Quantum entanglement and fixed point bifurcations in circuit QED.

G J Milburn

The University of Queensland

CENTRE FOR
QUANTUM COMPUTER
TECHNOLOGY
AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE

Andrew Doherty, UQ
Matthew Woolley, UQ
Charles Meaney, UQ.

Circuit QED.

Jahn-Teller $E \otimes \beta$ model.

Hamiltonian maps.

A transverse field Ising map.

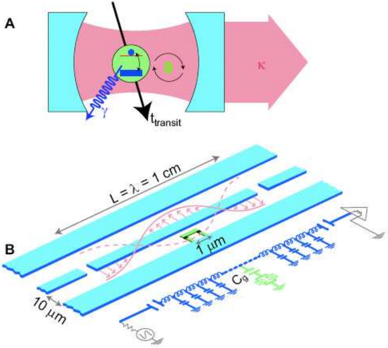
Motivation.

Can we use the technology developed for quantum computing to study quantum nonlinear systems?

- Superconducting implementations.
- Ion trap implementations

Circuit QED.

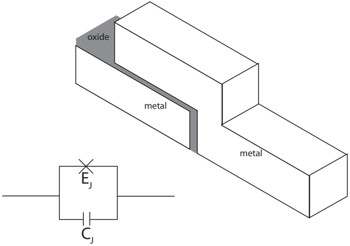
Superconducting qubits in a transmission line.



Girvin et al., (2003). and Blais, et al. (2004).

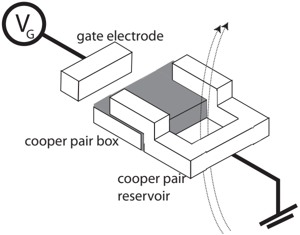
Circuit QED.

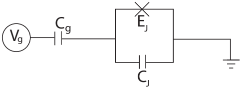
Superconducting tunnel junction.



Circuit QED.

The Cooper pair box.

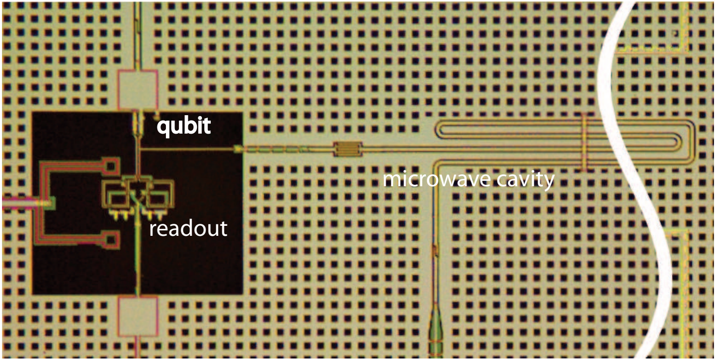




split junction $E_J(\phi_x)$.

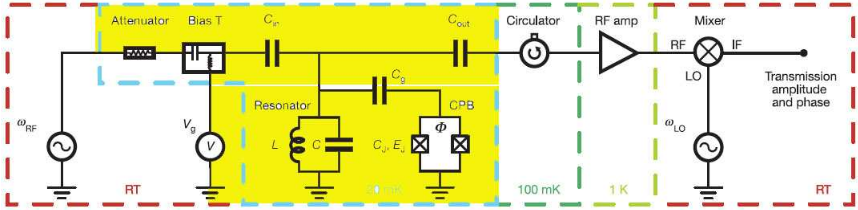
Circuit QED.

Microwave co -planar resonators.



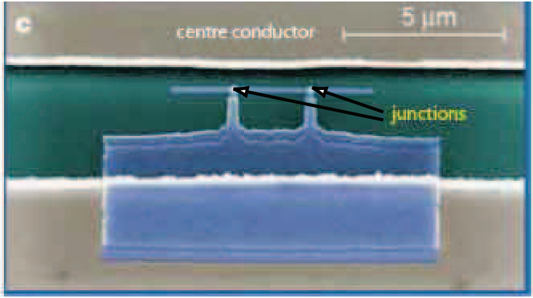
Circuit QED.

Effective Quantisation via equivalent circuit



Walraff *Nature*, (2004).

CQED



Walraff et al. *Nature* (2004)

The Hamiltonian.

$$H = 4E_c \sum_N (N - n_g(t))^2 |N\rangle\langle N| - \frac{E_J}{2} \sum_N |N\rangle\langle N+1| + |N+1\rangle\langle N|$$

$$\begin{aligned} E_C &= \frac{e^2}{2C_\Sigma} \\ n_g(t) &= \frac{C_g V_g(t)}{2e} \end{aligned}$$

$$V_g(t) = V_g^{(0)} + \hat{v}(t)$$

The Hamiltonian

Work in subspace, $N = 0, 1$.

$$H = H_{CPB} - 4E_C \delta\hat{n}_g(t)(1 - 2n_g^{(0)} - \bar{\sigma}_z)$$

$$H_{CPB} = -2E_C(1 - 2n_g^{(0)})\bar{\sigma}_z - \frac{E_J}{2}\bar{\sigma}_x$$

$$\bar{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad \bar{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\delta\hat{n}_g(t) \approx \frac{C_g}{2e}\hat{v}(t)$$

The Hamiltonian

$$H = \hbar\omega_c a^\dagger a + \frac{\hbar\epsilon}{2}\bar{\sigma}_z - \frac{\hbar\Delta}{2}\bar{\sigma}_x - \hbar g(a + a^\dagger)\bar{\sigma}_z$$

$\hbar\omega_c a^\dagger a$: cavity field

$$\hbar\epsilon = -2E_C(1 - 2n_g^{(0)})$$

$$\hbar\Delta = \frac{E_J \cos(\phi_e)}{2}$$

$$\hbar g = e\frac{C_g}{C_\Sigma}\sqrt{\frac{\hbar\omega_c}{Lc}}$$

Circuit QED

Rotating wave approximation: *Jaynes-Cummings*.

Diagonalise H_{CPB}

$$H = \hbar\omega_c a^\dagger a + \frac{\hbar\Omega}{2}\sigma_z - \hbar g(a\sigma_+ + a^\dagger\sigma_-)$$

$$\Omega = \sqrt{\Delta^2 + \epsilon^2}$$

Circuit QED

Dispersive limit: $\delta = \omega_c - \Omega \gg g$

Effective Hamiltonian in the interaction picture.

$$H_I = \frac{\hbar g^2}{2\delta} a^\dagger a \sigma_z$$

Beyond Jaynes -Cummings: the Jahn-Teller $E \otimes \beta$ model.

Circuit QED implementation:

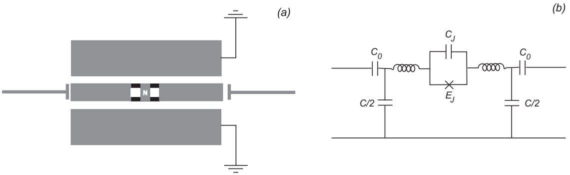


Figure: A scheme (a) and equivalent circuit (b), for a circuit QED implementation of a Jahn Teller model.

Coupling constant scales with $\alpha^{-1/2}$ (Devoret 2007).

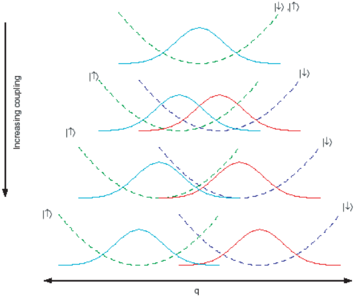
Beyond Jaynes -Cummings: the Jahn-Teller $E \otimes \beta$ model.

$$H = \hbar\omega_c a^\dagger a + \frac{\hbar\epsilon}{2}\bar{\sigma}_z - \frac{\hbar\Delta}{2}\bar{\sigma}_x - \hbar g(a + a^\dagger)\bar{\sigma}_z$$

Adiabatic potential:

$$V(x) = \frac{\omega_c^2}{2}\hat{x}^2 + \lambda\hat{x}\bar{\sigma}_z$$

Conditional displacement.



Beyond Jaynes -Cummings: the Jahn-Teller $E \otimes \beta$ model.

Semiclassical equations ($\epsilon = 0$):

$$\begin{aligned}
\dot{\alpha} &= -i\omega_c \alpha + i g s_z \\
\dot{s}_x &= 2g(\alpha + \alpha^*) s_y \\
\dot{s}_y &= \Delta s_z - 2g(\alpha + \alpha^*) s_x \\
\dot{s}_z &= -\Delta s_y
\end{aligned}$$

The Jahn-Teller $E \otimes \beta$ model.

Fixed points: $\dot{v} = 0$.
Critical value of coupling,

$$g_c = \sqrt{\frac{\Delta\omega_c}{4}}$$

If $g < g_c$ fixed points are $\alpha = 0, \quad s_x = s_y = 0, \quad s_z = \pm 1$

If $g > g_c$ fixed points are

$$\begin{aligned} s_y &= 0 \\ s_x &= \pm\frac{g_c^2}{g^2} \\ s_z &= \pm\sqrt{1 - \frac{g_c^2}{g^2}} \\ \alpha &= \frac{g}{\omega_c} s_z \end{aligned}$$

The Jahn-Teller $E \otimes \beta$ model.

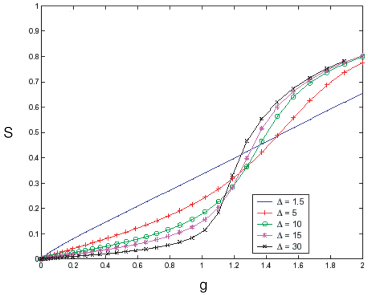
Fixed points $\leftrightarrow$ quantum ground state.

For $g < g_c$, $\quad |gs\rangle = |0\rangle \otimes |g\rangle$

For $g > g_c$, $\quad |gs\rangle = |-\alpha\rangle \otimes |\vec{n}_-\rangle + |\alpha\rangle \otimes |\vec{n}_+\rangle$

The Jahn-Teller $E \otimes \beta$ model.

Quantum entanglement in the ground state:



$S =$ entropy of reduced state of qubit.

Area preserving maps.

The King of Sweden and Professor Poincaré.

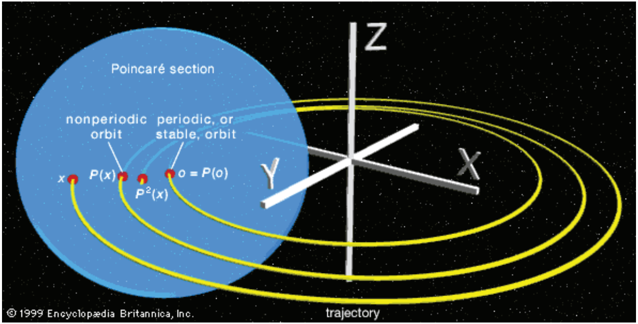
1885: Mathematical contest.

Is the solar system *dynamically stable* ?

Poincaré: *Méthodes Nouvelles de la Mécanique Céleste.*

Area preserving maps.

Beyond perturbation series to new geometric methods.



Studying the Poincaré map gives a complete characterization of the dynamics in a neighborhood of a periodic orbit.
The periodic orbit of the continuous dynamical system is stable if and only if the fixed point of the discrete dynamical system is stable.

Stroboscopic maps.

Periodically driven systems with a periodic Hamiltonian

$$H(t + T) = H(t)$$

Define discrete states

$$(q_n, p_n) = (q(t_0 + nT), p(t_0 + nT))$$

and *stroboscopic map*

$$(q_n, p_n) = F(q_{n-1}, p_{n-1})$$

Hamiltonian maps and Quantum computing.

Quantum description: unitary Floquet operator, $\hat{F}$, defines a *unitary dynamical map*:

$$|\psi_{n+1}\rangle = \hat{F}|\psi_n\rangle$$

$$H = T(\hat{p}) + V(\hat{q})\sum_n \delta(t-n)$$

Floquet map:

$$\begin{aligned} F &= U_T.U_V \\ &= e^{-iT(\hat{p})}e^{-iV(\hat{q})} \end{aligned}$$

A QC is a sequence of discrete unitary maps.

Quantum maps on a QC

Use a QC implementation, such as ion traps, to *hard wire* a quantum map.

Similar to approach of Plenio, Cirac and others to hard-wire a physical Hamiltonian flow.

Are here any *physically* interesting iterated unitary maps?

Example: transverse Ising map.

$$U(\chi, \theta) = e^{-iH_\chi} e^{-iH_\theta} = U(\chi) U(\theta)$$

$$H_\chi = \chi \sum_{n=1}^{N} \sigma_z^{(n)} \sigma_z^{(n+1)} \qquad \text{two qubit gates}$$

$$H_\theta = \theta \sum_{n=1}^{N} \sigma_x^{(n)} \qquad \text{one qubit gates}$$

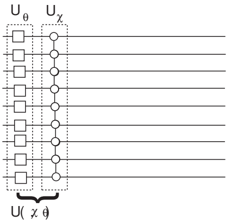
Iterated map:

$$[U(\chi) U(\theta)]^n$$

Example: transverse Ising map.

QC can implement

$$U(\chi,\theta) = e^{-iH_\chi} e^{-iH_\theta} = U(\chi)U(\theta)$$



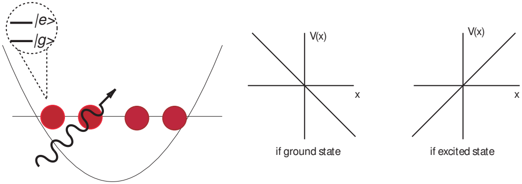
Example: transverse Ising map.

$$[U(\chi)U(\theta)]^n \neq e^{-inH_\theta - inH_\chi}$$

The iterated map is **not** an approximation to the transverse Ising dynamics.

Ion trap implementation.

Monroe et al. Science, 1996.
Linear potential seen by atom depends on internal state.



Effective Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\nu^2}{2}\hat{x}^2 + \chi(t)\hat{x}\sigma_z$$

$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$

Geometric phase gate.

pulse sequence eliminates vibrational motion.

- Sorenson, Molmer, (1999,2000),
- GJM, James and Schneider 2000,
- Lienfreid et al. 2003,
- Garcõa-Ripoll,Zoller, and Cirac, 2003

Key: use conditional displacements in phase space.

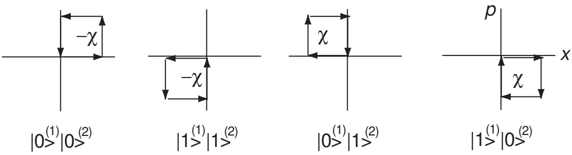
Ion trap implementation.

Use pulse sequence:

$$\begin{aligned} U_{int} &= e^{i\kappa_x \hat{X}\sigma_z^{(1)}} e^{i\kappa_p \hat{P}\sigma_z^{(2)}} e^{-i\kappa_x \hat{X}\sigma_z^{(1)}} e^{-i\kappa_p \hat{P}\sigma_z^{(2)}} \\ &= e^{-i\chi\sigma_z^{(1)}\sigma_z^{(2)}} \end{aligned}$$

$\chi = \kappa_x \kappa_p$.
No reference to vibrational degrees of freedom !
Effective Ising interaction.

Ion trap implementation.



area $= \pm\chi$

Example: transverse Ising map.

Find $\bar{H}$ where,

$$U(\chi, \theta) = e^{-iH_\chi} e^{-iH_\theta} = e^{-i\bar{H}}$$

Show that in the thermodynamic limit $\bar{H}$ is in the same universality class as the transverse field Ising model.

Example: transverse Ising map.

Use a Jordan-Wigner transformation on each unitary operator separately.

Step 1: define a_n,

$$\begin{aligned}\sigma_x^{(n)} &= 1 - 2a_n a_n^\dagger \\ \sigma_z^{(n)} &= a_n^\dagger + a_n \\ \sigma_y^{(n)} &= -i(a_n - a_n^\dagger)\end{aligned}$$

where

$$\begin{array}{lll}\{a_n^\dagger, a_n\} = 1, & a_n^2 = 0, & a_n^{\dagger 2} = 0, \\ [a_m^\dagger, a_n] = 0, & [a_m^\dagger, a_n^\dagger] = 0, & [a_m, a_n] = 0, m \neq n\end{array}$$

Example: transverse Ising map.

Step 2.

$$
\begin{aligned}
c_n &= e^{i\pi \sum_{j=1}^{n-1} a_j^\dagger a_j} a_n \\
c_n^\dagger &= a_n^\dagger e^{-i\pi \sum_{j=1}^{n-1} a_j^\dagger a_j}
\end{aligned}
$$

which obey fermionic anti-commutation relations.

Example: transverse Ising map.

$$\bar{H} = \Lambda_1 + \Lambda_2 + \Lambda_3$$

$$\begin{aligned}
\Lambda_1 &= \cos\theta \sin\chi [a_0 (c_n^\dagger c_{n+1}^\dagger - c_n c_{n+1}) \\
&\quad + \sum_{n,l} \frac{(a_{l+1} - a_{l-1})}{2} (c_n^\dagger c_{n+l}^\dagger - c_n c_{n+l})] \\
\Lambda_2 &= \ldots \\
\Lambda_3 &= \ldots
\end{aligned}$$

This has effective non-nearest neighbor interactions.

Example: transverse Ising map.

Does $\bar{H}$ fall into the same universality class as the transverse field Ising in thermodynamic limit?

YES !
can show,

$$a_l \leq ke^{-\mu l} \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty$$

where $\mu = |\ln(\sin\theta \sin\chi)|$.

Example: transverse Ising map.

Ising criticality occurs for $\theta = \pm\chi$.

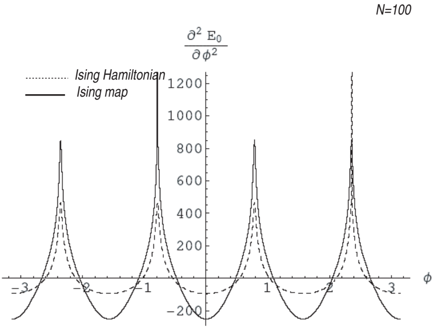
- Find the ground state of the effective hamiltonian.
- Look at second derivatives of the corresponding eigenvalue (quasi-energy)
- Singularities at Ising criticality points as N becomes large.

A many-body unitary map with a quantum phase transition, implemented on an ion trap QC.

Example: transverse Ising map.

Consider ground state of effective Hamiltonian, $\bar{H}$.

$$\phi = \arctan\left(\theta/\chi\right)$$



See Barjaktarevic, GJM, McKenzie Phys. Rev. A 70, (2004)

Conclusions.

- Circuit QED as a test-bed for quantum measurement and control.
- Circuit QED for quantum bifurcations in nonlinear Hamiltonian systems.
- Unitary maps are just as interesting as Hamiltonian flows.
- Ion traps to simulate interacted maps on many spins.