Information Theoretical Measures of Quantum Phase Transitions

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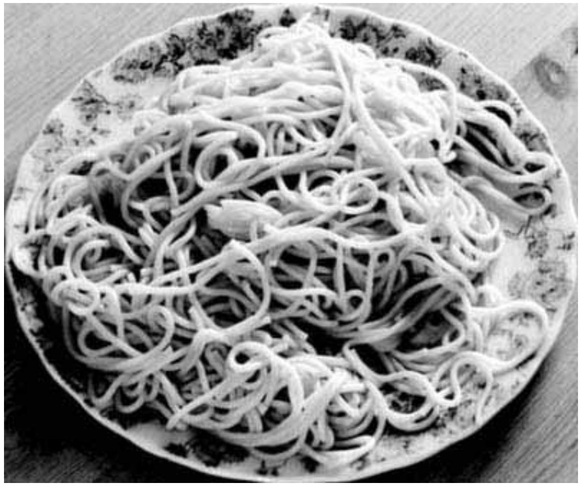
Motivation

Entanglement and fidelity measures identify and elucidate the nature of quantum phase transitions in correlated many-body condensed matter systems.

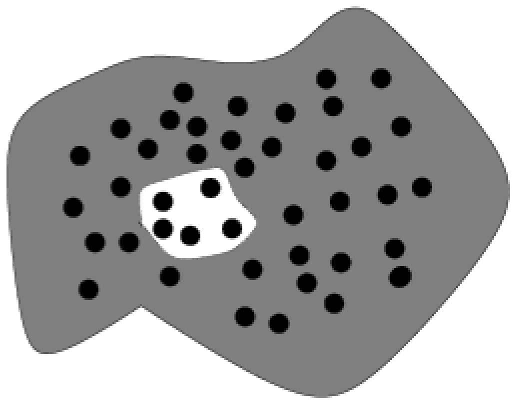
- Scaling of entanglement entropy.
- Identification of topological order.
- Discovery of hidden factorized states.
- Analysis of glassy phenomena.

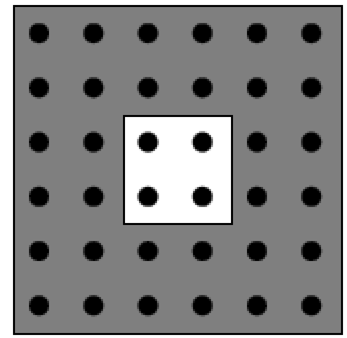
Entanglement and Fidelity



- Entanglement measures how "quantum" a state is. Typical entanglement measure is the von Neuman entropy of formation.
- Fidelity measures the overlap integral of two states.
- Both are measures of states, independent of the underlying Hamiltonian. Their scaling properties can be used to indentify phase transitions.

ENTANGLEMENT IN QUANTUM SYSTEMS





1. Start with a quantum many-body state $|\psi\rangle$ which you may obtain for example from exact diagonalization of a Heisenberg cluster.
2. Use this state to construct the corresponding density matrix, i.e. $\rho = |\psi\rangle\langle\psi|$.
3. Define the "inside" area A, and trace out the outside environment B out of the density matrix: $\rho_A = tr\rho_B$.
4. Use this reduced density matrix to calculate the von-Neumann entropy defined by $S = -\rho_A tr\rho_A$.
5. The initial state is maximally entangled if $S = \ln 2$ and minimally entangled if $S = 0$.

Two Examples

Classical:

$|1111\rangle$

“inside” area: sites 2 and 3 in the center.

$$\rho = |1111\rangle\langle 1111|$$

$$\rho_A = \langle 0|_1 \langle 0|_4 \rho |0\rangle_1 |0\rangle_4 + \langle 1|_1 \langle 1|_4 \rho |1\rangle_1 |1\rangle_4 = |11\rangle_{23}\langle 11|_{23}$$

$$S = -tr(|11\rangle\langle 11| \ln(|11\rangle\langle 11|)) = -tr 1 \ln(1) = 0$$

Not entangled

Quantum:

$\frac{1}{\sqrt{2}}(|1111\rangle + |0000\rangle)$

“inside” area: sites 2 and 3 in the center.

$$\rho_A = tr_B \rho = \frac{1}{2}(|11\rangle_{23}\langle 11|_{23} + |00\rangle_{23}\langle 00|_{23}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

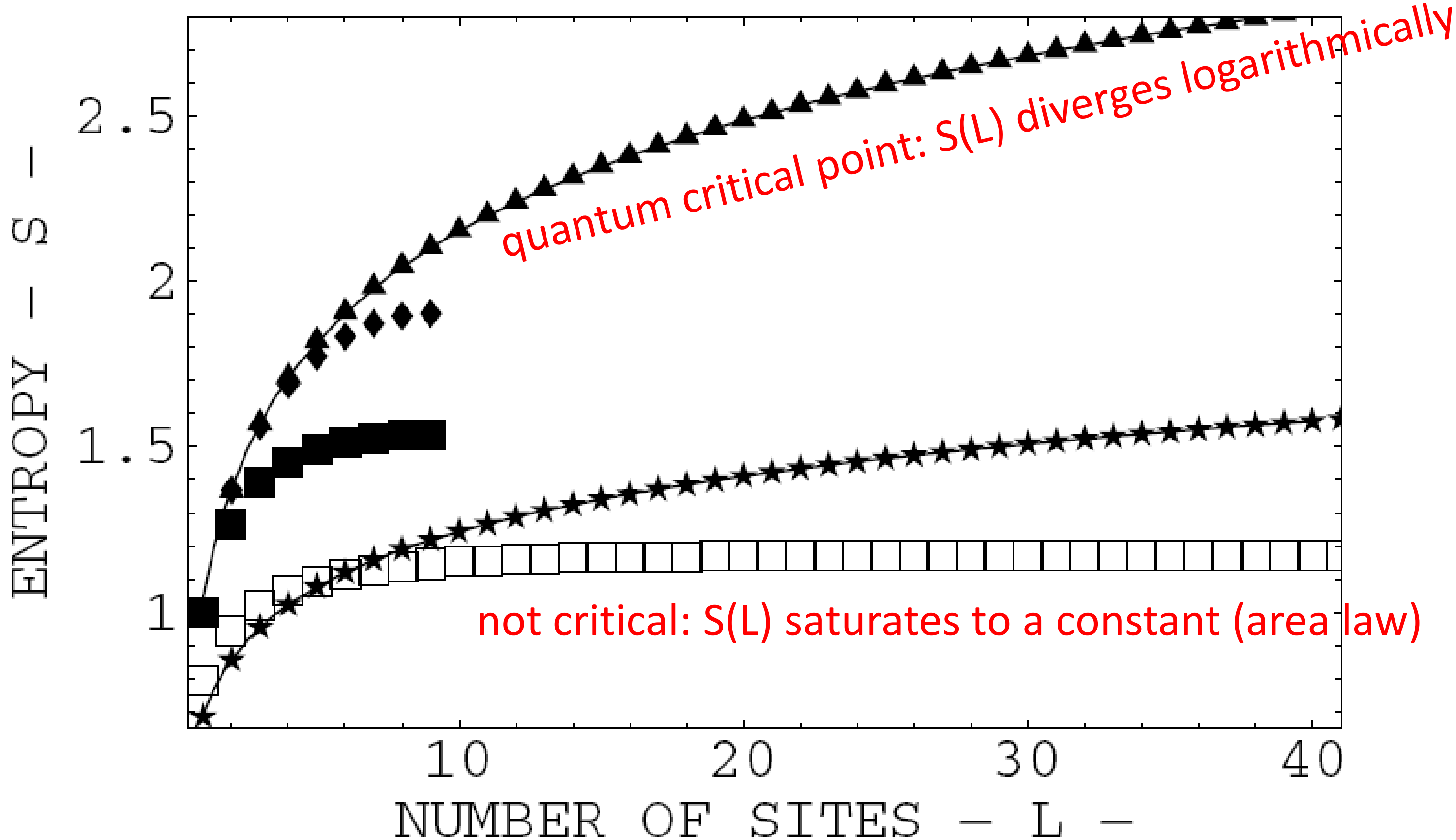
$$S = -tr\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ln \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -tr\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ln(1/2) & 0 \\ 0 & \ln(1/2) \end{pmatrix} = \ln(2)$$

Maximally entangled

Scaling with size of "Inside" Area

Example: XXZ quantum spin-1/2 chain

$$H_{XXZ} = \sum_{l=0}^{N-1} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \sigma_l^z \sigma_{l+1}^z - \lambda \sigma_l^z)$$



(Vidal, Latorre, Rico, Kitaev, PRL 90, 227902 (2003).)

Area Law

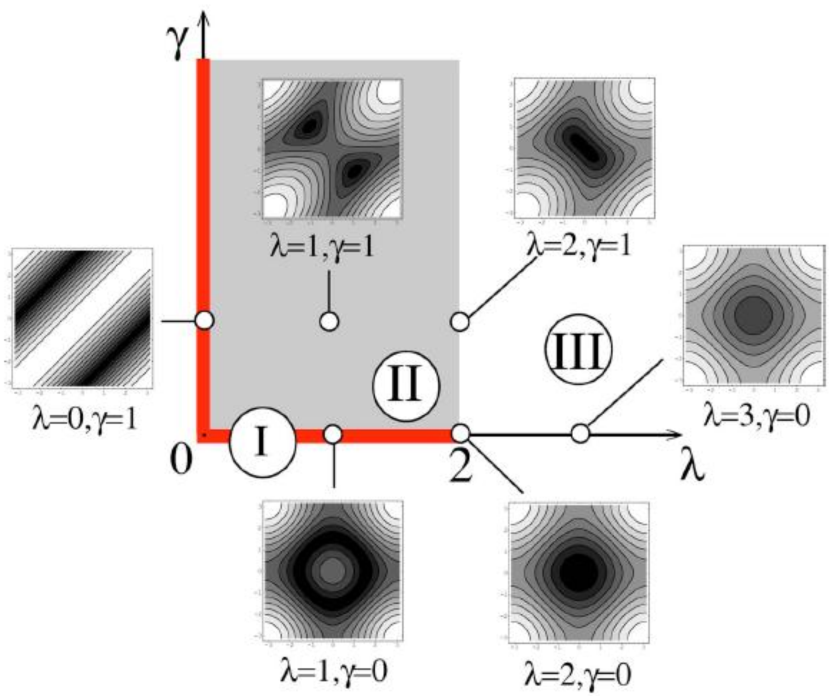
In d spatial dimensions the entanglement entropy as a function of the “inside” area asymptotically scales as:

$$S(L) \propto L^{d-1}$$

However, interesting exceptions to this “law” are observed at phase transitions and in systems with reduced phase space, i.e. nodal Fermi surfaces.

Scaling behavior of entanglement in a prototype many-body system (2D)

$$H = \sum_{\langle ij \rangle} [c_i^\dagger c_j - \gamma(c_i^\dagger c_j^\dagger + c_j c_i)] - \sum_i 2\lambda c_i^\dagger c_i$$

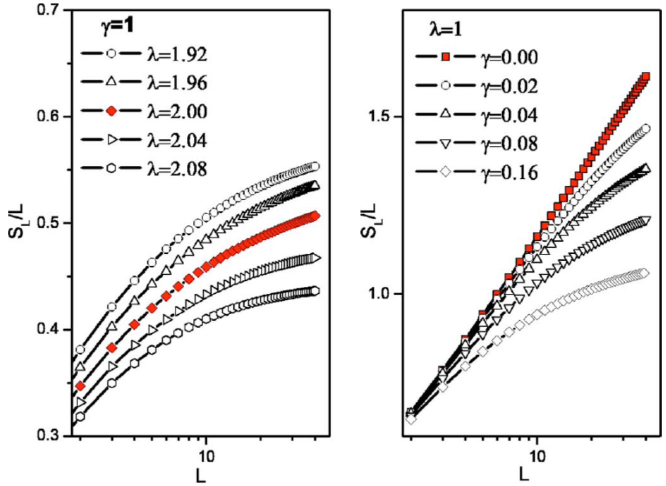


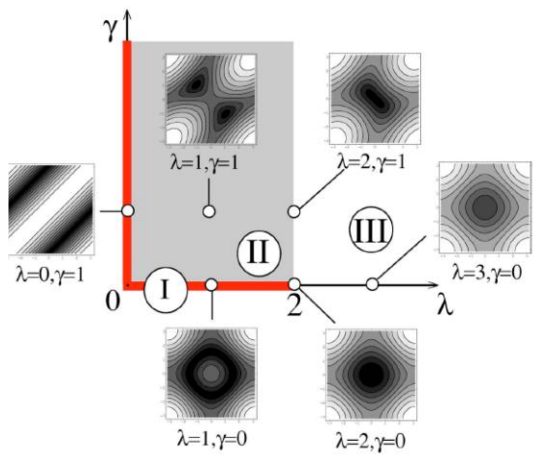
- Hamiltonian describes spinless fermions with pairing interaction γ and applied field λ.
- Phase diagram:

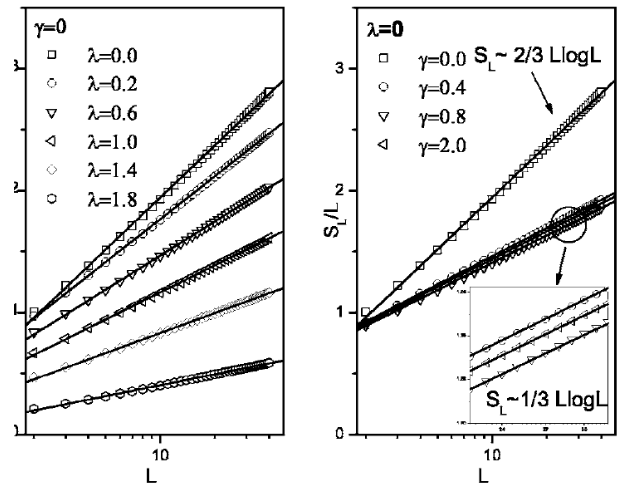
1. Quasi-free fermions (finite Fermi surface).
2. Nodal Superconductor (2 point nodes).
3. Band insulator (no Fermi surface).

(Li, Ding, Yu, Roscilde, Haas, PRB 74, 073103 (2006).)

Scaling of entanglement entropy







Scans across phase transitions confirm area law in phases II and III and super-area-law scaling at the phase boundaries.

Scans within phase I indicate violation of area law. Exact scaling result is confirmed at conformal point.

	S_L	$\bar{d}$	$g(0)$	$\langle c_i^\dagger c_j \rangle$
Phase I	$\sim(\log_2 L)L^{d-1}$	1	>0	Power-law decay
Phase II	$\sim L^{d-1}$	2	0	Power-law decay
Phase III	$\sim L^{d-1}$	d	0	Exp. decay

- area law holds in phases II and III, super-area-law detected in phase I.
- phases I and II are critical, i.e. they have power-law correlation functions.
- phase I has finite Fermi surface, phase II only has nodal points

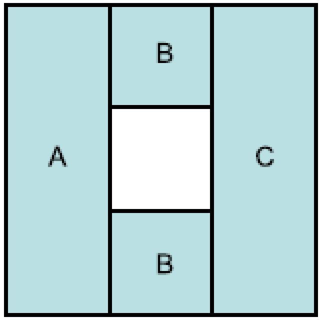
Finite density of states at Fermi surface is sufficient condition for critical phases to violate area law.

Detection of topological order

- Topological entanglement entropy:

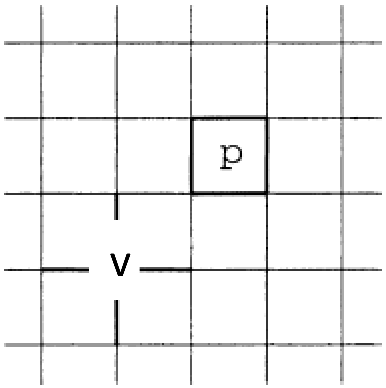
$$S_{\rm topo} \equiv S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$

(Kitaev & Preskill, PRL 96, 110404 (2006).)



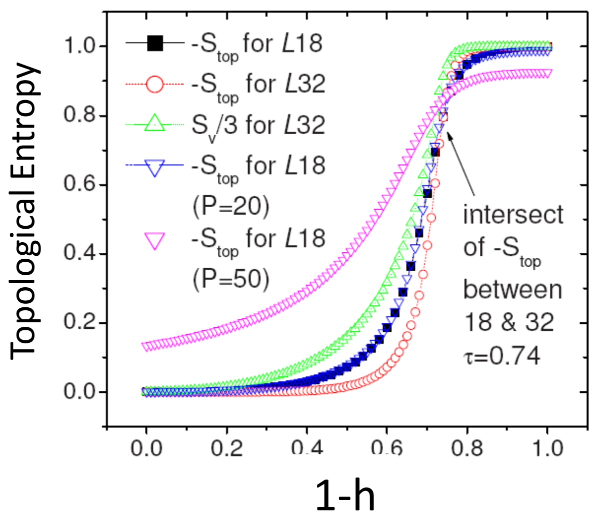
- Example: Kitaev's toric code:

$$H_{\rm TC} = -A\sum_{v}\prod_{j\in{\rm vertex}(v)}\sigma_j^z - B\sum_{p}\prod_{j\in{\rm plaquette}(p)}\sigma_j^x$$



Tension term (applied magnetic field) destroys loop condensate: transition from topologically ordered to paramagnetic phase.

$$H = H_{\rm TC} - h\sum \sigma_i^z$$



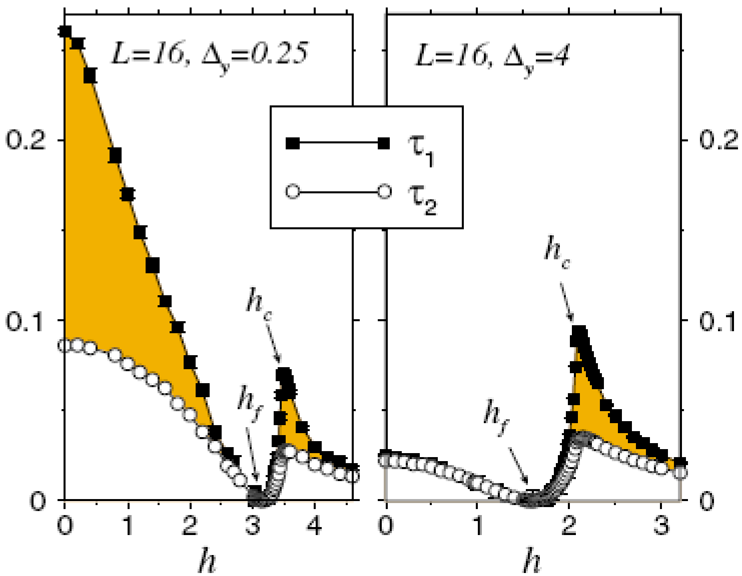
(Kitaev, Annals Physics 303, 30 (2003).)

(Hamma, Zhang, Haas, Lidar, PRB 77, 155111 (2008).)

Detection of hidden product states

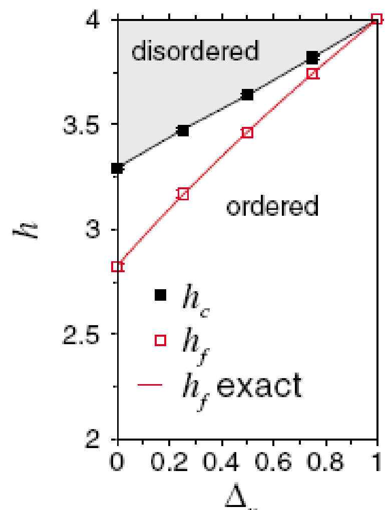
- Multipartite vs. Bipartite entanglement: numerical determination of τ_1 vs. τ_2.

$$\hat{\mathcal{H}}/J = \sum_{\langle ij \rangle} [\hat{S}_i^x \hat{S}_j^x + \Delta_y \hat{S}_i^y \hat{S}_j^y + \Delta_z \hat{S}_i^z \hat{S}_j^z] - \sum_i \boldsymbol{h} \cdot \hat{\boldsymbol{S}}_i$$



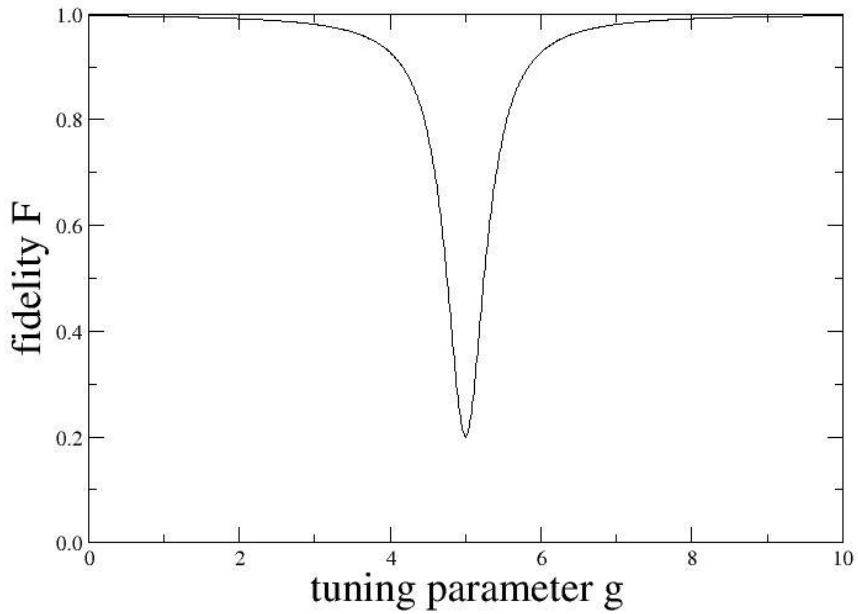
(Roscilde, Verrucchi, Fubini, Haas, Tognetti, PRL 94, 147108 (2005).)

- both entanglement measures go to zero at hidden factorized state (h_f).
- extrema at critical field (h_c), separating paramagnetic from antiferromagnetic phase.
- multipartite entanglement dominates over bipartite entanglement at h_c.



Fidelity in Quantum Systems

$$F(g) = \left| \langle \Psi(g) | \Psi(g+\Delta g) \rangle \right|$$



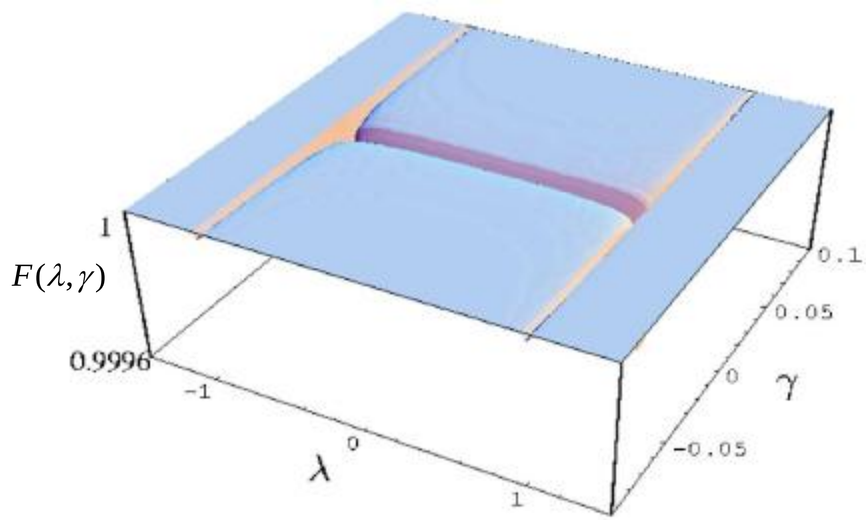
- fidelity measures overlap between wave functions infinitesimally separated in parameter space.
- fidelity is extensive in non-critical regimes: $F(g,L) \propto \exp(-L(\Delta g)^2/2)$
- fidelity is super-extensive in critical region: $F(g_c,L) \propto \exp(-L^2(\Delta g)^2/2)$

Critical scaling exponents can be extracted from analysis of F(g,L).

Fidelity in the quantum XY chain

$$\hat{H}(\gamma,\lambda) = -\sum_{i=-M}^{M}\left(\frac{1+\gamma}{2}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2}\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y + \frac{\lambda}{2}\hat{\sigma}_i^z\right)$$

- quantum phase transitions indicated by drop in fidelity.
- Ising transition at λ=±1.
- Anisotropy transition at γ=0.
- critical exponents in agreement with scaling theory.

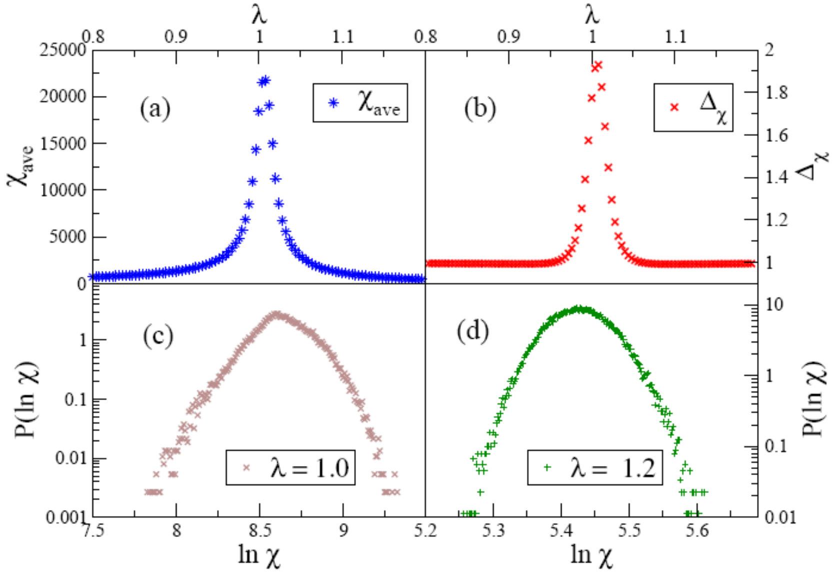


(Zanardi & Paunkovic, PRE 74, 031123 (2006).)

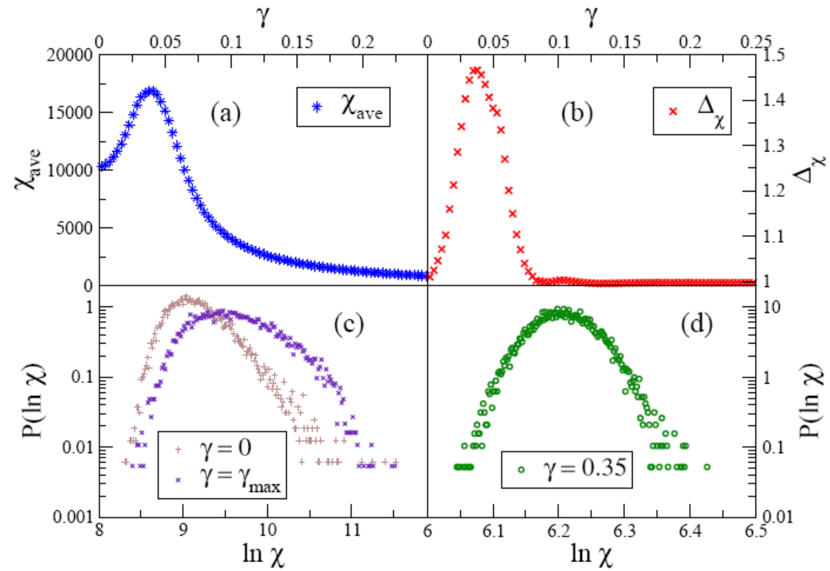
Fidelity in the random XY chain

$$H = -\sum_{i=0}^{L-1} \left(\frac{1+\gamma_i}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma_i}{2} \sigma_i^y \sigma_{i+1}^y + \lambda_i \sigma_i^z \right)$$

Ising transition:



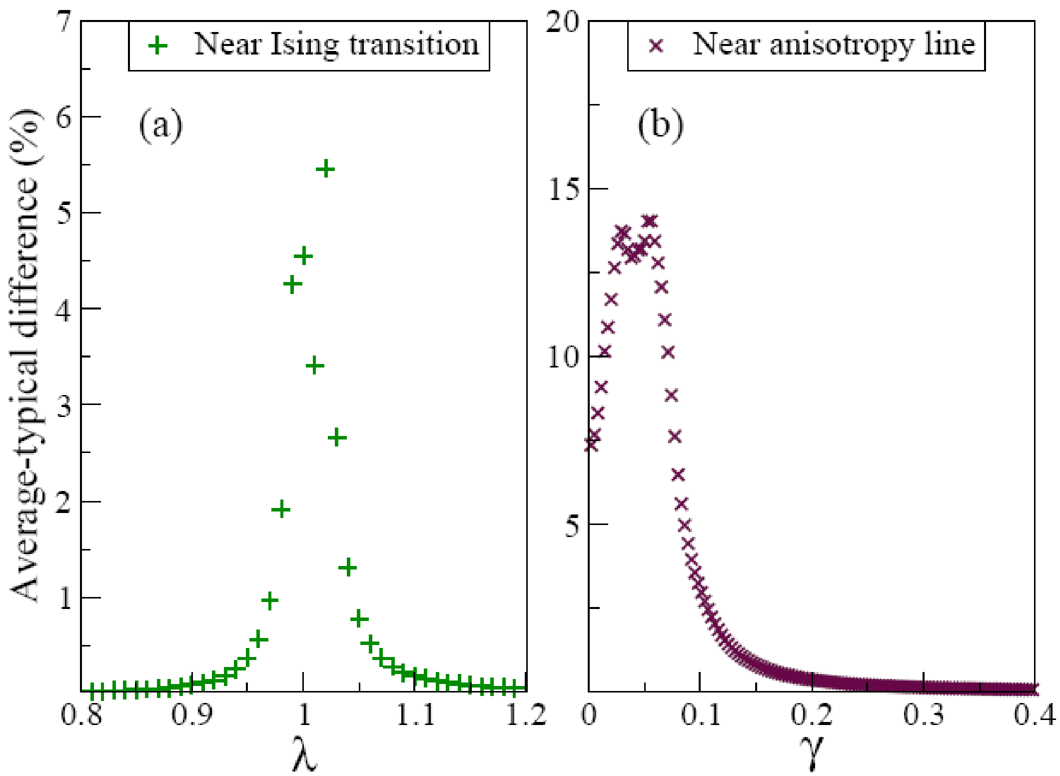
Anisotropy transition:



- fidelity susceptibility, $\chi(x) = \lim_{\delta x \to 0} \frac{-2 \ln F(x, x+\delta x)}{\delta x^2}$, is averaged over 50,000 realizations.
- broadening of critical regimes compared to clean case.
- signature of glassiness: asymmetric distribution of fidelity susceptibility.
- non-universal scaling exponent: Griffiths regime.

(Garnerone, Jacobson, Haas, Zanardi, cond-mat/0808.4140.

Griffiths regime



- average vs. typical fidelity susceptibility different close to the critical lines.
- special point: γ=0. Maps onto free fermions, hence Anderson localization.

(Garnerone, Jacobson, Haas, Zanardi, cond-mat/0808.4140.

Conclusions

- Entanglement and fidelity are useful measures to identify quantum phase transitions and exotic states in correlated matter.
- Deviations from canonical area law of entanglement are observed at quantum phase transitions and in critical phases.
- Entanglement measures can be used to identify topological order and factorized states.
- Fidelity can be used to extract critical exponents and to identify regimes with non-universal scaling properties, such as Griffiths phases.