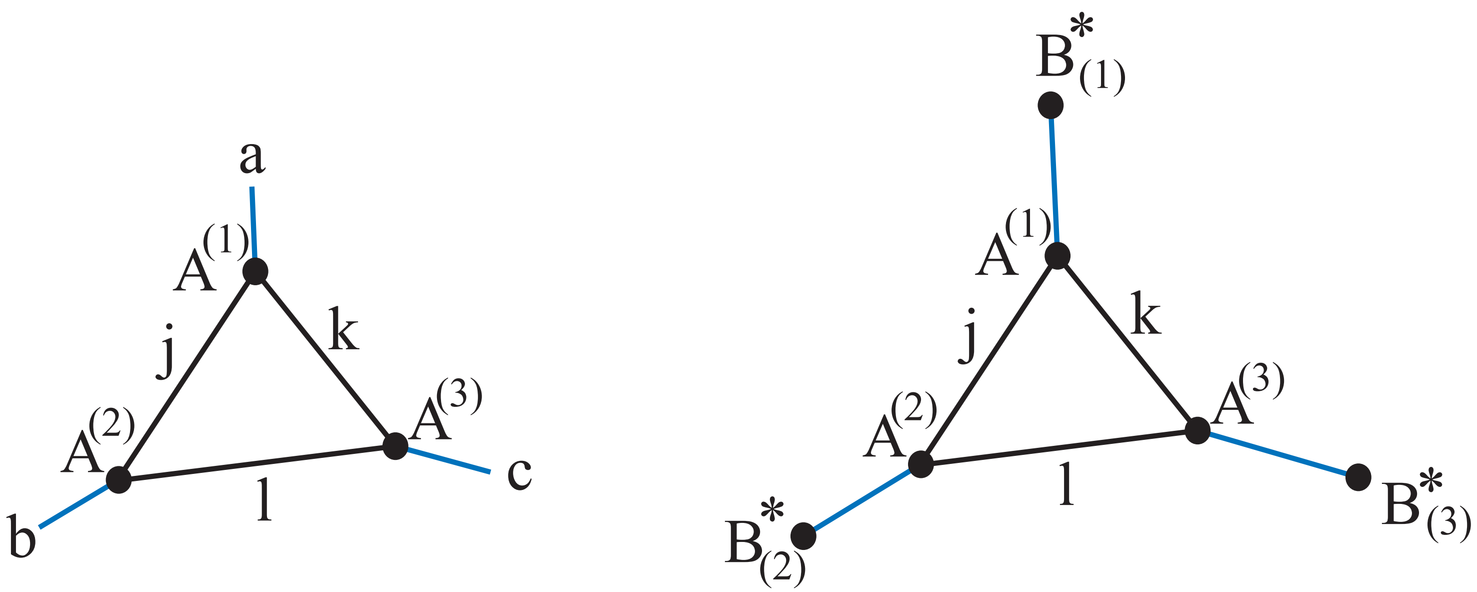
Requirements for classical simulation

- Predict probabilities for outcomes of complete measurements.

$$p = |\langle \text{local state} | \text{quantum resource} \rangle|^2$$

- Predict probabilities for outcomes of partial measurements (subset of qubits traced over).

Tensor networks



a) quantum state

b) state overlap <B|A>

tensor networks

$$|\psi\rangle = \sum_{abc} \left(\sum_{jkl} A^{(1)}_{ajk} A^{(2)}_{bjl} A^{(3)}_{ckl} \right) |a\rangle_1 |b\rangle_2 |c\rangle_3$$

Tensor networks

$$\underbrace{A_{ijklm}}_{\text{rank } r}, \quad i,j,k,l,m = 1 \,..\, \underset{\text{dimension}}{d}$$

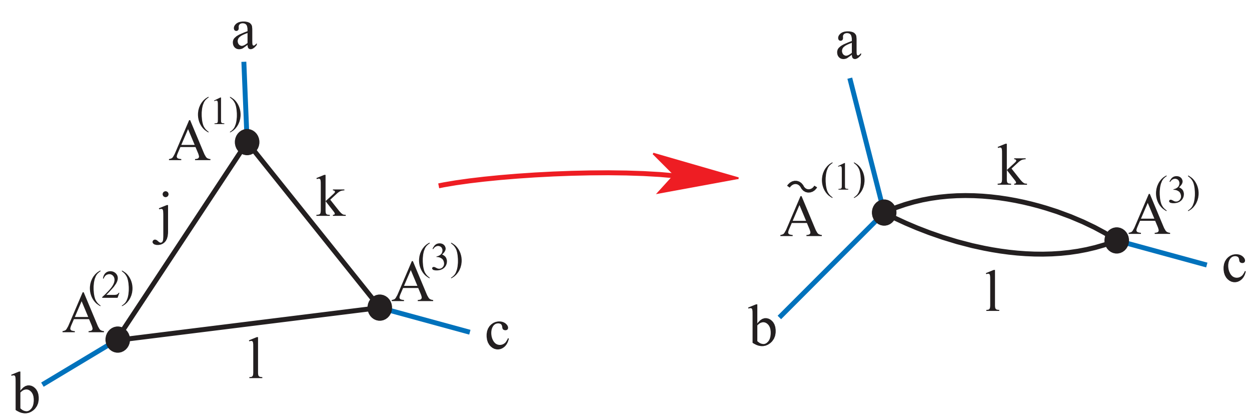
Number of components in A:

$$|A| = d^r. \tag{3}$$

- The rank of $A(v)$ equals the vertex degree $\deg(v)$.

Tensor networks

Task: Contract edges in the network graph.



This changes the degree of the remaining vertices.

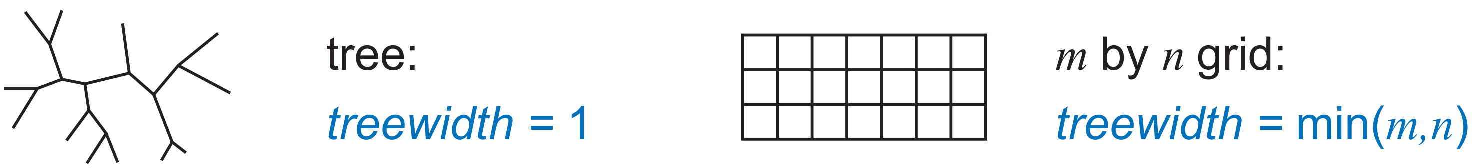
$$ {}_1\langle a| \otimes {}_2\langle b| \otimes {}_3\langle c|\psi\rangle = \left(\sum_{jkl} A^{(1)}_{ajk} A^{(2)}_{bjl} A^{(3)}_{ckl} \right) = \left(\sum_{kl} \tilde{A}^{(1)}_{abkl} A^{(3)}_{ckl} \right) $$

with $\tilde{A}^{(1)}_{abkl} = \sum_j A^{(1)}_{ajk} A^{(2)}_{bjl}$.

Graphs close to a tree

High vertex degrees in the contraction of edges in G can be avoided for tree-like graphs.

- The deviation of a graph from a tree is formalized by the *treewidth*.

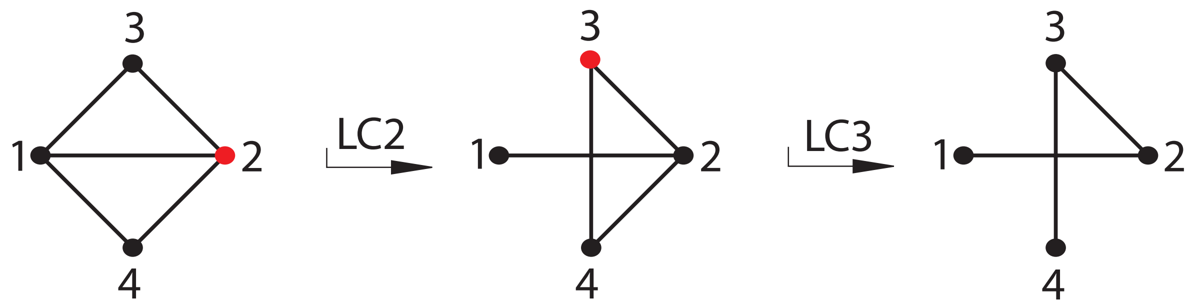


- MBQC can be efficiently simulated for graph states on *tree graphs* and graphs close to trees.

Graphs close to a tree

Theorem 1 (Markov & Shi, 05): *Consider a n-vertex graph G of tree width T. Then, a one-way quantum computation on $|G\rangle$ can be simulated in time $O(n)\exp(O(T))$.*

Tensor networks and entanglement



- Problem: Local complementation on a graph G leaves the computational power of the corresponding graph state $|G\rangle$ invariant but changes the treewidth of G.

- Remedy: rank width.

Tensor networks and entanglement

Theorem 2 (SDV05, VdN06): Be χ the rank width of an n-qubit graph state $|G\rangle$. *The complexity of classical MQC simulation on* $|G\rangle$ *is* Poly(n) exp(χ)*.*

Theorem 3 (VdN06): χ *is an entanglement monotone.*

- *Entanglement is necessary for hardness of the classical simulation.*
- Is entanglement also sufficient?

Part III:

Classical simulation of MQC on planar code states

Goals of Part III

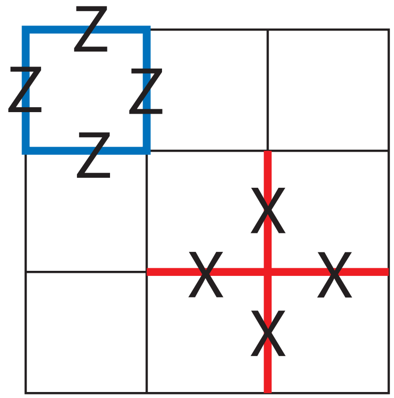
- MQC with the planar code state can be efficiently simulated classically, by mapping to the *planar* Ising model.

- What about entanglement in these states?

- MQC with a universal 2D cluster state can also mapped to the Ising model: planar + magnetic field.

S. Bravyi and R. Raussendorf, PRA 76, 022304 (2007).

Definition of the planar code state



$|K\rangle$

- Qubits live on the edges.
- The planar code state is a stabilizer state. Stabilizer operators associated with the sites and plaquettes of the lattice.

Why consider a planar code state?

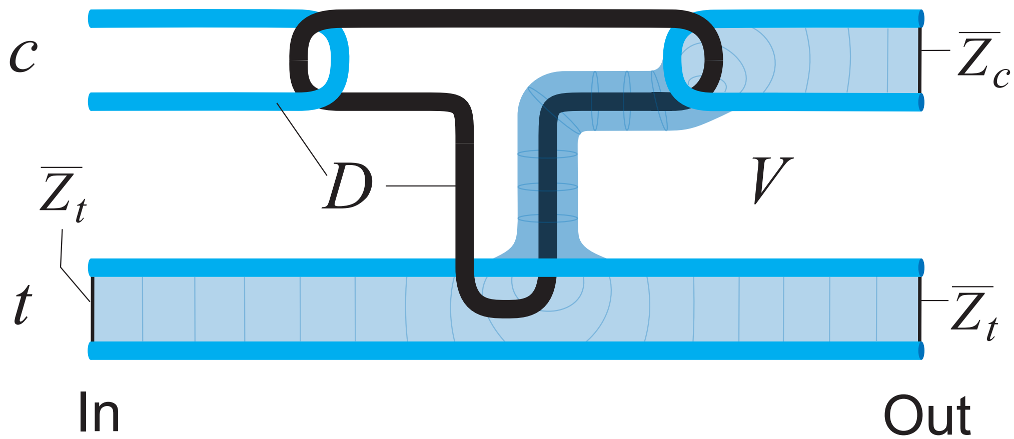
- Planar code states and cluster states are closely related.

- $|K\rangle$ obeys entropy area law.

- $|K\rangle$ shows topological order.

2D local FTQC

Combine cluster states and planar code states to obtain this:



- Fault-tolerant universal quantum computation in 2D local architecture.
- Threshold: 0.75×10^{-2} for each source in an error model with preparation, gate, storage and measurement errors.

R.Raussendorf and J. Harrington, Phys. Rev. Lett. 98, 190504 (2007).

Our Results

Theorem 4A: *Complete local measurements on a planar code state can be simulated efficiently classically.*

Theorem 4B: *Suppose that at each step j of MQC the sets of measured and unmeasured qubits E_j, $\overline{E}_j$ are connected. Then, partial local measurements on a planar code state can be simulated efficiently classically.*

S. Bravyi and R. Raussendorf, PRA 76, 022304 (2007).

Connection with Ising model

Task: compute overlap between $|K\rangle$ and local state $|\Psi\rangle = \bigotimes_{(ij)} |\phi_{jk}\rangle$.

$|K\rangle$ written in the computational basis $\{|x_1, x_2, .., x_n\rangle, x_k = \pm 1\}$:

$$|K\rangle = \sum_{x \in \mathcal{L}_0} |x\rangle;$$

where $\mathcal{L}_0 = \{x :\ B_p\,|x\rangle = |x\rangle, \forall\, p\}$. Then

$$\langle \Psi | K \rangle = \Lambda \sum_{x \in \mathcal{L}_0} \exp\left(\sum_{(jk)} \beta_{jk} x_{jk} \right),$$

where $\exp(2\beta_{ij}) = \langle \phi_{ij}| + 1\rangle / \langle \phi_{ij}| - 1\rangle$.

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where $\exp(2\beta_{ij}) = \langle \phi_{ij}| + 1\rangle / \langle \phi_{ij}| - 1\rangle$.

Now solve the constraint $x \in \mathcal{L}_0$:

$$x_{ij} = \sigma_i \sigma_j, \quad (\sigma_k = \pm 1 \text{ for all sites } k).$$

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$$|K\rangle = \sum_{x \in \mathcal{L}_0} |x\rangle;$$

where $\mathcal{L}_0 = \{x : B_p |x\rangle = |x\rangle, \forall p\}$. Then

$$\langle \Psi | K \rangle = \frac{\Lambda}{2} \sum_{\{\sigma\}} \exp\left(\sum_{(jk)} \beta_{jk} \sigma_i \sigma_j \right) =: Z[\beta],$$

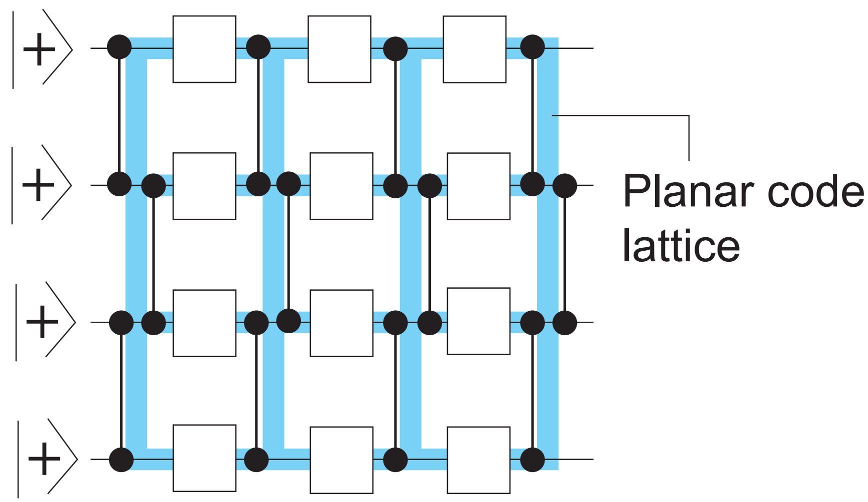
where $\exp(2\beta_{ij}) = \langle \phi_{ij}| + 1\rangle / \langle \phi_{ij}| - 1\rangle$.

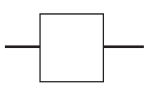
$Z[\beta]$ is the partition function of the Ising model.

Connection with the circuit model

Compute partition function by transfer matrix method:

$$\langle \Psi | K \rangle = \frac{\Lambda}{2} \langle \hat{+} | T^{(z)}_{L+1} T^{(x)}_{L+1} T^{(z)}_{L} \, .. \, T^{(z)}_{2} T^{(x)}_{1} T^{(z)}_{1} | \hat{+} \rangle .$$



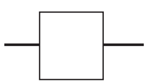
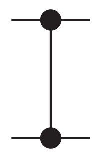
 : $T^{(x)}_{l,p} = \exp\left(\beta_{h(l,p)} X_p\right)$,  : $T^{(z)}_{l,p} = \exp\left(\gamma_{v(l,p)} Z_p Z_{p+1}\right)$

Mapping to non-interacting fermions

Map Pauli operators X_p, $Z_p \otimes Z_{p+1}$ to Majorana fermions c_l, with $\{c_k, c_l\} = 2\delta_{kl} I$ (Jordan-Wigner transformation):

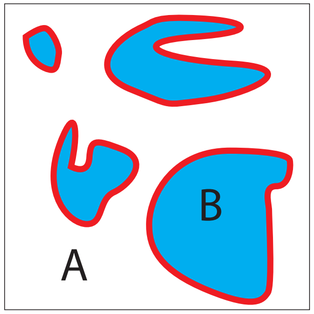
$$\begin{aligned} c_{2p} &= X_1 X_2 .. X_{p-1} Y_p, \\ c_{2p-1} &= X_1 X_2 .. X_{p-1} Z_p, \end{aligned}$$

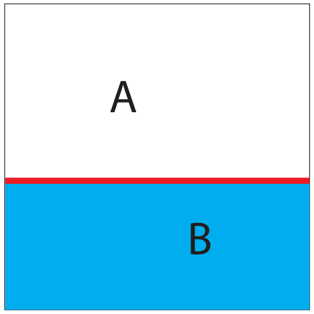
Then,

: $T^{(x)}_{l,p} = \exp\left(i\beta c_{2p-1} c_{2p}\right)$, : $T^{(z)}_{l,p} = \exp\left(i\gamma c_{2p} c_{2p+1}\right)$

$T^{(x)}$, $T^{(z)}$ are *quadratic* in $\{c_l\}$ $\rightarrow$ efficiently simulatable.

Entanglement in surface code states





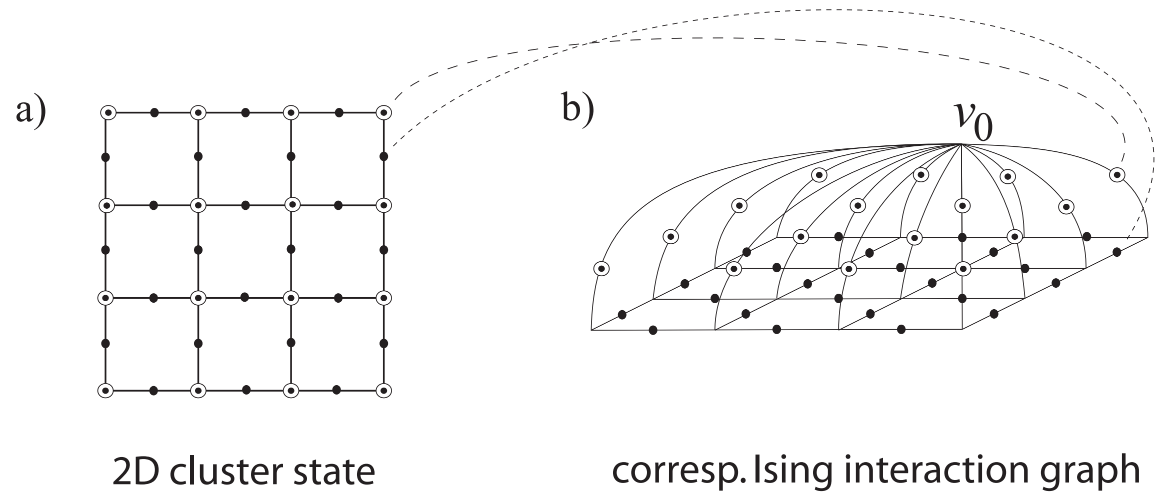
- In surface code states bi-partite entanglement proportional to length of boundary between parties, thus large.

- Classical simulation nevertheless efficient.

Large entanglement not sufficient for hardness of classical simulation.

The 2D cluster state

- 2D cluster state $|\mathcal{C}\rangle$ is universal for MQC.



$$\langle \Psi|\mathcal{C}\rangle \sim \sum_{\{\sigma_j|\, j\neq v_0\}} \exp\left(\sum_{(jk)|\, j,k\neq v_0} \beta_{jk}\sigma_j\sigma_k + \sum_{j\neq v_0} \beta_{j0}\sigma_j \right)$$

- *Planar Ising model with magnetic fields.*

(Barahona 82: $\geq$ NP-hard)

Summary

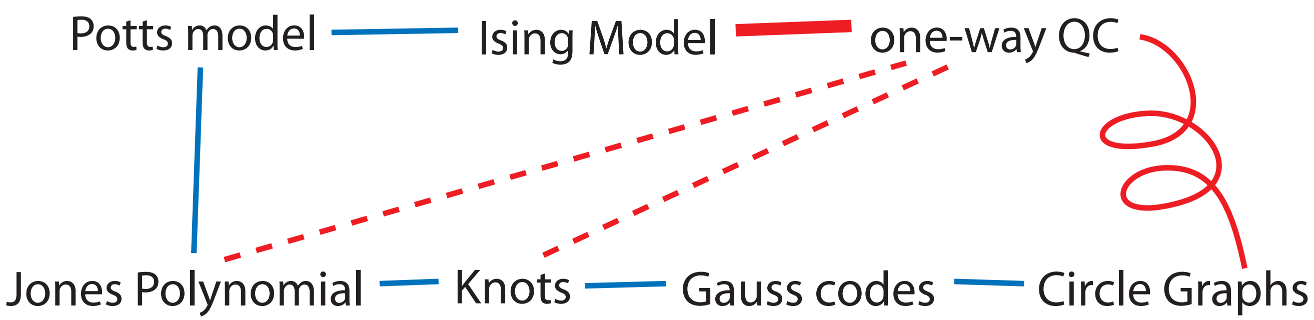
- MQC on planar code state can be efficiently simulated classically, by mapping to the planar Ising model.

- MQC on a universal 2D cluster state also described by the Ising model, but interaction graph is non-planar.

- Large entanglement in the resource state is necessary but *not sufficient* for universal MQC & hardness of classical simulation.

\+ Base camp for exploring graph theory from a quantum information perspective.

Open problems



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