# Complexity of the quantum adiabatic algorithm

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# Introduction

- What is the **“Quantum Adiabiatic Algorithm”**?
- Motivation for studying the **complexity** of the Quantum Adiabatic Algorithm for **much larger sizes** than has been studied before.
- The **Monte Carlo** method that will be used to do this.
- **Results** for the a particular problem (Exact Cover).
- **Conclusions**.

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The best known is **Shor's factoring algorithm** which factors an integer of $n$ bits in a time which is **polynomial** in $n$, as opposed to the best classical algorithm which take a time of order $\exp(c\, n^{1/3})$ .

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**NP-Hard.**

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Could a quantum computer solve **typical** instances of NP-Hard problems with just **polynomial complexity**, i.e.

$$\text{complexity} \propto N^{\sigma}\,,$$

for some value of $\sigma$?

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Add a "driver Hamiltonian", which is simple and does not commute with $\mathcal{H}_P$. The simplest is a "transverse field" $\mathcal{H}_D = -h \sum_i \sigma_i^x$.

The total Hamiltonian is

$$\mathcal{H} = [1 - \lambda(t)]\, \mathcal{H}_D + \lambda(t)\, \mathcal{H}_P \,,$$

where the **"control parameter"** $\lambda(t)$ varies from **0** at $t = 0$ to **1** at $t = \mathcal{T}$, the **running time**, or **complexity**.

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At $t = \mathcal{T}$, just have $\mathcal{H}_P$. If the evolution is adiabatic, the system is in the ground state of $\mathcal{H}_P$ and **the problem is solved**.

# Quantum Adiabatic Algorithm

The **Quantum Adiabatic Algorithm** is less demanding on the hardware than algorithms like Shor's.

The QAA **gradually** evolves the Hamiltonian, which is represened by the connections in the computer, whereas Shor's algorithm proceeds by a series of **discrete** unitary transformations.

It is easier to avoid interference between the bits and to maintain quantum coherence if changes are made gradually, rather than in a series of discrete jumps.

Here there is **real interest in the quantum computing community** in building a quantum computer which uses the QAA.

Even if one can build one **will it be more efficient than a classical computer** for NP-hard problems?

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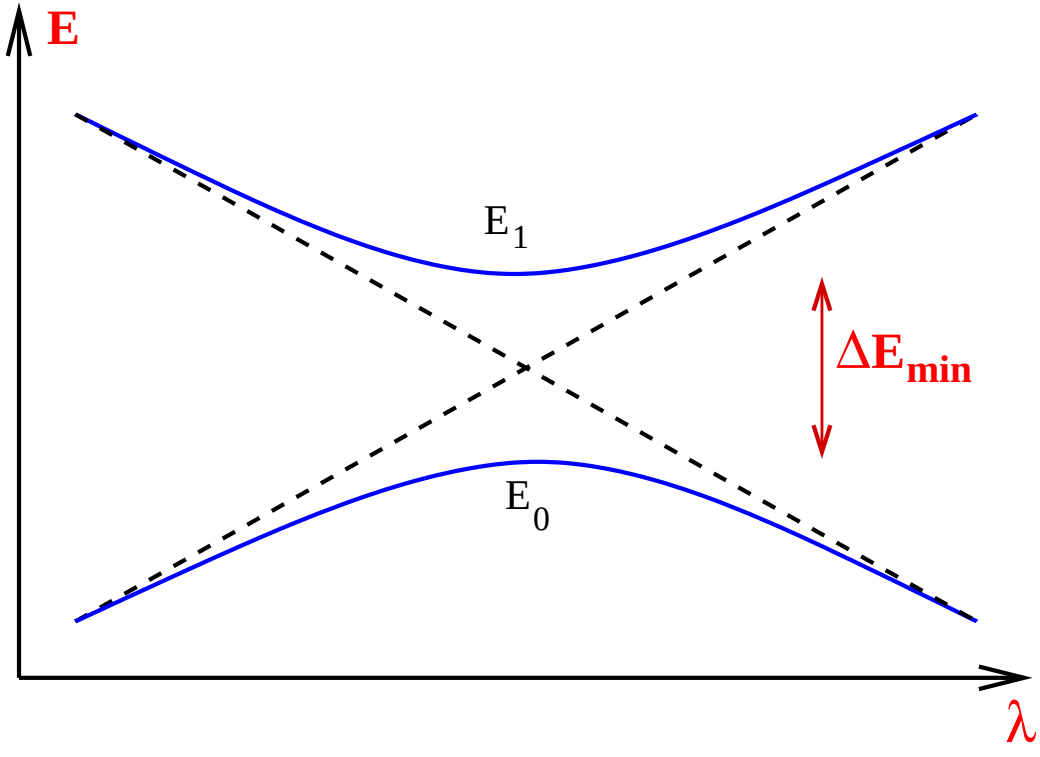


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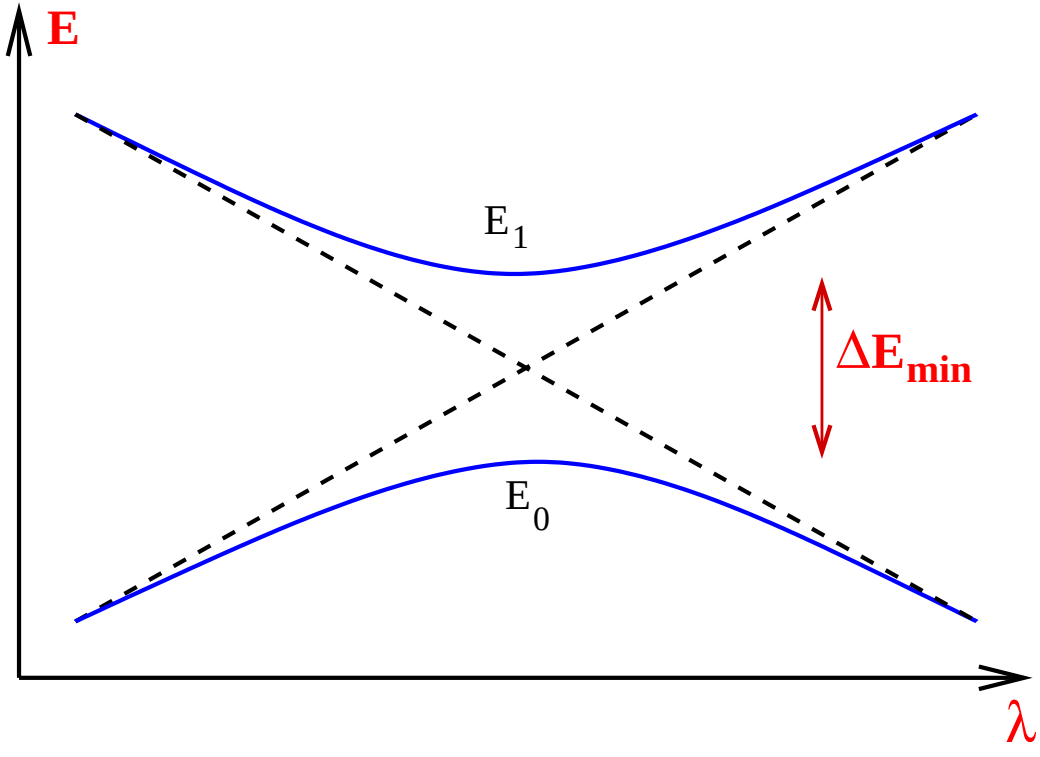


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**Landau-Zener theory**. To stay in ground state, time $\propto (\Delta E_{\mathrm{min}})^{-2}$.

# Quantum Phase Transition

As $\lambda(t)$ is varied the system is likely to go through a **Quantum Phase Transition** where the gap will be particularly small.

Hence we are, effectively interested in:

**The Size Dependence of the Energy Gap at a Quantum Phase Transition**

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$\Longrightarrow$ **“Monte Carlo" methods**

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Working through the details, one ends up with copies of the system at different values of imaginary time $\tau$ where $0 \leq \tau < \beta$. One discretizes imaginary time (Trotter decomposition) into $L_\tau$ "time slices" separated by the time-slice width $\Delta\tau$. We have

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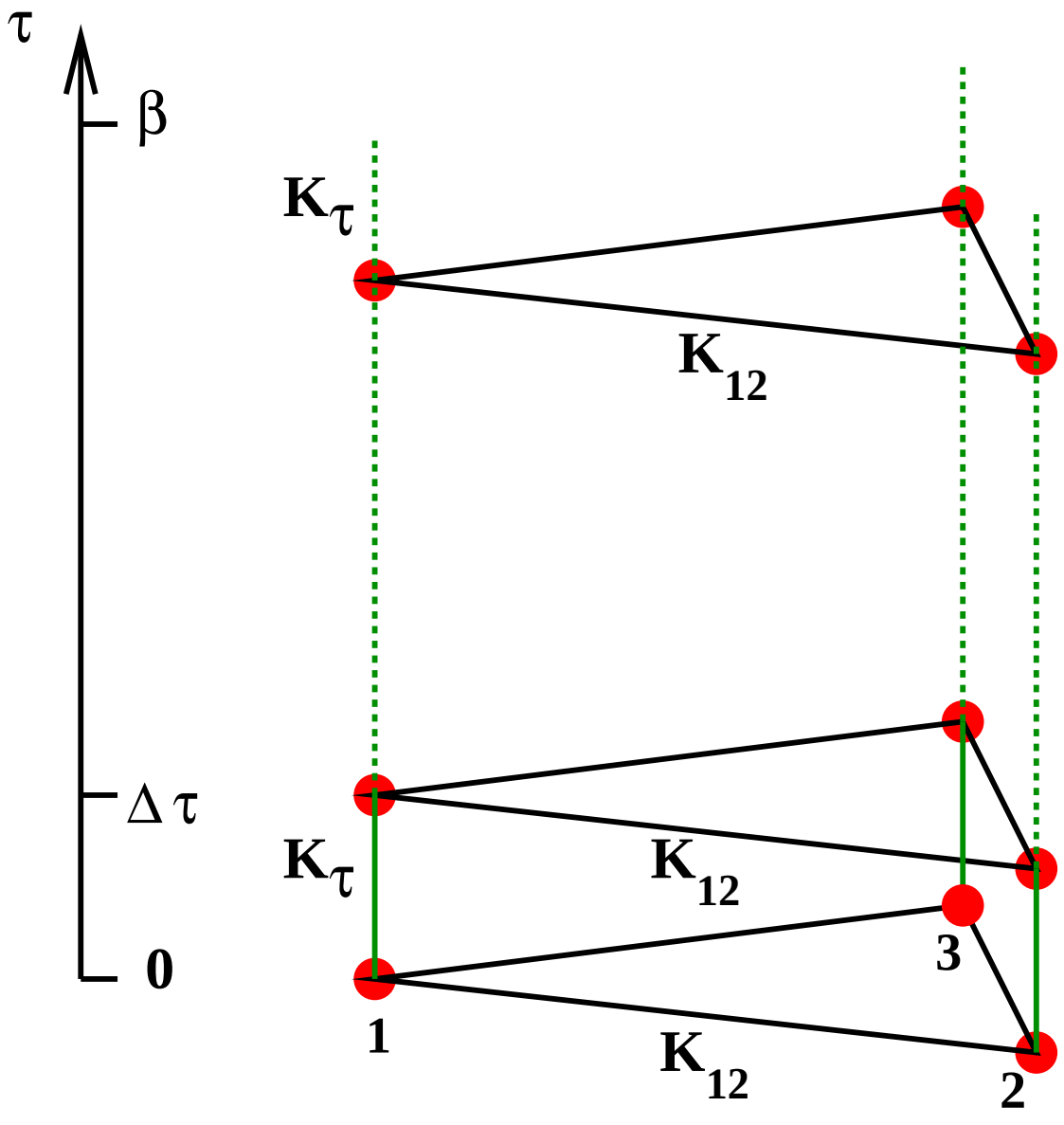
The exact quantum mechanical Hamiltonian is reproduced in the limit $\Delta\tau \to 0$. However, we shall argue that this limit is not necessary for our purposes.

# Quantum Monte Carlo: II




Trotter decomposition in QMC.

At each time slice 3 sites are shown. An independent spin $\sigma_i^z(\tau)$ lives at each site and each of the $L_\tau$ time slices. If spins $i$ and $j$ have an interaction in $\mathcal{H}_P$, then, each time slice, these spins interact with a coupling $K_{ij}$, the same for each slice. Spins on the same site but at neighboring time slices are coupled by an interaction $K_\tau$, again the same for all slices. (Details on next slide.)

The slice at time $\tau = \beta$ is identified with the slice at $\tau = 0$ (i.e. we have periodic boundary conditions in the imaginary time direction).

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2. couplings between different spins at the same site but neighboring time slices arising from the driver Hamiltonian

$$\mathcal{H}_D = -\sum_i \sigma_i^x \Longrightarrow -\sum_{m=0}^{L_\tau - 1} K_\tau \sigma_i^z(\tau_m)\sigma_i^z(\tau_{m+1})$$

   where $e^{-2K_\tau} = \tanh(\Delta\tau\, h)$.

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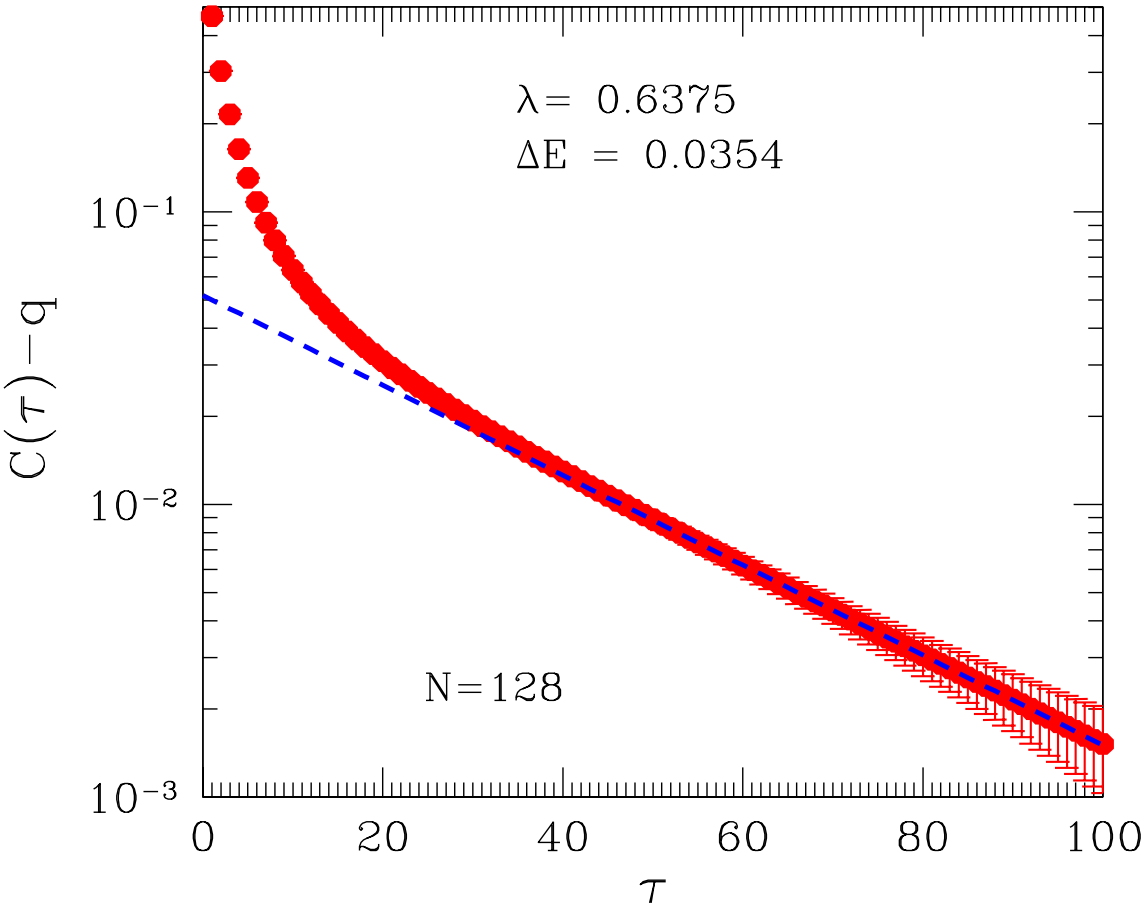
Hence, at large $\tau$, we have

$$\boxed{C(\tau) = q + \frac{1}{N} \sum_{i=1}^{N} \left| \langle 0 | \sigma_i^z | 1 \rangle \right|^2 e^{-(E_1 - E_0)\tau} ,}$$

where $q = N^{-1} \sum_i \langle \sigma_i^z \rangle^2$. (See next slide for some results.)

# Sample results for $C(\tau)$




Results for the time dependent correlation function against $\tau$ for one instance of the Exact Cover problem with $N = 128$ near the location of the minimium gap. Note that the vertical axis is logarithmic. Fitting to the straight line region gives a slope (equal to the gap $\Delta E$) equal to $0.0354$.

We took $L_\tau = 300, \Delta\tau = 1$, so $T^{-1} \equiv \beta = 300$. Hence the condition $T \ll \Delta E$ is well satisfied.

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$$\begin{aligned} \mathcal{H}_\mathrm{P} &= \frac{1}{8}\sum_{\alpha=1}^{M}\Big(5 - \sigma_{\alpha_1}^z - \sigma_{\alpha_2}^z - \sigma_{\alpha_3}^z + \sigma_{\alpha_1}^z\,\sigma_{\alpha_2}^z \\ &+ \sigma_{\alpha_2}^z\,\sigma_{\alpha_3}^z + \sigma_{\alpha_3}^z\,\sigma_{\alpha_1}^z + 3\,\sigma_{\alpha_1}^z\,\sigma_{\alpha_2}^z\,\sigma_{\alpha_3}^z\Big), \end{aligned} \tag{1}$$

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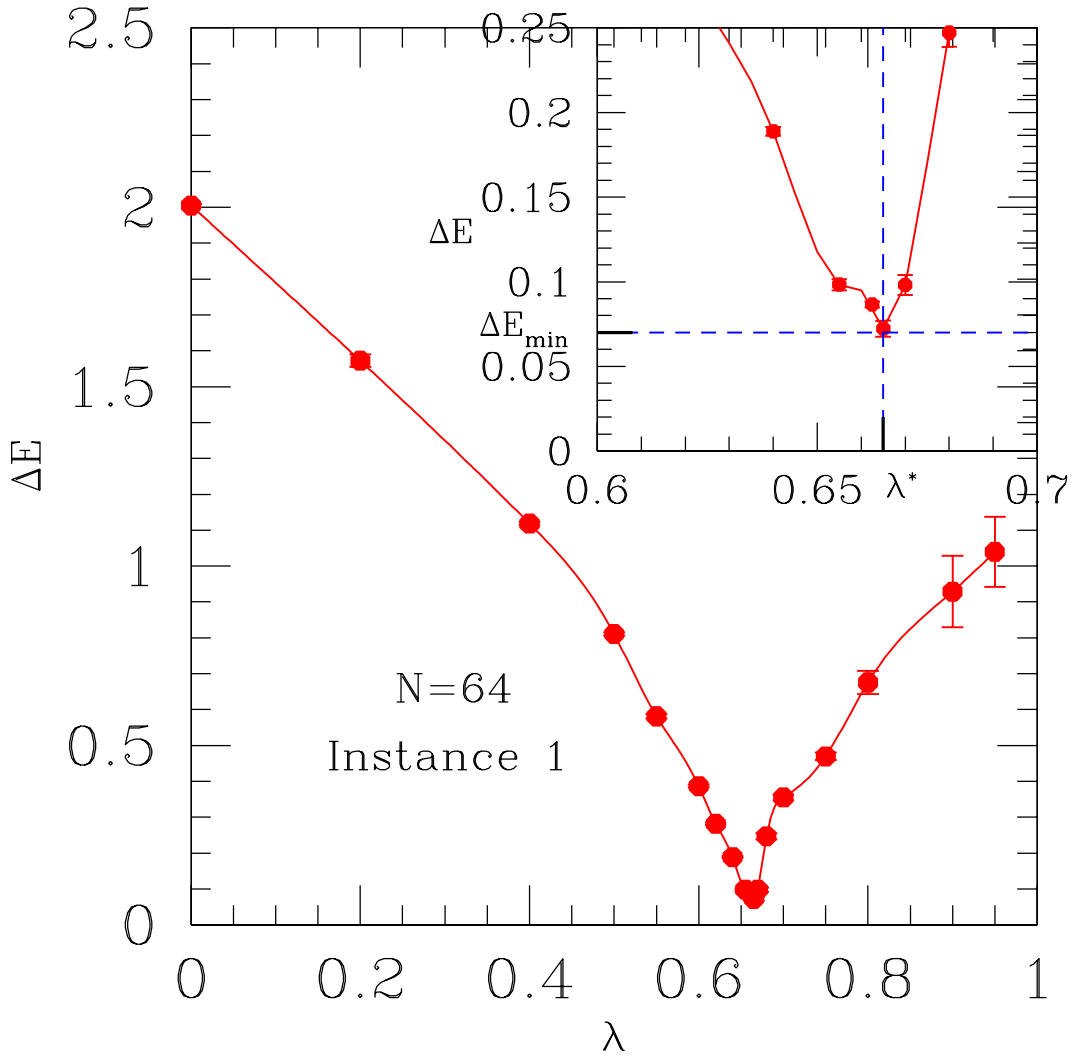
Following Farhi et al. we take instances with a **"Unique Satisfying Assignment"** (USA). To find these with reasonable probability, we adjust the ratio $M/N$ for each size $N$.

# Dependence of gap on $\lambda$




Results for the gap to the first excited state $\Delta E$ as a function of the control parameter $\lambda$ for one instance with $N = 64$. The gap has is finite for $\lambda = 0$ (this is due to the driver Hamiltonian, $\sum_i \sigma_i^x$). It is also finite for $\lambda = 1$ because we chose instances with this property (Unique Satisfying Assignment). There is a minimum of the gap at an intermediate value of $\lambda$, presumably close to a

**quantum phase transition.**

We compute $\Delta E_{\mathrm{min}}$ for many (50) instances for several different sizes, $N = 16, 32, 64, 128$.

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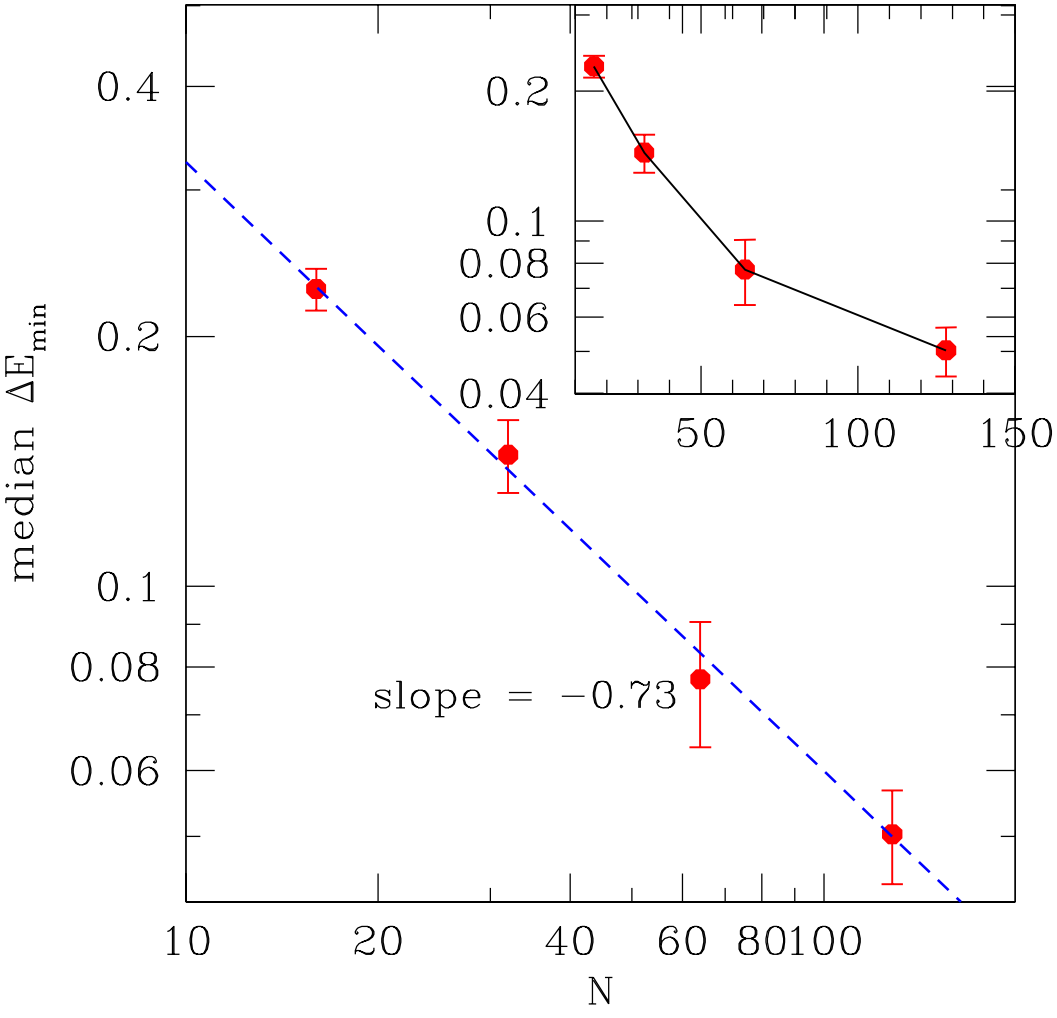


50 instances for each size.

A log-log plot of the **median** of the minimum gap as a function of the number of bits $N$ up to $N = 128$. From the satisfactory straight line fit, it is seen that the median $\Delta E_{\mathrm{min}}$ *decreases as a power law*,

$$\text{median } \Delta E_{\mathrm{min}} \propto N^{-\mu},$$

for these sizes, with

$$\mu = 0.73 \pm 0.06.$$

The inset shows a log-linear plot. The pronounced curvature shows that the behavior is *not exponential* for this range of sizes.

Expect $\text{complexity} \propto N^{2\mu}$ (if matrix element effects are small).

# Imaginary time discretization

**Note:** The discretization of imaginary time does not affect the way the complexity varies with $N$, though it does affect the precise value of the energy gap for given $N$ and $\lambda$. Once the relaxation time $(\Delta E)^{-1}$ is much larger than the "lattice spacing" $\Delta\tau$ the lattice discretization is unimportant. Hence, whether the minimum gap varies exponentially with $N$ or as a power law **will not depend on the value of $\Delta\tau$**.

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**Example:** Exact solution of the Ising model in two dimensions. The magnetization tends to zero as $T \to T_c^-$, like $(T_c - T)^\beta$. With a lot of work, this can be calculated on different lattices, e.g. square and triangular. The value of $T_c$ depends on the lattice (it is **"non-universal**) but $\beta = 1/8$, the same for all lattice structures, i.e. it is **"universal"**.

# Imaginary time discretization

**Note:** The discretization of imaginary time does not affect the way the complexity varies with $N$, though it does affect the precise value of the energy gap for given $N$ and $\lambda$. Once the relaxation time $(\Delta E)^{-1}$ is much larger than the "lattice spacing" $\Delta\tau$ the lattice discretization is unimportant. Hence, whether the minimum gap varies exponentially with $N$ or as a power law **will not depend on the value of $\Delta\tau$**.

In the theory of continuous phase transitions this concept of **"universality"** is well established. Universality means that some (universal) quantities like "critical exponents" don't depend on microscopic details such as the lattice structure. Other (non-universal) quantities, such as the location of the critical point, do depend on details.

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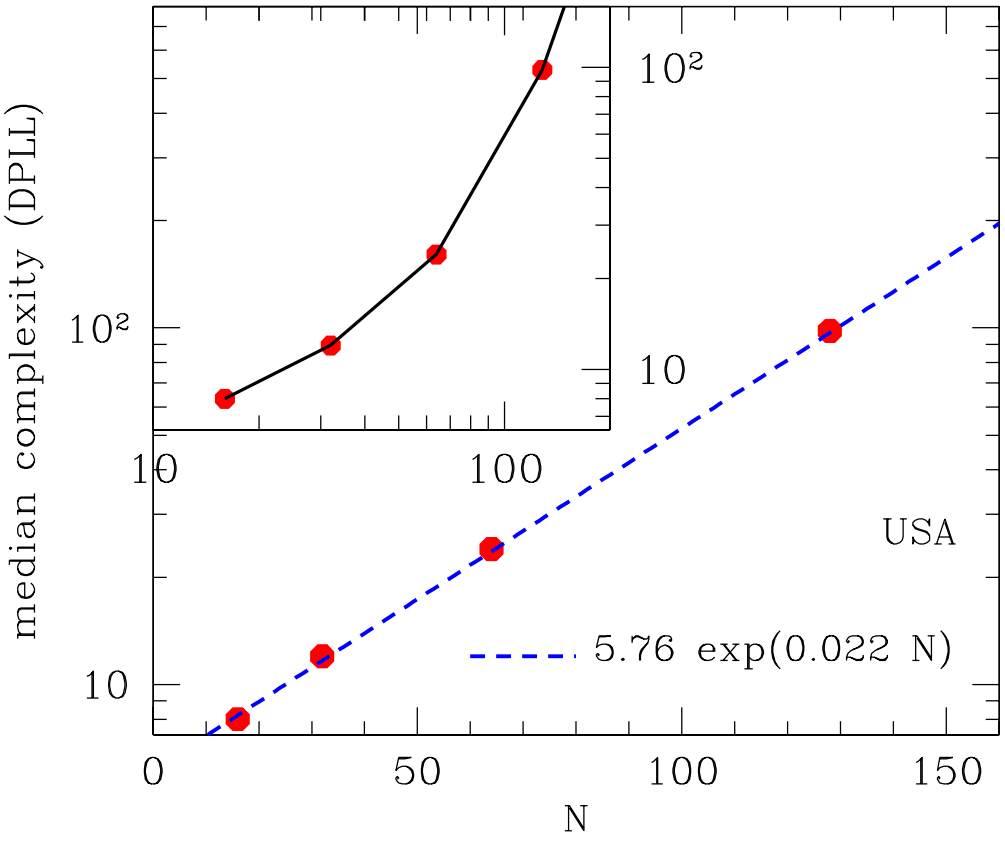
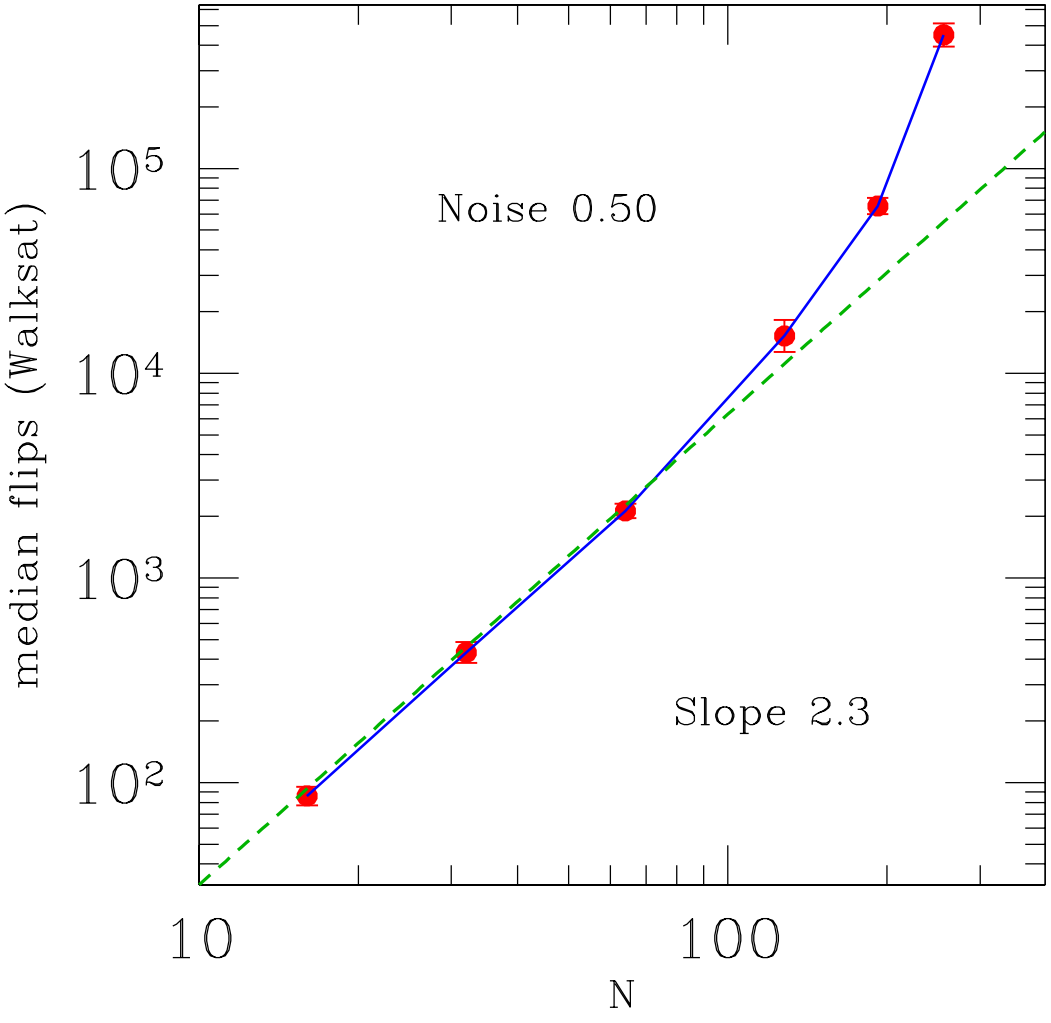
**Note:** One can simulate the $\Delta\tau \to 0$ limit, but this is more complicated.

# Classical Algorithms: I



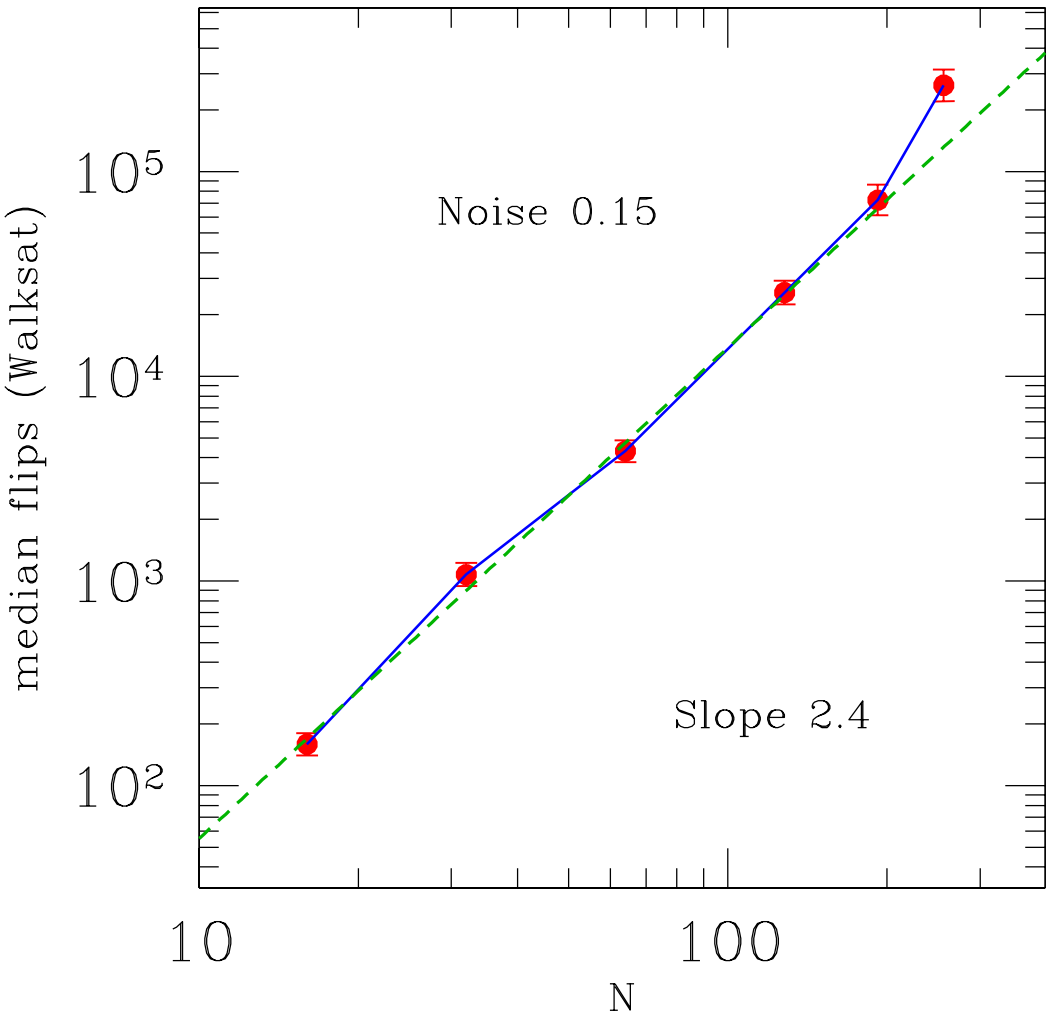

A commonly used classical algorithm for satisfiability problems is the Davis Putnum algorithm. This is guaranteed to correctly say whether or not there is a satisfying assignment. The figure shows the complexity for the instances used in the QMC simulations. It is clearly exponential for the range of sizes studied.

# Classical Algorithms: II




A classical algorithm which is more analgous to QAA is WALKSAT, a local **heuristic** search algorithm. Like simulated annealing, it includes “uphill" moves in a stochastic way. Using the default value of the “noise parameter" the complexity for the QAA instances with USA crosses over from power-law to (presumably) exponential for $N \gtrsim 100$. (But note the QMC is so far only for $N \leq 128$).

# Classical Algorithms: IIb




Adjusting the noise parameter, the crossover to exponential behavior is pushed to larger sizes $N \gtrsim 200$. (Remember: the QMC is so far only for $N \leq 128$).

# Conclusions

- Using Quantum Monte Carlo simulations (QMC) we have been able to study the complexity of the Quantum Adiabative Algorithm (QAA) for the Exact Cover problem with a Unique Satsifying Assignment (USA) for much larger sizes (up to **128**) than in earlier work (**20–24**).

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Thank you