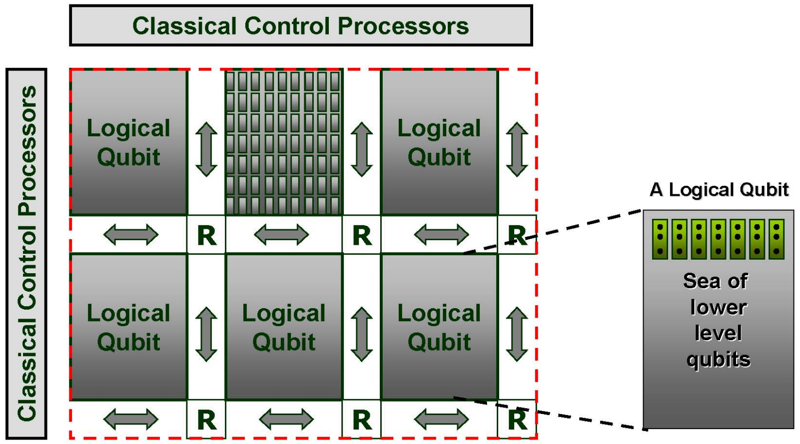
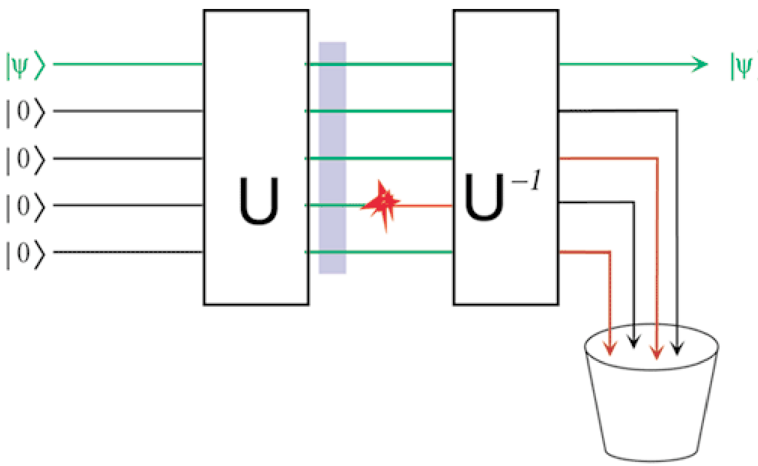
Decoherence in Quantum Walks and Quantum Computers

Andrew Hines

Fault-tolerant quantum computation



- *Fault-tolerant-threshold*: Using concatenation, for a given error/noise model and code, can determine a threshold for the noise strength such that the computation is still successful



- Local, Markovian error model (Aharonov & Ben-Or 99, Knill 2004)
  - Locality: No correlations between environments of different qubits, except through gates;
  - Markovian: The environment is renewed each time step, no correlations between environments at different times;
  - threshold in terms of a probability of error at each circuit location.

- Local, non-Markovian error model (Burkhard & Terhal 05, Aliferis, Gottesman & Preskill 06)
  - Starts from a Hamiltonian formulation
  - threshold in terms of an operator norm on system-bath interaction term in Hamiltonian

- Long-range correlated, non-Markovian error model (Aharonov, Preskill & Kitaev 06)
  - Extends above to non-local environments - qubits share environments even when not interacting
  - interaction terms between environment and *pairs* of qubits (long-range); must decay faster than $1/r^D$

Fault-tolerant quantum computation

- Assumptions are implicit in any model: we need to know what assumptions are physically relevant, does QEC need to be changed, & want thresholds in a language experimentalists can understand

  - ie. system+environment described by an effective Hamiltonian, based on an Uv cut-off
$$\mathcal{H} = H^{QC}(\Omega_0) + V(\Omega_0) + H^{Env}(\Omega_0)$$
  - the interaction which determines the threshold is dependent upon this cut-off

$$V(\Omega_0) = \sum_k A_k^\alpha \hat{\tau}_k^\alpha + \sum_{jk} B_k^{\alpha\beta} \hat{\tau}_j^\alpha \hat{\tau}_k^\beta + \ldots$$

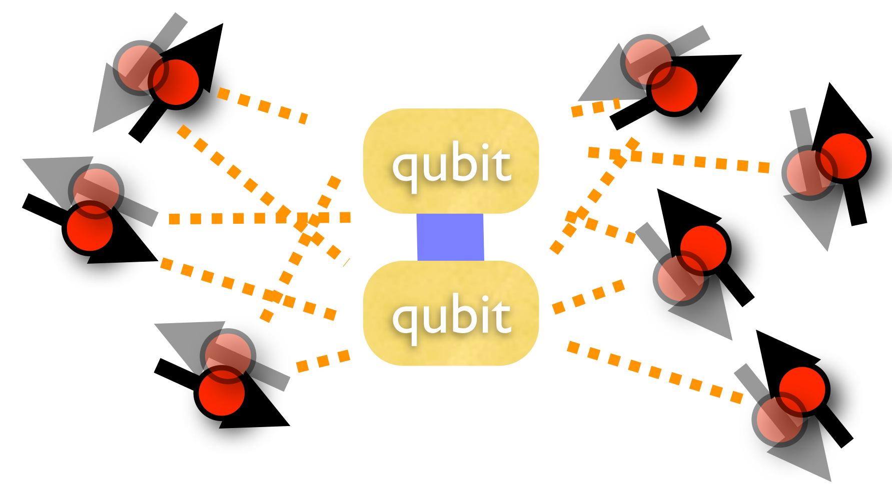
- Such questions are being addressed in Hamiltonian formulation (*Novais, Baranger, PRL 06,07*)

- we want to understand the dynamics of decoherence from quantum environments

Quantum Environments

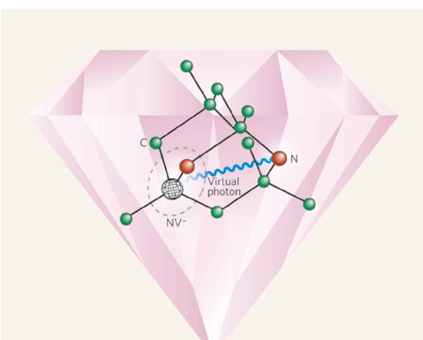
- oscillator baths (delocalised modes): *Feynman-Vernon/Caldeira-Leggett*

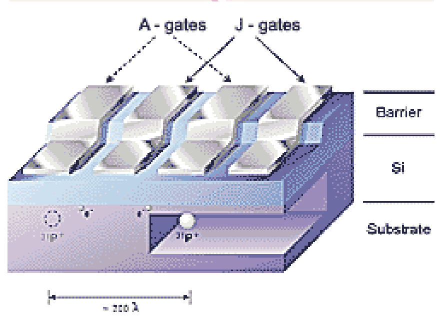
$$H_{osc} = \sum_{q=1}^{N_o} \left(\frac{p_q^2}{m_q} + m_q \omega_q^2 x_q^2 \right)$$

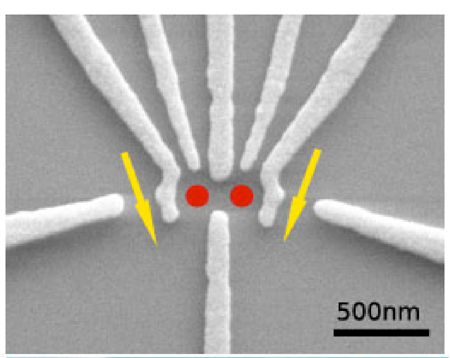


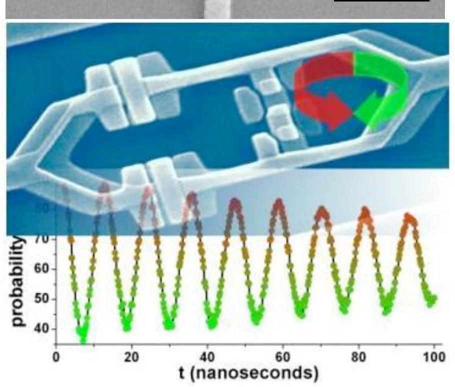
- ***spin baths*** (localised modes): *Prokofiev-Stamp*

$$H_{sp} = \sum_{k=1}^{N_o} \mathbf{h}_k \cdot \vec{\sigma}_k + \sum_{k,k'} V_{kk'}^{\alpha\beta} \sigma_k^{\alpha} \sigma_{k'}^{\beta}$$









- not noise sources; complex bath dynamics, back-action, environment mediated interactions - correlated errors

- effectiveness of quantum error correction in presence of correlated errors; qubit register interacting with common environment

- quantum environments; entanglement & monogamy relations, quantum phase transitions, chaotic dynamics - in case of realistic environments (mesoscopic)

- specific architectures relevant to spin baths: Quantum dots & NV-centres in diamond; Two-level fluctuators (charge traps) in ion-traps.

Quantum Walks

- Originally invented as a way of developing new algorithms for quantum information processing, quantum walks are also of key interest in the simulation of many-body systems.
- Describes the dynamics of a particle on some mathematical graph.
- Hamiltonians describing a ‘quantum walker’ can be mapped to a vary large class of Hamiltonians describing quantum information processing systems.
- Use as an approach to understanding the effects of quantum environments on quantum information processing

Quantum Walk Hamiltonians

The quantum walk is defined by the topology of the graph upon which the system walks, and by the `on-site' and 'inter-site' terms in the Hamiltonian.

1. Simple quantum walk

$$
\begin{aligned}
\hat{H}_S &= -\sum_{ij} \Delta_{ij}(t) \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_i \hat{c}_j^\dagger \right) + \sum_j \epsilon_j(t) \hat{c}_j^\dagger \hat{c}_j \\
&= -\sum_{ij} \Delta_{ij}(t) \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_i \hat{c}_j^\dagger \right) + \sum_j \epsilon_j(t) \hat{c}_j^\dagger \hat{c}_j
\end{aligned}
$$

Each node of the graph, labeled by an integer j, corresponds to the quantum state denoting the location of the 'particle'

$$|j\rangle = \hat{c}_j^\dagger |0\rangle$$

2. Composite quantum walk

The composite walker differs in that it has `internal' degrees of freedom, which can function in various ways.

These internal variables are assumed to be under the control of the operator. For example, Feynman's original model of a quantum computer is a special case of a composite quantum walk.

$$
\begin{aligned}
\hat{H}_C = & -\sum_{ij} \left(F_{ij}\left(\mathcal{M}_{ij}; t\right) \hat{c}_i^\dagger \hat{c}_j + H.c. \right) + \\
& \sum_j G_j\left(\mathcal{L}_j; t\right) \hat{c}_j^\dagger \hat{c}_j + \hat{H}_0\left(\{\mathcal{M}_{ij}, \mathcal{L}_j\}\right)
\end{aligned}
$$

A variety of mappings

APH and PCE Stamp, *Phys. Rev. A* **75**, 062321 (2007)

Single-excitation encoding

- N qubits = N nodes
- A walk in physical space

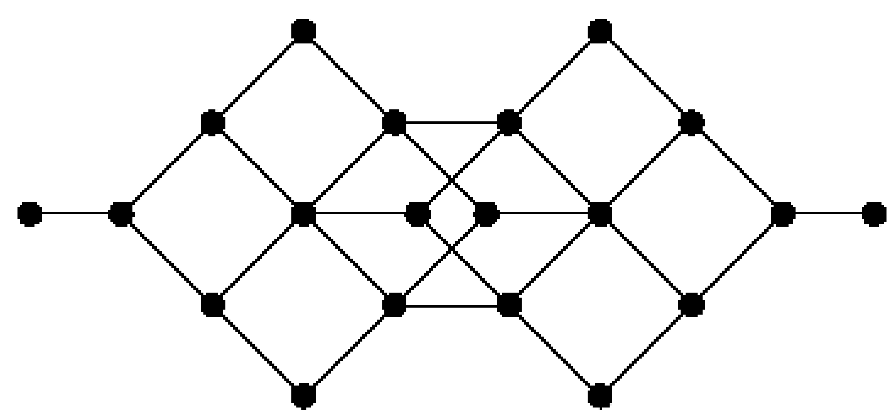
$$|j\rangle \equiv |\downarrow\downarrow \dots \underset{j^{th}\ \text{spin}}{\downarrow\uparrow\downarrow} \dots \downarrow\rangle$$

$$\hat{H} = -\sum_{\langle i,j\rangle} \Delta_{ij}(t) \left(\hat{\sigma}_i^+ \hat{\sigma}_j^- + \hat{\sigma}_j^+ \hat{\sigma}_i^-\right) + 2\sum_j \epsilon(t) \left(1 + \hat{\sigma}_j^z\right)$$

- require only two-qubit operations to represent any quantum walk.

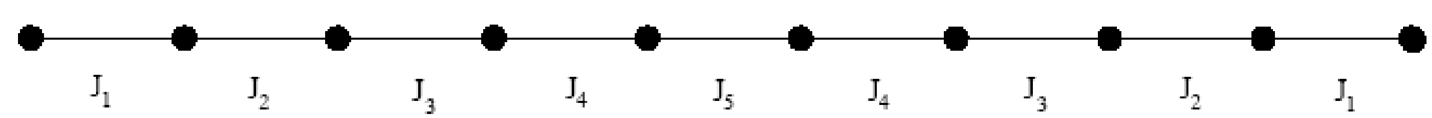
Spin chains

The above Hamiltonian corresponds to the XY-model, and for a 1*D* chain we have a walk on the line in the single excitation subspace. Considering a higher number of excitations, with each state encoding a node results in interesting graphs for the quantum walk.



Graph of walk corresponding to a spin-chain with 6 spins, in 3 excitation subspace.

This graph can be "collapsed" to a biased walk on the line, where nodes refer to "column" subspaces.



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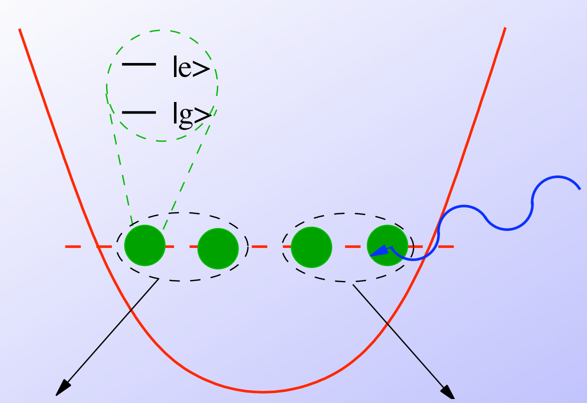
- require only two-qubit operations to represent any quantum walk.

Trotter form to simulate the Hamiltonian evolution:

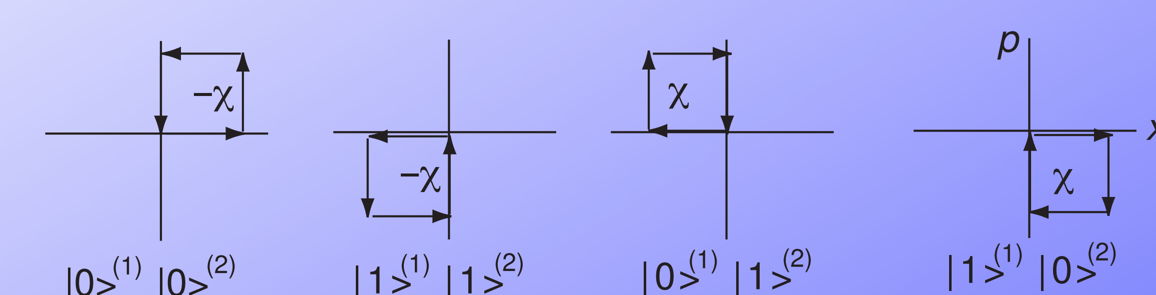
$$\hat{H} = \sum_{\langle i,j\rangle} \hat{h}_{ij} \qquad \hat{h}_{ij} = \Delta_{ij}\left(\hat{\sigma}_i^+\hat{\sigma}_j^- + \ldots\right.$$

$$e^{-i\hat{H}t} \approx \left[\prod_{\langle i,j\rangle} e^{-i\hat{h}_{ij}t/M}\right]^M$$

Ion-trap implementation



Geometric phase gate: alternate laser pulses upon ions generate the desired two-qubit terms



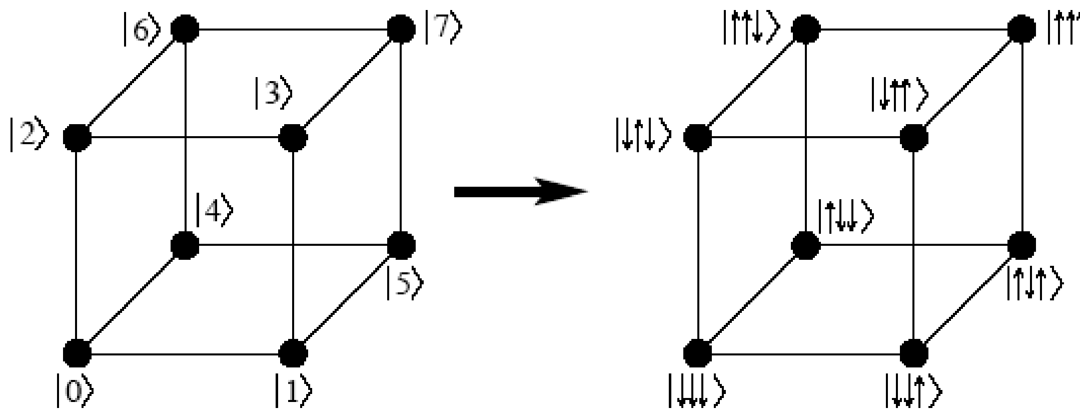
Binary-expansion encoding

APH and PCE Stamp, *Phys. Rev. A* **75**, 062321 (2007)

- N qubits = 2^N nodes
- A walk in information space

$$|j\rangle \equiv |\bar{z} = |z_1 z_2 \dots z_N\rangle, \;\; z_k = \begin{cases} \downarrow \equiv 0 \\ \uparrow \equiv 1 \end{cases}$$

<u>Example</u>: Hypercube walk mapped to a set of qubits



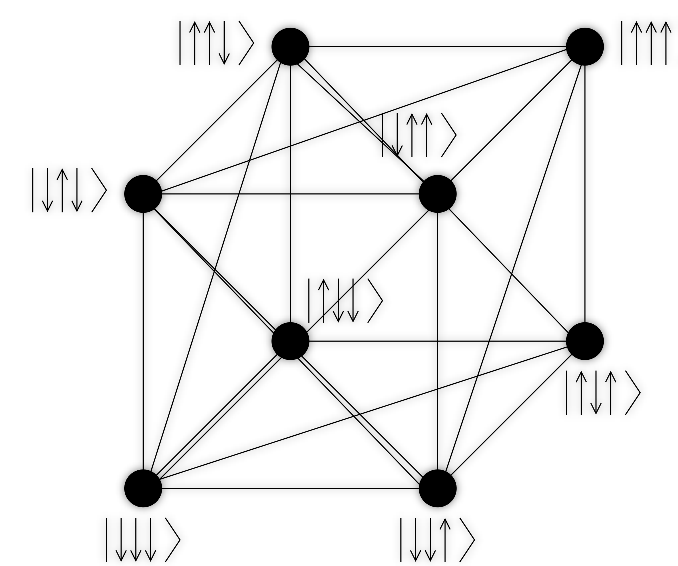
$$\hat{H} = \Delta_0 \sum_{ij} \hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i \equiv \Delta_0 \sum_k \hat{\sigma}_k^x$$

Static qubit Hamiltonians to quantum walks

$$\hat{H} = \sum_{n=1}^{N} (\varepsilon_n \hat{\sigma}_n^z + \delta_n \hat{\sigma}_n^x) - \sum_{i,j} \chi_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^x + \sum_{i<j} V_{ij}^{\perp} \hat{\sigma}_i^x \hat{\sigma}_j^x + V_{ij}^{\parallel} \hat{\sigma}_i^z \hat{\sigma}_j^z \equiv - \sum_{ij} \Delta_{ij} \hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i + \sum_{j=0}^{2^N} \epsilon_j \hat{c}_j^\dagger \hat{c}_j$$

$$\Delta_{ij} = \begin{cases} \delta_a + \sum_c (-1)^{j_c} \chi_{ca} & \text{if } i_a \neq j_a \text{ and } i_b = j_b \; \forall b \neq a \\ V_{ab}^{\perp} & \text{if } i_a \neq j_a \text{ and } i_b \neq j_b \text{ and } j_c = i_c \; \forall c \neq a, b \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_j = \sum_{a=1}^{N} (-1)^{j_a} \varepsilon_a + \sum_{a,b} (-1)^{j_a + j_b} V_{ab}^{\parallel}.$$



Binary-expansion encoding

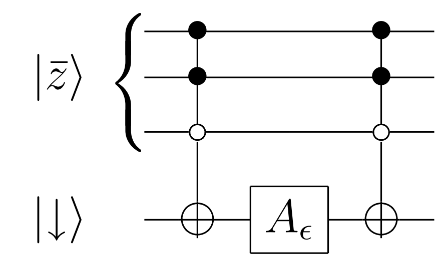
APH and PCE Stamp, *Phys. Rev. A* **75**, 062321 (2007)

Circuit Simulation

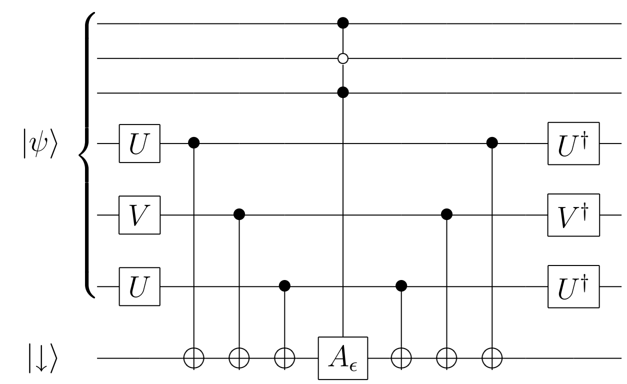
Using the Trotter formula, one can construct quantum circuits to simulate an arbitrary quantum walk

$$c_{\bar{z}}^{\dagger} c_{\bar{z}} \equiv |\bar{z}\rangle\langle\bar{z}| = \bigotimes_{k=1}^{M} |z_k\rangle\langle z_k| = \bigotimes_{k=1}^{M} \mathbb{P}_{z_k} = \prod_{k=1}^{M} (1 - (-1)^{z_k} \hat{\tau}_k^z)$$

$$U(\epsilon) = e^{-i\hbar\epsilon|\bar{z}\rangle\langle\bar{z}|}$$



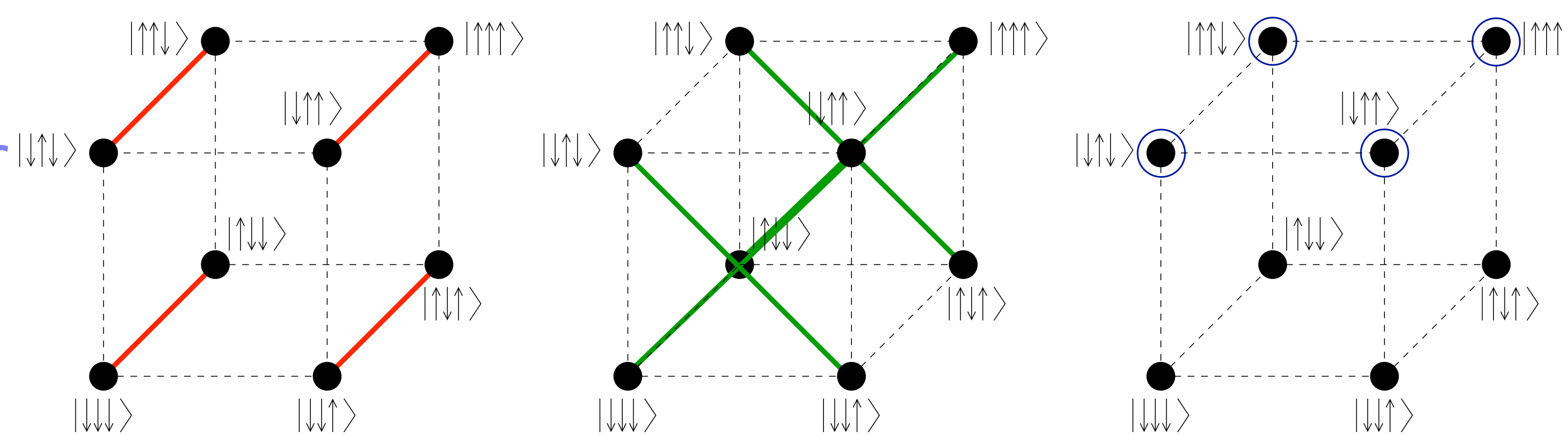
$$\begin{aligned} |\bar{z}\rangle\langle\bar{w}| + |\bar{w}\rangle\langle\bar{z}| &= \prod_{k=1}^{M} (\mathbb{P}_k^{z_k})^{\delta(z_k - w_k)} \delta(1 - z_k - w_k)\tau_k^{+} \delta(1 + z_k - w_k)\tau_k^{-}, \\ &= \prod_{k=1}^{M} (\mathbb{P}_k^{z_k})^{\delta(z_k - w_k)} \left(\tau_k^x + i(z_k - w_k)\tau_k^y\right)^{1-\delta(z_k - w_k)} + \text{h.c.}, \end{aligned}$$



$$\exp[-i\hbar\epsilon \mathbb{P}_1^{\uparrow}\mathbb{P}_1^{\downarrow}\mathbb{P}_1^{\uparrow}\hat{\tau}_4^x\hat{\tau}_5^y\hat{\tau}_6^x]$$

Dynamic qubit networks to gates

Fundamental gates in a universal set as variants of a quantum walk on the hypercube



$$R_x^{(1)}(\gamma) = \exp(-i\gamma\hat{\tau}_1^x/2), \qquad V_{23}^{\perp}(\chi) = \exp(i\chi\hat{\tau}_2^x\hat{\tau}_3^x), \qquad R_z^{(2)}(\theta) = \exp(-i\theta\hat{\tau}_2^z/2)$$

Decoherence in Quantum Walks

- Restricted to simple (Markovian) models; mainly discrete-time

$$\rho_{n+1} = (1-p)\hat{U}\rho_n\hat{U}^\dagger + p\sum_k \hat{M}_k(\hat{U}\rho_n\hat{U}^\dagger)\hat{M}_k^\dagger$$

- focus has been quantum-to-classical transition of walk characteristics; "intentional" decoherence

- Decoherence mechanism will depend upon how the quantum walk is implemented; while unitary dynamics may be the same, open-system dynamics will depend upon the system in question; how it interacts with its environment

- want a use a Hamiltonian description allowing the incorporation of realistic couplings to environments that exist in Nature

diagonal

$$H_{int} = \sum_{j\alpha} U_j(X_\alpha) n_j$$

non-diagonal

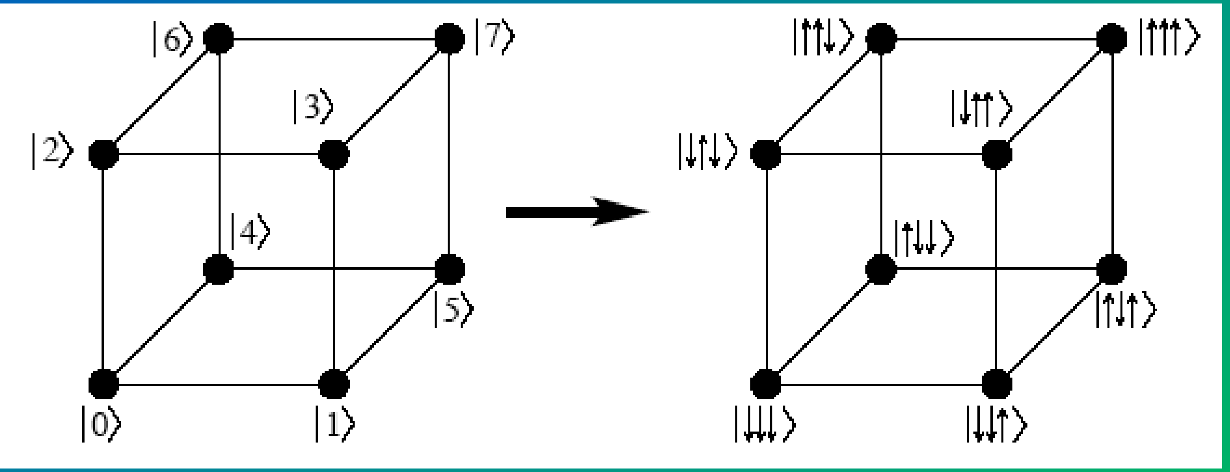
$$H_{int} = \sum_{ij,\alpha} V_{ij}(X_\alpha)[\hat{c}_i^\dagger \hat{c}_j + H.c.]$$

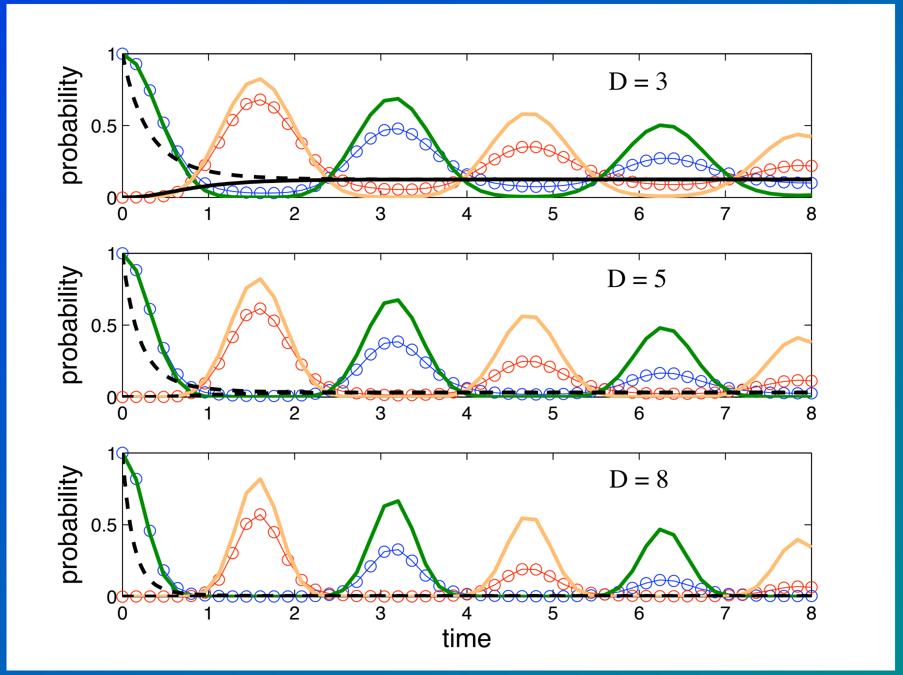
- Consider two simple examples.

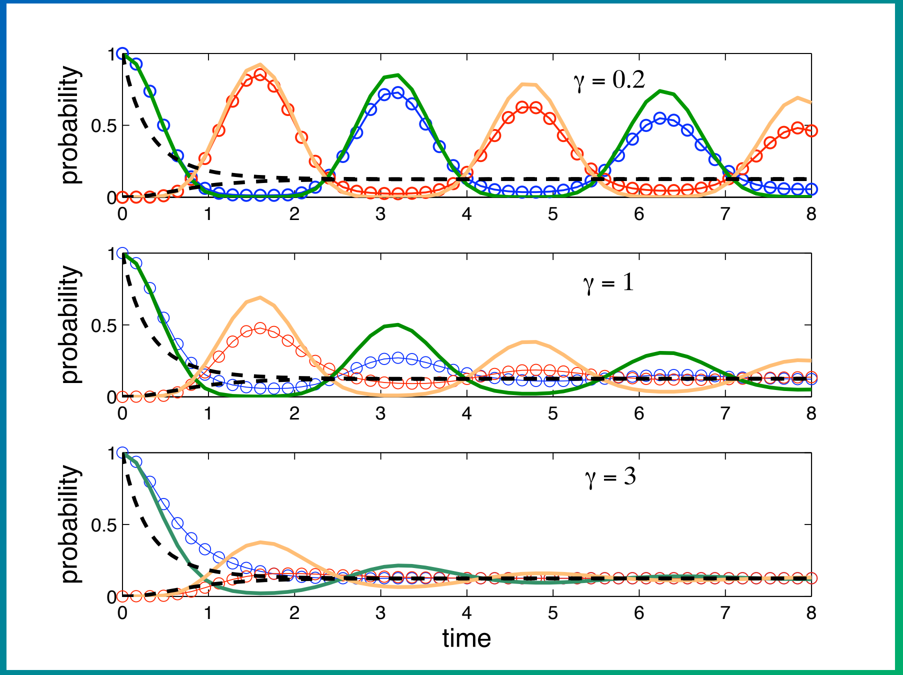
Quantum walk on the hypercube

$$\hat{H}^{HC} = -\Delta_0 \sum_{[j,k]} \left(\hat{c}_j^\dagger \hat{c}_k + \hat{c}_j \hat{c}_k^\dagger \right)$$

$$= -\Delta_0 \sum_k \hat{\tau}_k^z$$







- Markovian environment described by master equation:

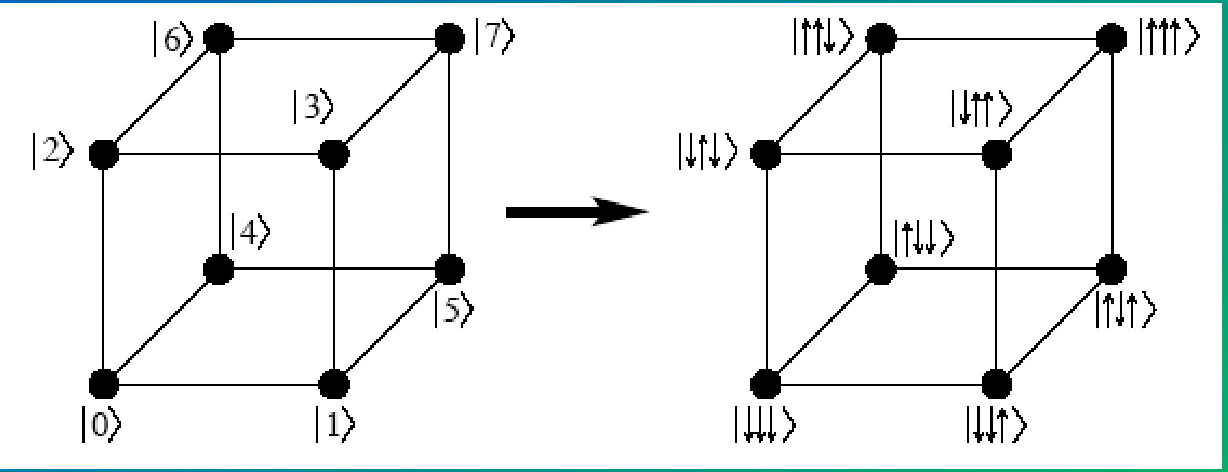
$$\frac{d\rho(t)}{dt} = -i[\hat{H}, \rho(t)] + \gamma \sum_k \mathcal{D}[\hat{M}_k]\rho(t)$$

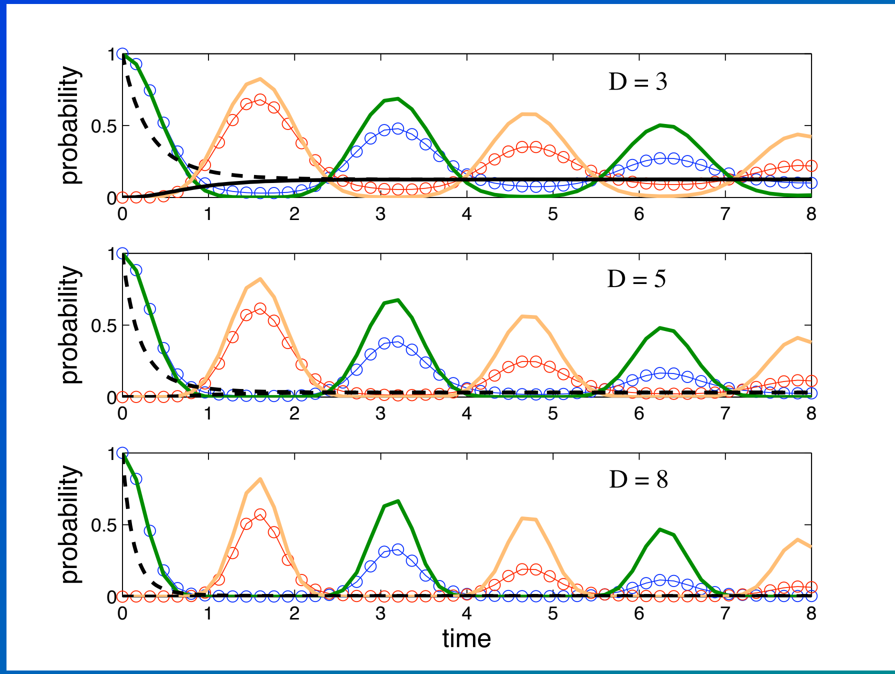
- the coupling to the environment is described by the M_k's
- this coupling, and the decoherence, will depend upon ***how*** the walk is ***implemented***.
- Compare the
  - "physical" walk $\hat{M}_k = |k\rangle\langle k|$
  - "information space" walk $\hat{M}_k = \hat{\tau}_k^z / 2$

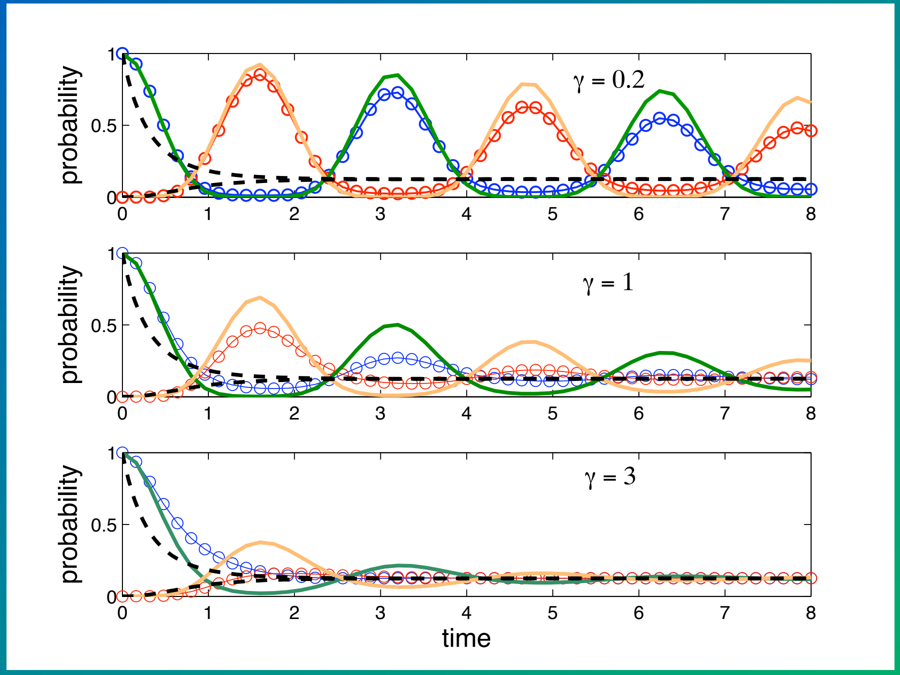
Quantum walk on the hypercube

$$P_{\vec{z}0}(t) = \cos^{2n_\downarrow}(2\Delta_o t)\sin^{2n_\uparrow}(2\Delta_o t)$$

Alagic & Russell, PRA, **72**, 062304 (2005)
APH & Stamp, to appear, CJP.







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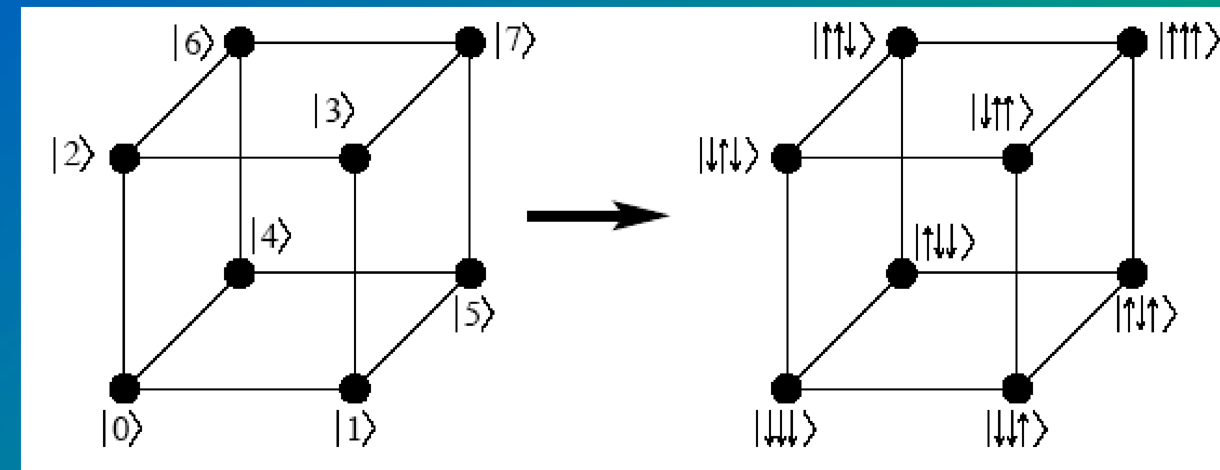
- this coupling, and the decoherence, will depend upon ***how*** the walk is ***implemented***.
- Compare the
  - "physical" walk $\hat{M}_k = |k\rangle\langle k|$ (site)
  - "information space" walk $\hat{M}_k = \hat{\tau}^z_k/2$ (qubit)

Quantum walk on the hypercube

$$\hat{H} = -\Delta_0 \sum_{[j,k]} \left(\hat{c}_j^\dagger \hat{c}_k + \hat{c}_j \hat{c}_k^\dagger \right)$$

$$= -\Delta_0 \sum_{k} \hat{\tau}_k^z$$



Decoherence can be “useful”

Kendon & Tregenna, PRA, **67**, 042315 (2005)
Drezgic, APH & Sarovar, in preparation.

- Certain decoherence can produce useful characteristics in quantum walks - mixing times (uniform distribution), hitting times
- “Engineered” decoherence - non-unitary dynamics - as opposed to an environment.

Interesting to consider: $\sum_d \mathcal{D}[\vec{v} \cdot \vec{\sigma}_d]\rho$

and the effect upon the characteristics of interest in algorithms (mixing, hitting times). This is equivalent to changing the basis the QW is implemented in, while the decoherence remains the same.

Hypercube coupled to an oscillator bath

$$\mathcal{H} = H_o^{HC} + V + H_{osc}$$

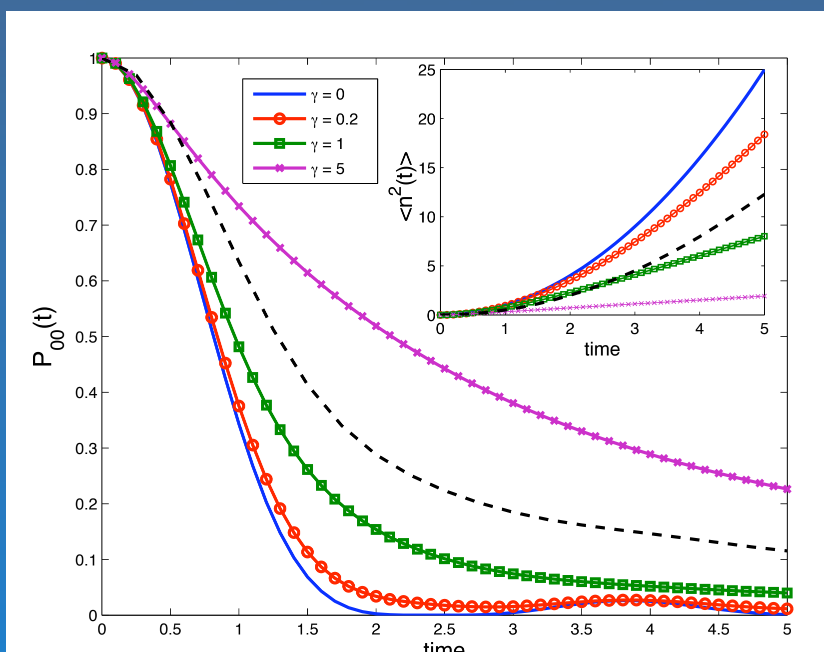
$$H_{osc} = \sum_{q=1}^{N_o} \left(\frac{p_q^2}{m_q} + m_q \omega_q^2 x_q^2 \right)$$

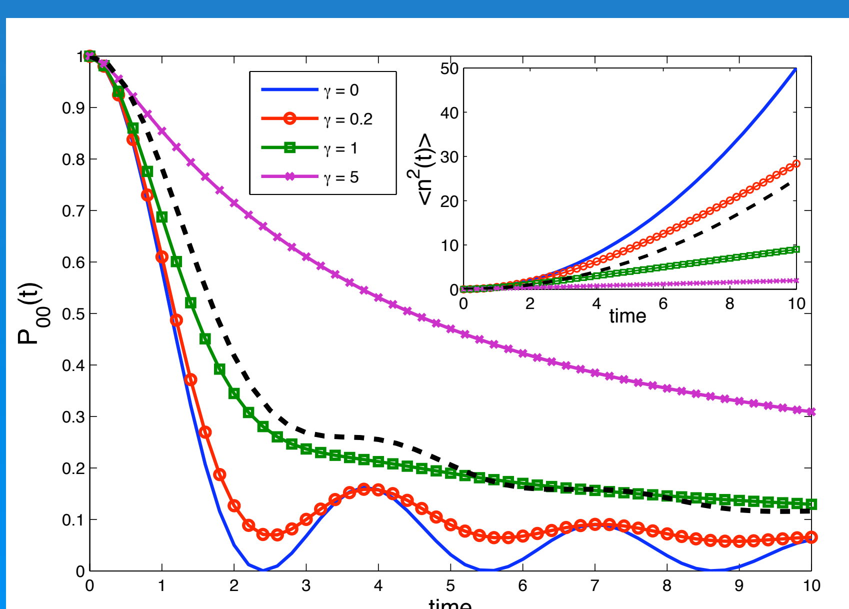
- Diagonal coupling to qubits $$V = \sum_{q=1}^{N_o} v_n^z(q) \hat{\tau}_n^z x_q$$
- Results in an extra inter-qubit coupling in the effective Hamiltonian

$$V_{nm}^{zz}(\tilde{\Omega}_C) = \int_{\tilde{\Omega}_C}^{\Omega_o} \frac{d\omega}{\pi} \frac{J_{nm}^{zz}(\omega)}{\omega}$$

Quantum walk on the hyperlattice

$$\hat{H}_o^{HL} = -\Delta_o \sum_{ij} (\hat{c}_i^\dagger \hat{c}_j + H.c.) \equiv \sum_{\mathbf{p}} \epsilon_o(\mathbf{p}) \hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}}$$





- Free quantum walk, uniform hopping, is trivially solvable; for walker initially at origin,
- the occupation probability is: $P_n^0(t) = \prod_{\mu=1}^{d} J_{n_\mu}(2\Delta_0 t)$
- quantum evolution: classical evolution:

$P_0^0(t) \propto 1/t^d$ $P_0^0(t) \propto 1/t^{d/2}$

$\langle n^2(t) \rangle = \sum_{\vec{n}} n^2 P_{\vec{n}}^0(t) \propto t^2$ $\langle n^2(t) \rangle \propto t$

- to left, results for decoherence, modeled via master equation with $\hat{M}_k = |k\rangle\langle k|$, shows classical behavior, as decoherence increases, before we see a quantum Zeno effect

Quantum walk on the hyperlattice

N.Prokof'ev and P.C.E.Stamp, PRA 74, 020102(R) (2006)

We compare this with the results when we couple ***non-diagonally*** to a ***spin-bath***:

$$\hat{H} = \Delta_0 \sum_{\langle jk \rangle} \left\{ \hat{c}_j^\dagger \hat{c}_k \cos\left(\sum_n \alpha_n \hat{\sigma}_n^x + H.c. \right) \right\}$$

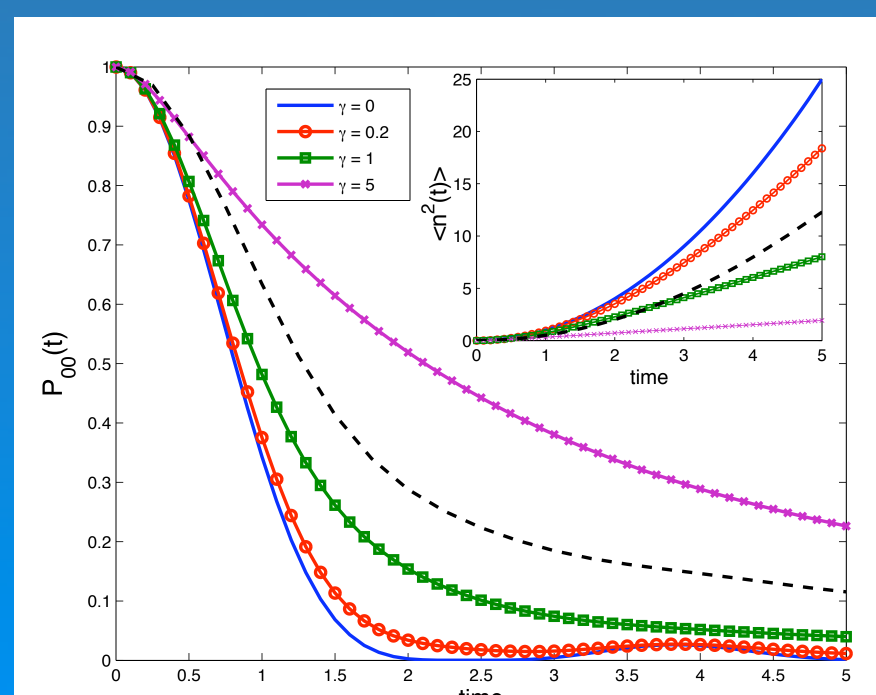
Transitions of the walker cause bath spins to flip.

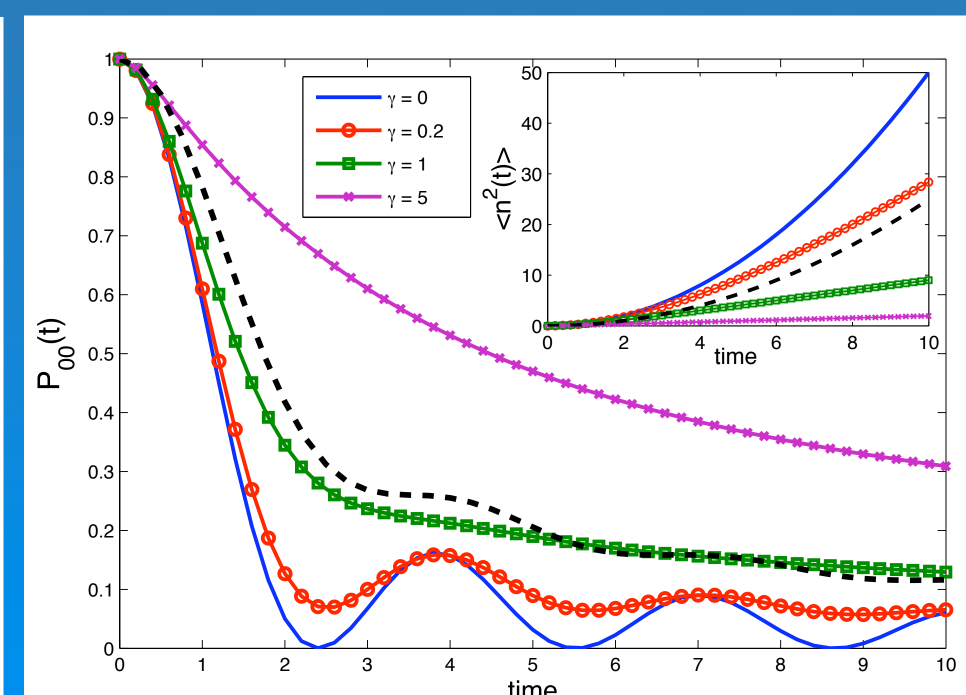
- decoherence strength quantified by $\lambda = \sum_k \alpha_k^2 \gg 1$ (*strong* decoherence)
- this system is exactly solvable: $P_{\vec{n}}^0(t) = \int_o^{2\pi} \frac{d\varphi}{2\pi} \prod_{\mu=1}^{d} J_{n_\mu}^2 (z \cos\varphi), \quad z = 2\Delta_0 t$
- characteristic behavior:

Spin-bath

$$P_0^0(z \to \infty) \propto 1/t$$

$$\langle n^2(t) \rangle = \frac{d}{2} (\Delta_0 t)^2$$





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- characteristic behavior:

Spin-bath	Free quantum	Classical
$P_0^0(z\to\infty) \propto 1/t$	$P_0^0(t) \propto 1/t^d$	$P_0^0(t) \propto 1/t^{d/2}$
$\langle n^2(t)\rangle = \frac{d}{2}(\Delta_0 t)^2$	$\langle n^2(t)\rangle = \sum_{\vec{n}} n^2 P_{\vec{n}}^0(t) \propto t^2$	$\langle n^2(t)\rangle \propto t$

This exact solution shows that the density matrix has one component showing quasi-localization with another showing coherent ballistic dynamics, far from the origin!

Final thoughts

- ***The graph over which the quantum walk takes place can be represented using different sets of basis states;***
- ***there is no general principle forcing environmental couplings to distinguish different nodes or transition directions in these different encodings.***
- ***In the design of quantum computers and certain search algorithms, the above result shows the importance of investigating quantum walks for which environmental couplings do not distinguish different 'position' (or 'nodal') states in the information space in which the quantum walk is encoded.***
- ***motivate considering encodings which couple non-diagonally to the environement.***