Resilient Quantum Computation in Correlated Environments

Eduardo Novais[1], E. R. Mucciolo[2] and Harold U. Baranger[1]

1- Department of Physics, Duke University, Durham NC 27708-0305.

2- Department of Physics, University of Central Florida, Orlando FL 32816-2385.

December 2006.

Part I

Prologue

Quantum computation?

Richard Feynman em
"The Feynman Lectures on Computations".

*"But, we are going to be even more ridiculous later and consider **bits written on one atom** instead of the present 10^{11} atoms. Such nonsense is very entertaining to professors like me. I hope you will find it interesting and entertaining also."*

Quantum computation?

Richard Feynman em
"The Feynman Lectures on Computations".

"But, we are going to be even more ridiculous later and consider ***bits written on one atom*** *instead of the present* 10^{11} *atoms. Such nonsense is very entertaining to professors like me. I hope you will find it interesting and entertaining also."*

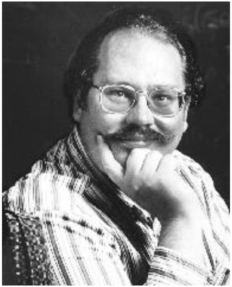
Quantum computation?

Richard Feynman em
"The Feynman Lectures on Computations".

"But, we are going to be even more ridiculous later and consider ***bits written on one atom*** *instead of the present* 10^{11} *atoms. Such nonsense is very entertaining to professors like me. I hope you will find it interesting and entertaining also."*

20 years later ...

- has (at this moment) some advantages over classical computation:
  1. factorization of large numbers,
  2. search in a list of entries,
  3. simulation of quantum systems.
- is (in principle) possible with error correction;



Peter Shor in an e-mail to Leonid Levin.

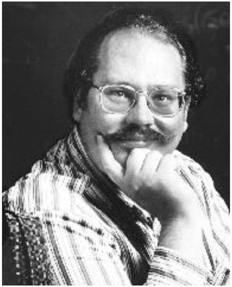
"I'm trying to say that you don't need a machine which handles amplitudes with great precision. If you can build quantum gates with accuracy of 10^{-4}*, and put them together in the right fault-tolerant way, quantum mechanics says that you should be able to factor large numbers."*

20 years later ...

- has (at this moment) some advantages over classical computation:

  1. factorization of large numbers,
  2. search in a list of entries,
  3. simulation of quantum systems.

- is (in principle) possible with error correction;



Peter Shor in an e-mail to Leonid Levin.

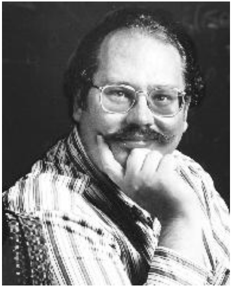
"I'm trying to say that you don't need a machine which handles amplitudes with great precision. If you can build quantum gates with accuracy of 10^{-4}*, and put them together in the right fault-tolerant way, quantum mechanics says that you should be able to factor large numbers."*

20 years later ...

- has (at this moment) some advantages over classical computation:

  1. factorization of large numbers,
  2. search in a list of entries,
  3. simulation of quantum systems.

- is (in principle) possible with error correction;



Peter Shor in an e-mail to Leonid Levin.

"I'm trying to say that you don't need a machine which handles amplitudes with great precision. If you can build quantum gates with accuracy of 10^{-4}*, and put them together in the right fault-tolerant way, quantum mechanics says that you should be able to factor large numbers."*

The future looks bright...

What we have:

- quantum computers have a purpose;
- it is believed that they are theoretically possible.

What we want to know now:

?

The future looks bright...

What we have:

- quantum computers have a purpose;
- it is believed that they are theoretically possible.

What we want to know now:

When can I play
Quantum-Quake?

Building a Quantum Computer is a hard task...

DiVincenzo's criteria for a quantum computer

1. a physical system *scalable* and with well defined qubits,
2. to be able to *initialize* the state,
3. *decoherence* times larger than quantum gates operation times,
4. a *complete* set of quantum gates,
5. the ability to *measure* individual qubit.

- I used three important terms:
  1. qubits,
  2. quantum gates, and
  3. decoherence.
- we also need:
  1. entanglement.

Building a Quantum Computer is a hard task...

DiVincenzo's criteria for a quantum computer

1. a physical system *scalable* and with well defined qubits,
2. to be able to *initialize* the state,
3. *decoherence* times larger than quantum gates operation times,
4. a *complete* set of quantum gates,
5. the ability to *measure* individual qubit.

- I used three important terms:
  1. qubits,
  2. quantum gates, and
  3. decoherence.
- we also need:
  1. entanglement.

Building a Quantum Computer is a hard task...

DiVincenzo's criteria for a quantum computer

1. a physical system *scalable* and with well defined qubits,
2. to be able to *initialize* the state,
3. *decoherence* times larger than quantum gates operation times,
4. a *complete* set of quantum gates,
5. the ability to *measure* individual qubit.

- I used three important terms:
  1. qubits,
  2. quantum gates, and
  3. decoherence.
- we also need:
  1. entanglement.

Building a Quantum Computer is a hard task...

DiVincenzo's criteria for a quantum computer

1. a physical system *scalable* and with well defined qubits,
2. to be able to *initialize* the state,
3. *decoherence* times larger than quantum gates operation times,
4. a *complete* set of quantum gates,
5. the ability to *measure* individual qubit.

- I used three important terms:
  1. qubits,
  2. quantum gates, and
  3. decoherence.
- we also need:
  1. entanglement.

Building a Quantum Computer is a hard task...

DiVincenzo's criteria for a quantum computer

1. a physical system *scalable* and with well defined qubits,
2. to be able to *initialize* the state,
3. *decoherence* times larger than quantum gates operation times,
4. a *complete* set of quantum gates,
5. the ability to *measure* individual qubit.

- I used three important terms:
  1. qubits,
  2. quantum gates, and
  3. decoherence.
- we also need:
  1. entanglement.

What is entanglement?

E. Schrödinger *(Cambridge Philosophical Society)*

"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled."

Entanglement: the best and the worst for a quantum computer

- *entanglement* is the new ingredient of quantum computation.
- *decoherence* is the unavoidable entanglement of the computer and the environment.

What is entanglement?

E. Schrödinger *(Cambridge Philosophical Society)*

"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled."

Entanglement: the best and the worst for a quantum computer

- *entanglement* is the new ingredient of quantum computation.
- *decoherence* is the unavoidable entanglement of the computer and the environment.

What is entanglement?

E. Schrödinger *(Cambridge Philosophical Society)*

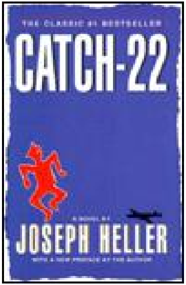
"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled."

Entanglement: the best and the worst for a quantum computer

- *entanglement* is the new ingredient of quantum computation.
- *decoherence* is the unavoidable entanglement of the computer and the environment.

the *"catch-22"* in quantum computation

- catch-22: the solution creates the problem.



low decoherence X interaction and control.

" Okay, let me see if I've got this straight. In order to be grounded, I've got to be crazy, and i must be crazy to be flying, but if i ask to be grounded, that means I'm not crazy anymore and have to keep flying."

Part II

The problem

Protecting quantum information from decoherence

- There are 3 “strategies” (design for different physical limits):

1- decoherence free subspaces;

D. A. Lidar, I. L. Chuang and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).

2- dynamical decoupling;

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).

L.Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

3- quantum error correction.

P. Shor, Phys. Rev. A **52**, 2493 (1995).
A. Steane, Phys. Rev. Lett. **77**, 793 (1996).
E. Knill, R. Laflamme and W. H. Zurek, Science **279**, 342 (1998);

D. Aharonov and M. Ben-Or, arXiv:quant-ph/9906129.

Protecting quantum information from decoherence

- There are 3 “strategies” (design for different physical limits):

1- decoherence free subspaces;

D. A. Lidar, I. L. Chuang and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).

2- dynamical decoupling;

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).

L.Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

3- quantum error correction.

P. Shor, Phys. Rev. A **52**, 2493 (1995).
A. Steane, Phys. Rev. Lett. **77**, 793 (1996).
E. Knill, R. Laflamme and W. H. Zurek, Science **279**, 342 (1998);

D. Aharonov and M. Ben-Or, arXiv:quant-ph/9906129.

Protecting quantum information from decoherence

- There are 3 “strategies” (design for different physical limits):

1- decoherence free subspaces;

D. A. Lidar, I. L. Chuang and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).

2- dynamical decoupling;

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).

L.Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

3- quantum error correction.

P. Shor, Phys. Rev. A **52**, 2493 (1995).
A. Steane, Phys. Rev. Lett. **77**, 793 (1996).
E. Knill, R. Laflamme and W. H. Zurek, Science **279**, 342 (1998);

D. Aharonov and M. Ben-Or, arXiv:quant-ph/9906129.

Protecting quantum information from decoherence

Wojciech Zurek

Quantum error correction is the most versatile.
Will be used some way or another.

3- quantum error correction.

P. Shor, Phys. Rev. A **52**, 2493 (1995).
A. Steane, Phys. Rev. Lett. **77**, 793 (1996).
E. Knill, R. Laflamme and W. H. Zurek, Science **279**, 342 (1998);
D. Aharonov and M. Ben-Or, arXiv:quant-ph/9906129.

A major result of QEC: “threshold theorem”

“threshold theorem”

Provided the noise strength is below a critical value, quantum information can be protected for arbitrarily long times.

Hence, the computation is said to be *fault tolerant* or *resilient*.

Dorit Aharonov, Phys. Rev. A 062311 (2000).

Quantum to classical phase transition in a noisy QC.

threshold theorem

resilient quantum computation

strong entanglement between qubits

''low temperature regime"

noise quantum computer

simulated efficiently by a Turing machine

weak entanglement between qubits

''high temperature regime"

error probability of a qubit (ε)

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the “theorem” is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the “theorem” is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the "theorem" is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the “theorem” is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

R. Alicki *et al*, Phys. Rev. A **65**, 062101 (2002).

Dynamical description of quantum computing: Generic nonlocality of quantum noise

This shows that the implicit assumption of quantum error correction theory, independence of noise and self-dynamic, fails in long time regimes.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the "theorem" is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

B. M. Terhal and G. Burkard, Phys. Rev. A **71** 012336 (2005).

Fault-tolerant quantum computation for local non-Markovian noise

A third type of decoherence exists which is essentially troublesome in our analysis. This is the example of a single qubit, or spin, couple to a bosonic bath.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the "theorem" is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

What we will consider

- work with a microscopic Hamiltonian:
  1. show how to calculate probabilities,
  2. derive the effects of correlations.
- describe when a correlated environments can be included in the traditional derivation of the threshold theorem.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the “theorem” is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

What we will consider

- work with a microscopic Hamiltonian:
  1. show how to calculate probabilities,
  2. derive the effects of correlations.
- describe when a correlated environments can be included in the traditional derivation of the threshold theorem.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the "theorem" is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

What we will consider

- work with a microscopic Hamiltonian:
  1. show how to calculate probabilities,
  2. derive the effects of correlations.
- describe when a correlated environments can be included in the traditional derivation of the threshold theorem.

Is this the whole story? Fortunately NOT.

Fortunately NOT

- the "theorem" is derived using error models (NOT Hamiltonians),
- implicitly assumes that perturbation theory works,
- correlated environments are usually not considered.

What we will consider

- work with a microscopic Hamiltonian:
  1. show how to calculate probabilities,
  2. derive the effects of correlations.
- describe when a correlated environments can be included in the traditional derivation of the threshold theorem.

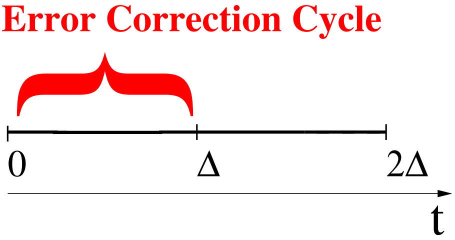
Part III

Quantum Error Correction

How does error correction work?

Executive summary

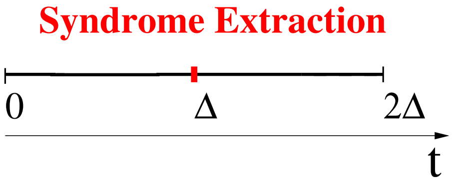
1. encode the information in a large Hilbert space;
2. let the system evolves;
3. extract the "syndrome";
4. correct the system;
5. start again;



How does error correction work?

Executive summary

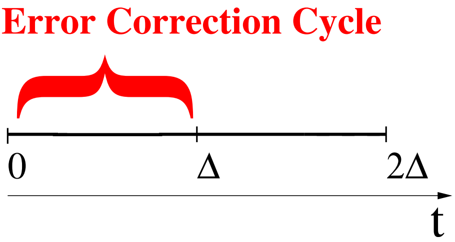
1. encode the information in a large Hilbert space;
2. let the system evolves;
3. extract the “syndrome”;
4. correct the system;
5. start again;



How does error correction work?

Executive summary

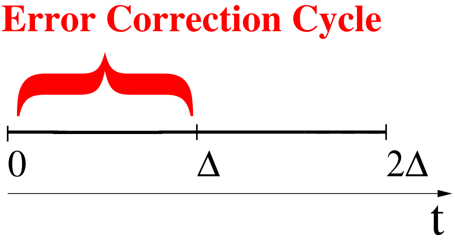
1. encode the information in a large Hilbert space;
2. let the system evolves;
3. extract the "syndrome";
4. correct the system;
5. start again;



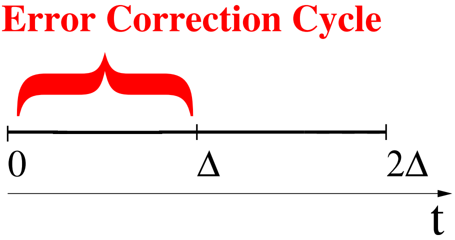
How does error correction work?

What are the consequences?

1. QEC reduces the probability of a cycle with an error,
2. QEC reduces the decoherence of a cycle with no-error.



How does error correction work?



What are the consequences?

1. QEC reduces the probability of a cycle with an error,
2. QEC reduces the decoherence of a cycle with no-error.

How this translates to the quantum evolution?

- consider and environment controlled by a “free” Hamiltonian:

$$H_0$$

- and an interaction:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{1}$$

where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.

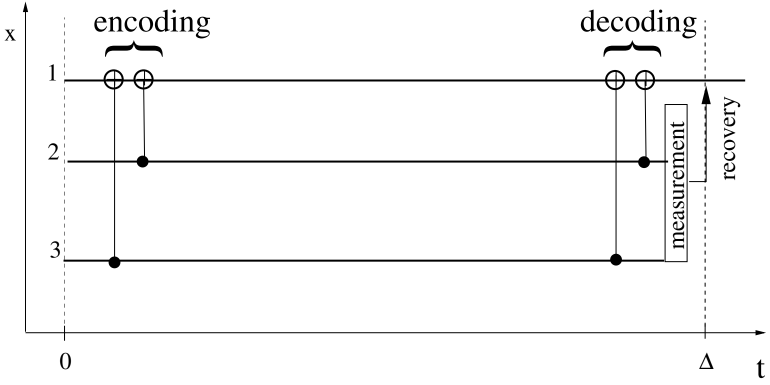
time evolution of encoded qubits

- usually the quantum evolution in the interaction picture is:

$$\begin{aligned}
\hat{U}(\Delta,0) &= T_t e^{-i\frac{\lambda}{2}\sum_{\mathbf{x}}\int_0^\Delta dt \vec{f}(\mathbf{x},t)\vec{\sigma}(\mathbf{x})} \\
&= 1 - i\frac{\lambda}{2}\sum_{\mathbf{x}}\int_0^\Delta dt \vec{f}(\mathbf{x},t)\vec{\sigma}(\mathbf{x}) \\
&- \frac{\lambda^2}{4}\sum_{\substack{\mathbf{x},\mathbf{y} \\ \alpha,\beta=\{x,y,z\}}}\int_0^\Delta dt_1 \int_0^{t_1} dt_2 f^\alpha(\mathbf{x},t_1) f^\beta(\mathbf{y},t_2)\sigma^\alpha(\mathbf{x})\sigma^\beta(\mathbf{y}) + \ldots
\end{aligned}$$

- when the syndrome is extracted only a set of terms is kept to the next QEC cycle.
- the evolution for long times in non-unitary.

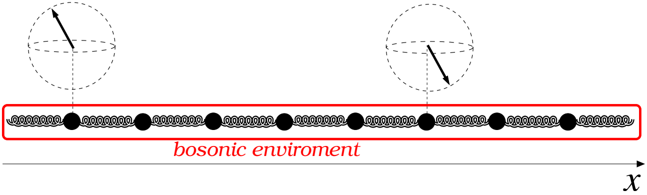
Example: a QEC that protects against phase flips



- encoded words:

$$
\begin{aligned}
|\bar{\uparrow}\rangle &= \left(|\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)/2 \\
|\bar{\downarrow}\rangle &= \left(|\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right)/2
\end{aligned}
$$

Example: the spin-boson model (ohmic dissipation)

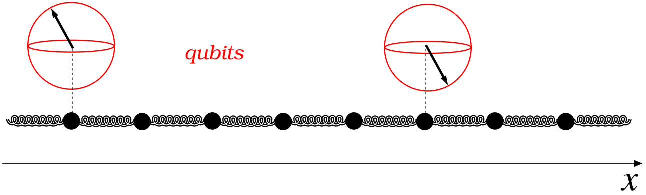


$$H = \frac{v_b}{2}\int_{-\infty}^{\infty} dx\,[\partial_x \phi(x)]^2 + [\Pi(x)]^2 + \sqrt{\frac{\pi}{2}}\lambda \sum_n \partial_x \phi(n)\,\sigma_n^z$$

where ϕ and $\Pi = \partial_x \theta$ are canonical conjugate variables, σ_n^z act in the Hilbert space of the qubits, v_b is the velocity of the bosonic excitations, and $\hbar = k_B = 1$. The bosonic modes have an ultraviolet cut-off, Λ, that defines the short-time scale of the field theory, $t_{uv} = (\Lambda v_b)^{-1}$.

bosonic field notation

Example: the spin-boson model (ohmic dissipation)



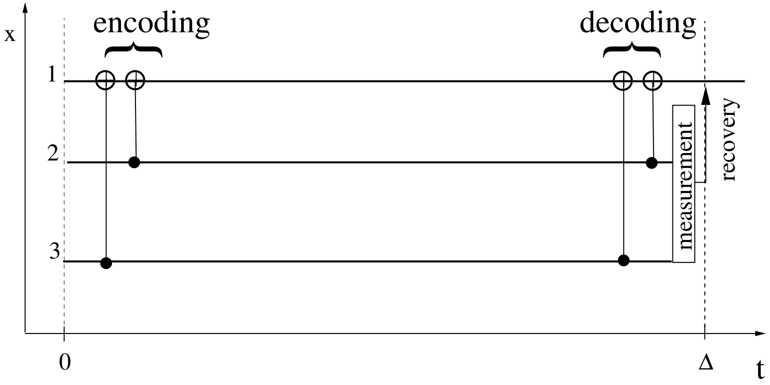
$$H=\frac{v_{\mathrm{b}}}{2}\int_{-\infty}^{\infty}dx\left[\partial_x\phi\left(x\right)\right]^2+\left[\Pi\left(x\right)\right]^2+\sqrt{\frac{\pi}{2}}\lambda\sum_n\partial_x\phi\left(n\right)\sigma_n^z$$

where ϕ and $\Pi=\partial_x\theta$ are canonical conjugate variables, σ_n^z act in the Hilbert space of the qubits, v_{b} is the velocity of the bosonic excitations, and $\hbar=k_B=1$. The bosonic modes have an ultraviolet cut-off, Λ, that defines the short-time scale of the field theory, $t_{\mathrm{uv}}=\left(\Lambda v_{\mathrm{b}}\right)^{-1}$.

bosonic field notation

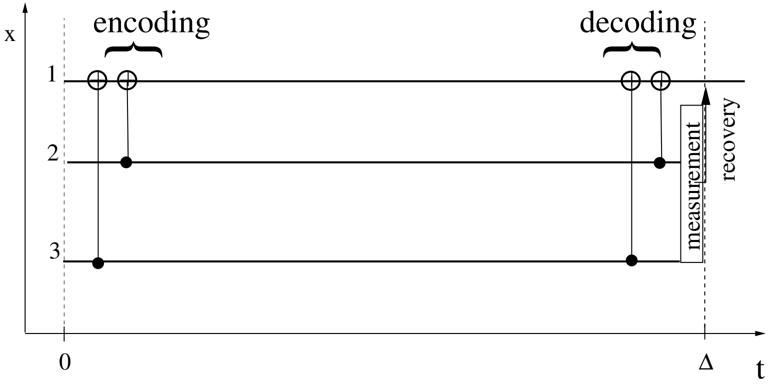
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



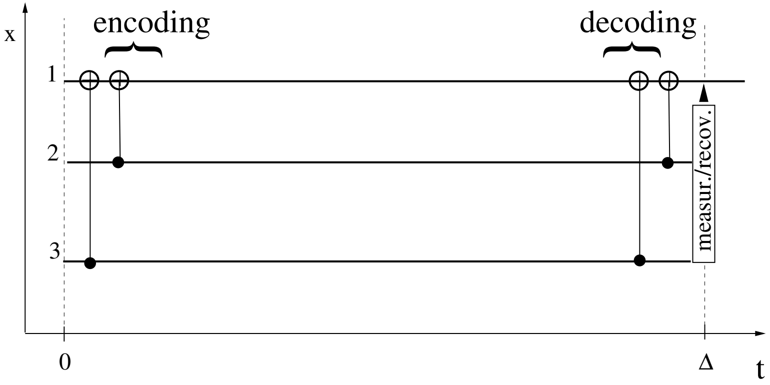
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



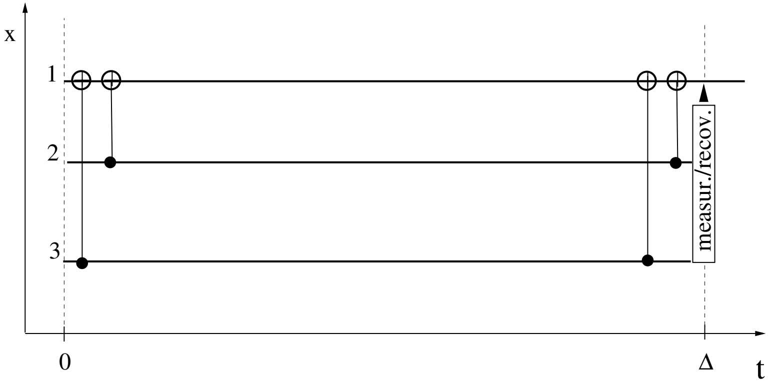
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



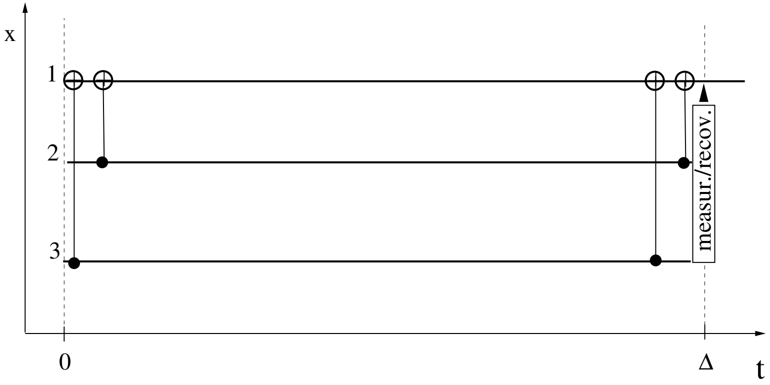
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}\left(\Delta,0\right)=\exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



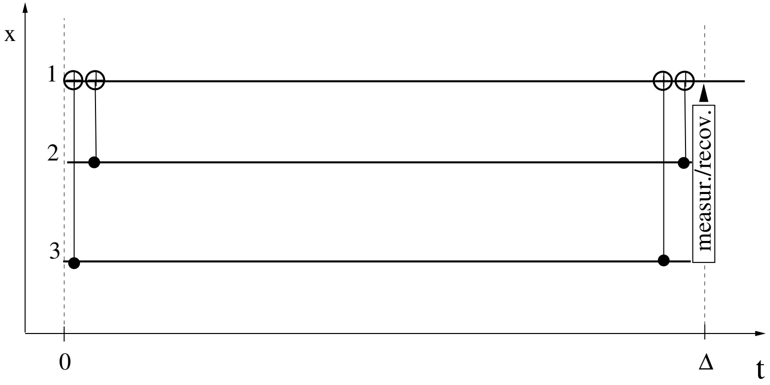
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



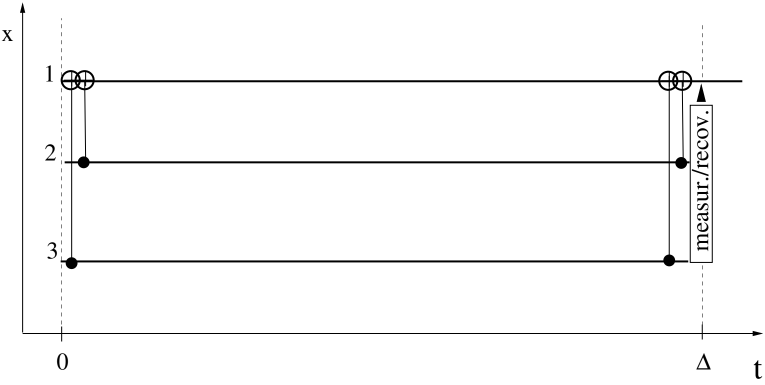
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



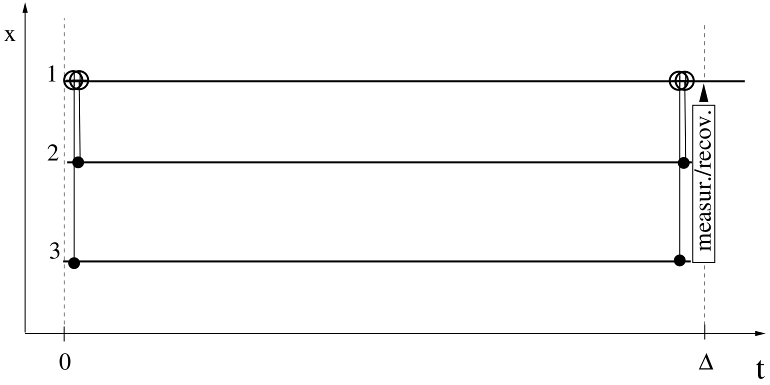
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



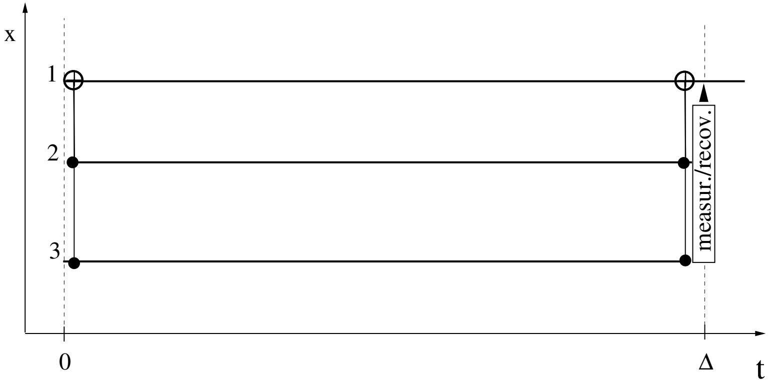
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



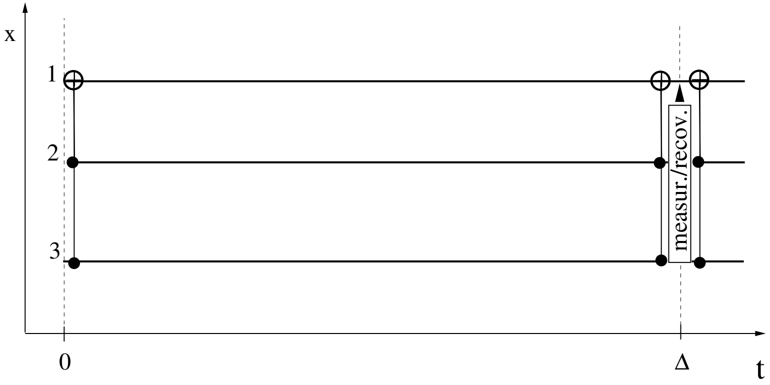
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



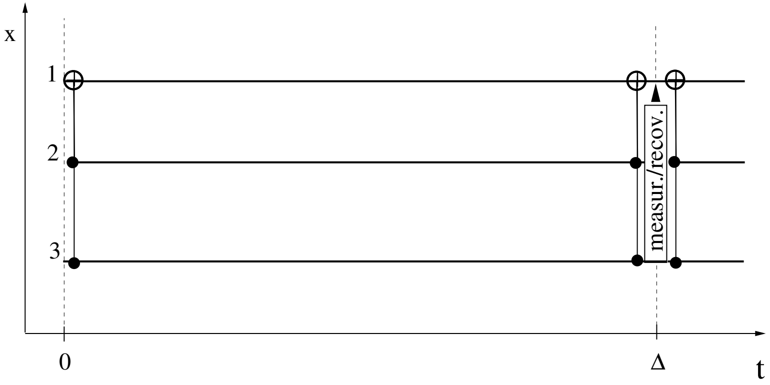
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0)=\exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



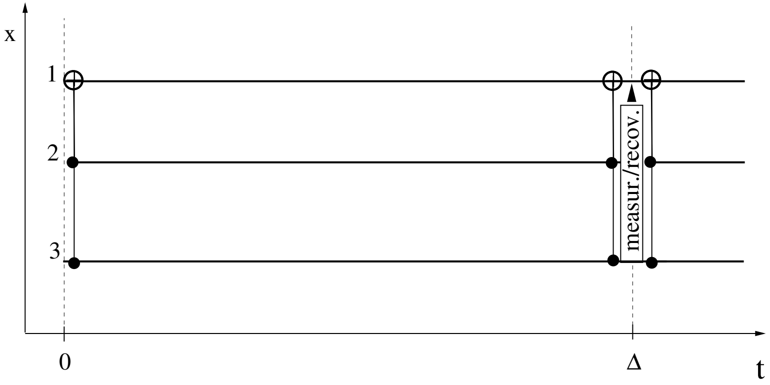
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



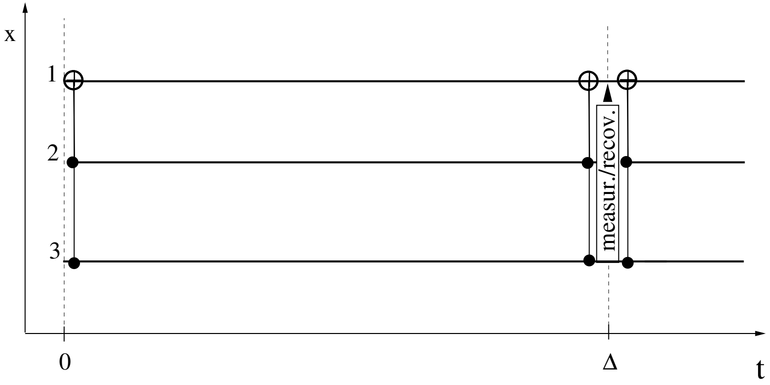
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



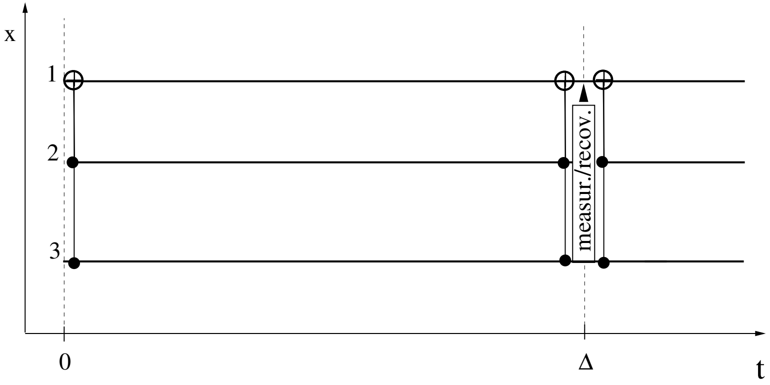
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



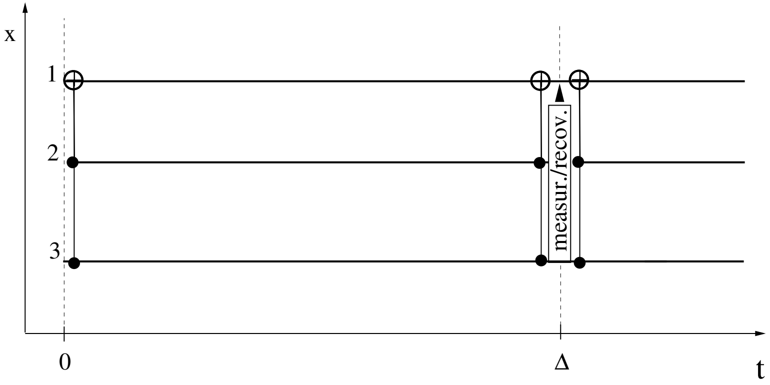
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



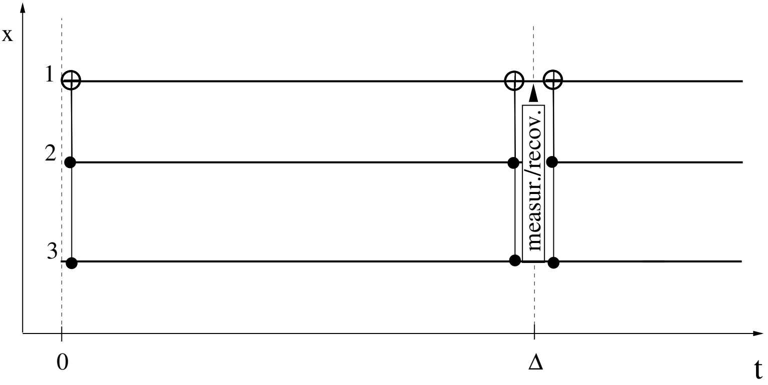
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



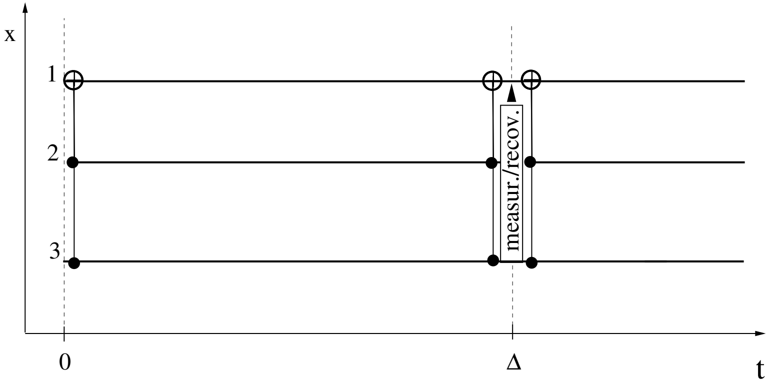
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



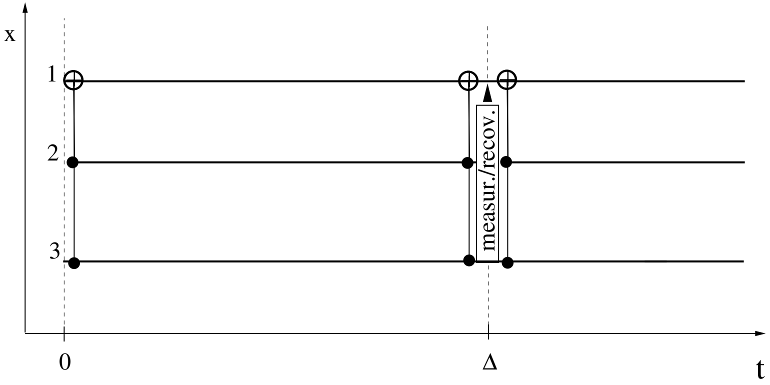
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



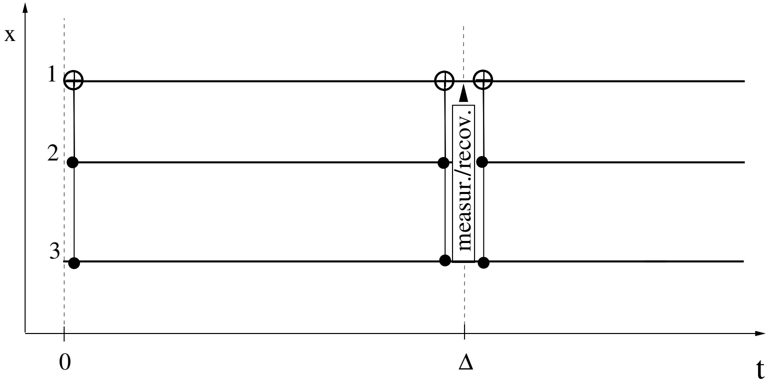
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



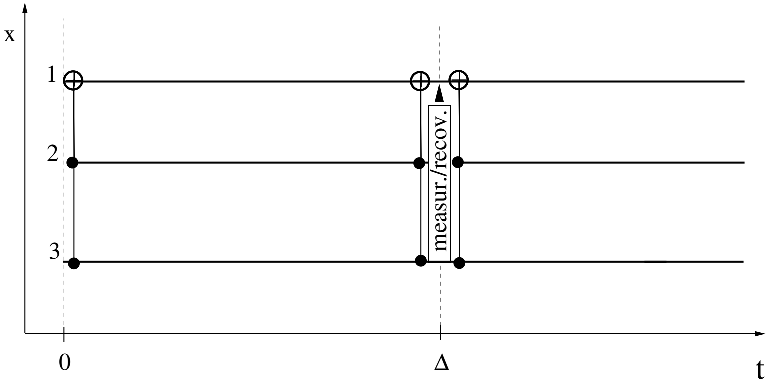
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



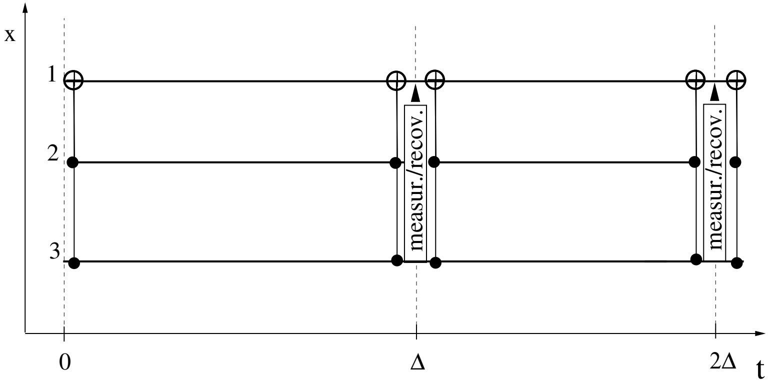
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



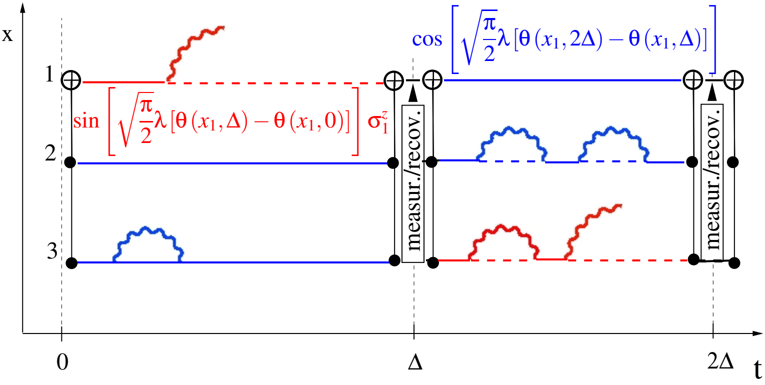
Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right) - \theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



Example: time evolution of encoded qubits

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



Example: time evolution of encoded qubits

- evolution with "no errors":

$$\upsilon_0(\Delta,0) = \eta_1\eta_2\eta_3 I - i\nu_1\nu_2\nu_3\bar{Z},$$

- evolution with "one error":

$$\begin{aligned}
\upsilon_1(\Delta,0) &= i\nu_1\eta_2\eta_3 I - \eta_1\nu_2\nu_3\bar{Z}, \\
\upsilon_2(\Delta,0) &= i\eta_1\nu_2\eta_3 I - \nu_1\eta_2\nu_3\bar{Z}, \quad \text{or} \\
\upsilon_3(\Delta,0) &= i\eta_1\eta_2\nu_3 I - \nu_1\nu_2\eta_3\bar{Z},
\end{aligned}$$

where:

$$\begin{aligned}
\eta_j &= \cos\left[\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\right], \\
\nu_j &= \sin\left[\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\right].
\end{aligned}$$

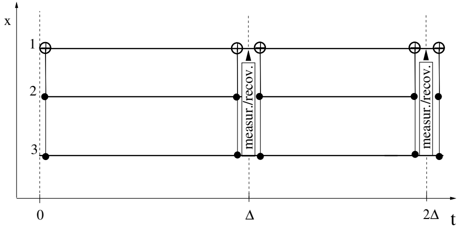
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



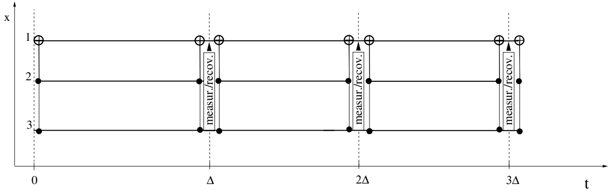
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$\left|\psi\left(N\Delta\right)\right\rangle = \ldots.\upsilon_j\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \left\langle\psi\left(N\Delta\right)\middle|\psi\left(N\Delta\right)\right\rangle \\ &= \left\langle\psi_0\right|\ldots.\upsilon_j^2\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k^2\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle\end{aligned}$$



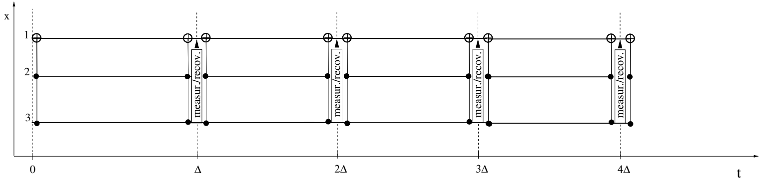
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



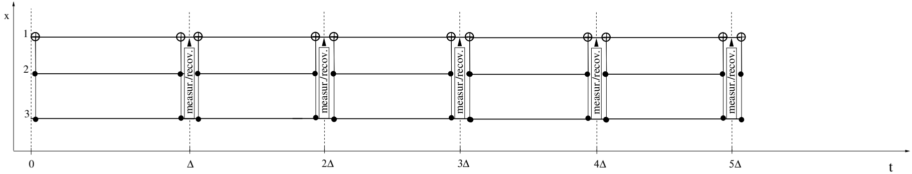
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



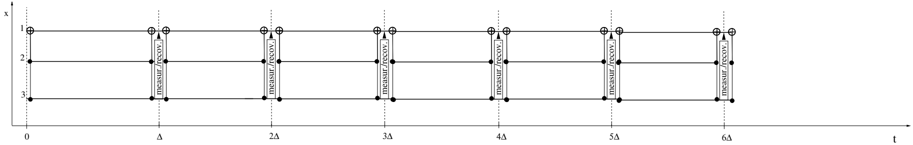
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



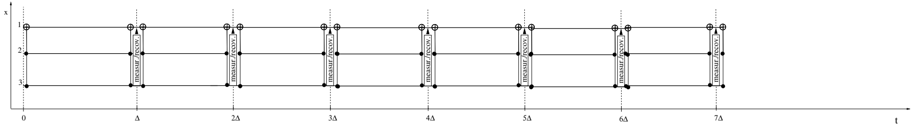
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



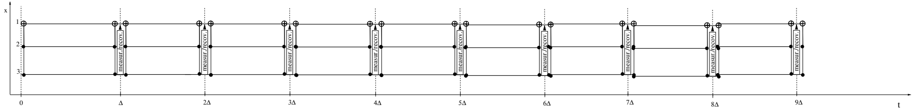
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle \;=\; \ldots.\upsilon_j\left((n+1)\Delta, n\Delta\right)\ldots.\upsilon_k\left((m+1)\Delta, m\Delta\right)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2\left((n+1)\Delta, n\Delta\right)\ldots.\upsilon_k^2\left((m+1)\Delta, m\Delta\right)\ldots|\psi_0\rangle\end{aligned}$$



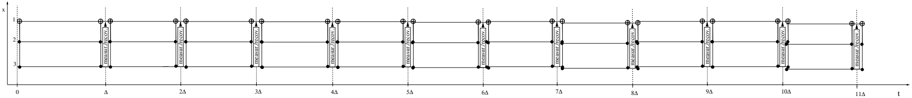
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$\left|\psi\left(N\Delta\right)\right\rangle = \ldots.\upsilon_j\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \left\langle\psi\left(N\Delta\right)\right|\psi\left(N\Delta\right)\rangle \\ &= \left\langle\psi_0\right|\ldots.\upsilon_j^2\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k^2\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle\end{aligned}$$



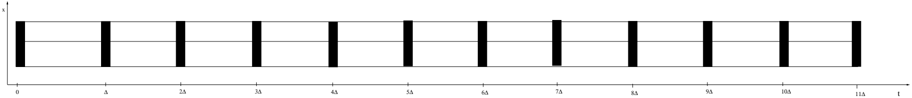
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = ...\upsilon_j((n+1)\Delta, n\Delta)....\upsilon_k((m+1)\Delta, m\Delta)...|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|....\upsilon_j^2((n+1)\Delta, n\Delta)....\upsilon_k^2((m+1)\Delta, m\Delta)...|\psi_0\rangle\end{aligned}$$



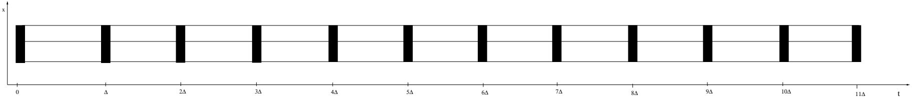
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = ...\upsilon_j((n+1)\Delta, n\Delta)....\upsilon_k((m+1)\Delta, m\Delta)...|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|....\upsilon_j^2((n+1)\Delta, n\Delta)....\upsilon_k^2((m+1)\Delta, m\Delta)...|\psi_0\rangle\end{aligned}$$



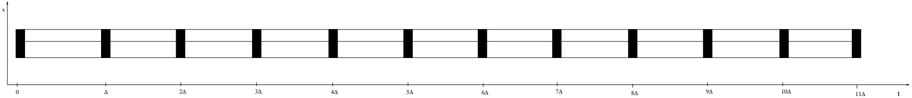
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$\left|\psi\left(N\Delta\right)\right\rangle = \ldots.\upsilon_j\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \left\langle\psi\left(N\Delta\right)|\psi\left(N\Delta\right)\right\rangle \\ &= \left\langle\psi_0\right|\ldots.\upsilon_j^2\left(\left(n+1\right)\Delta, n\Delta\right)\ldots.\upsilon_k^2\left(\left(m+1\right)\Delta, m\Delta\right)\ldots\left|\psi_0\right\rangle\end{aligned}$$



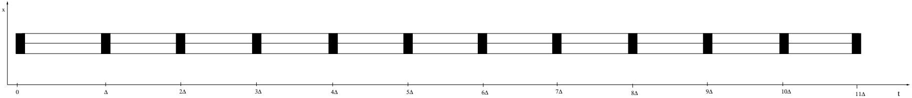
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



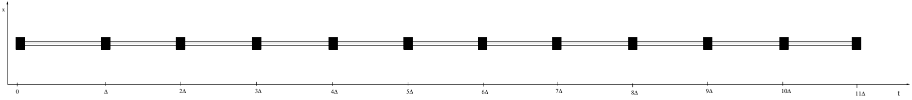
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



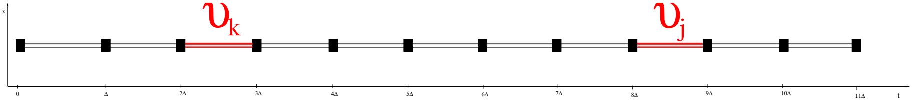
Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots.\upsilon_j((n+1)\Delta, n\Delta)\ldots.\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

- the probability of this history is

$$\begin{aligned}\mathcal{P} &= \langle\psi(N\Delta)|\psi(N\Delta)\rangle \\ &= \langle\psi_0|\ldots.\upsilon_j^2((n+1)\Delta, n\Delta)\ldots.\upsilon_k^2((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle\end{aligned}$$



Example: consequences for the probability of events

- by the end of a QEC cycle

$$\upsilon_0^2 \sim 1 - \frac{3\varepsilon}{2} - \sum_{j=1}^{3} \frac{\pi\lambda^2\Delta^2}{2} : [\partial_t \theta(j,0)]^2 : ,$$

$\boxed{\textit{uncorrelated probability}}$

$$\upsilon_{j=\{1,2,3\}}^2 \sim \frac{\varepsilon}{2} + \frac{\pi\lambda^2\Delta^2}{2} : [\partial_t \theta(j,0)]^2 : .$$

with $\varepsilon = \lambda^2 \ln\left[1 + (\Lambda v_b \Delta)^2\right]/2$.

Example: consequences for the probability of events

- by the end of a QEC cycle

$$\upsilon_0^2 \sim 1 - \frac{3\varepsilon}{2} - \sum_{j=1}^{3} \frac{\pi\lambda^2\Delta^2}{2} :\left[\partial_t \theta\left(j,0\right)\right]^2: ,$$

$\boxed{\textit{coarse-grained operators - correlated part}}$

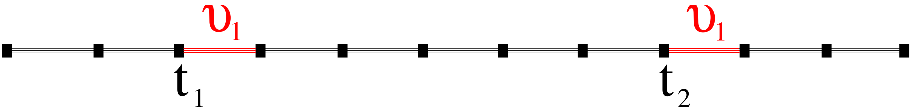
$$\upsilon_{j=\{1,2,3\}}^2 \sim \frac{\varepsilon}{2} + \frac{\pi\lambda^2\Delta^2}{2} :\left[\partial_t \theta\left(j,0\right)\right]^2: .$$

with $\varepsilon = \lambda^2 \ln\left[1 + \left(\Lambda v_b \Delta\right)^2\right]/2$.

Example: consequences for the probability of events

example: 1- consider that at times $t_1 < t_2$
2- the syndromes gave errors at the first qubit,
3- for simplicity assume the qubits very far apart.

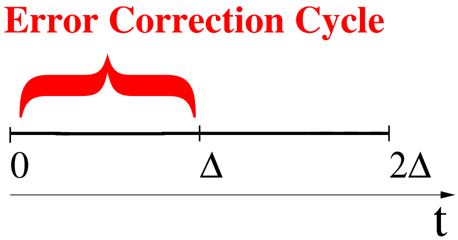
$$\begin{aligned}
\mathcal{P} &= \langle \psi(N\Delta) | \psi(N\Delta) \rangle \\
&= \langle \psi_0 | \ldots \upsilon_1^2 (t_2 + \Delta, t_2) \ldots \upsilon_1^2 (t_1 + \Delta, t_1) \ldots | \psi_0 \rangle \\
&\approx \left(\frac{\varepsilon}{2}\right)^2 + \frac{\pi^2 \lambda^4 \Delta^4}{4} \left\langle : [\partial_t \theta (x_1, t_2)]^2 :: [\partial_t \theta (x_1, t_1)]^2 : \right\rangle + O\left(\lambda^6\right) \\
&\approx \left(\frac{\varepsilon}{2}\right)^2 + \frac{\lambda^4 \Delta^4}{8 (t_1 - t_2)^4} + O\left(\lambda^6\right)
\end{aligned}$$



What did we learn about correlated environments?

E. Novais and Harold U. Baranger.
Phys. Rev. Lett. 97, 040501 (2006).

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.

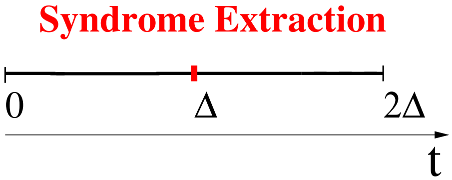


- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of "dynamical decoupling" of the logical qubit.

What did we learn about correlated environments?

E. Novais and Harold U. Baranger.
Phys. Rev. Lett. 97, 040501 (2006).

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.

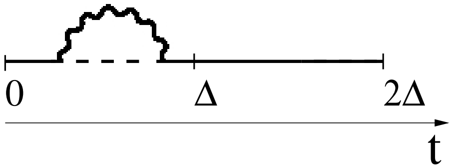


- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of "dynamical decoupling" of the logical qubit.

What did we learn about correlated environments?

E. Novais and Harold U. Baranger. Phys. Rev. Lett. 97, 040501 (2006).

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.

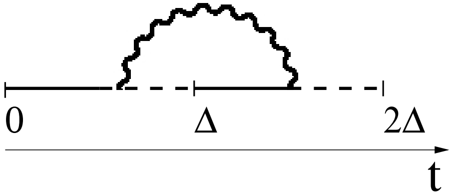


- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of “dynamical decoupling” of the logical qubit.

What did we learn about correlated environments?

E. Novais and Harold U. Baranger.
Phys. Rev. Lett. 97, 040501 (2006).

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.

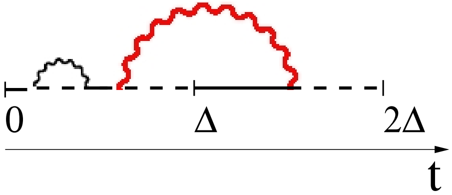


- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of "dynamical decoupling" of the logical qubit.

What did we learn about correlated environments?

E. Novais and Harold U. Baranger.
Phys. Rev. Lett. 97, 040501 (2006).

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.

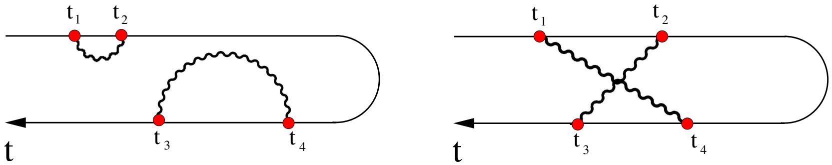


- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of “dynamical decoupling” of the logical qubit.

How does error correction help?

- for instance, in an ohmic environment the propagator for a qubit is $\sim \frac{1}{\left(t_i - t_j\right)^2}$
- typically this gives terms for the density matrix like

$$\int dt_1 \int dt_2 \int dt_3 \int dt_4 \frac{1}{\left(t_1 - t_2\right)^2 \left(t_3 - t_4\right)^2}$$

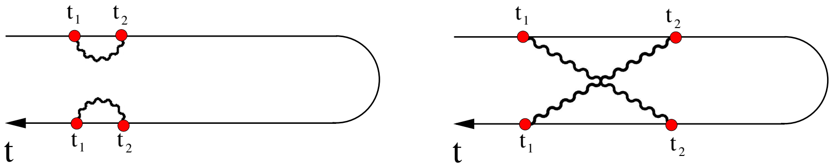


- that is why decoherence would grow as $\ln t$ for an ohmic bath.
- that is what happens inside a QEC cycle.

How does error correction help?

- however in a QEC evolution, for long times we "know" where errors occurred in the coarse-grain variables.

$$\int dt_1 \int dt_2 \frac{1}{(t_1 - t_2)^2 (t_1 - t_2)^2}$$



QEC steers a very peculiar evolution

decoherence for long times may be very different from the original microscopic Hamiltonian.

Part IV

The threshold theorem in a correlated environment

What is our approach?

The old trick of ...

reducing the problem to a known one.

Executive summary

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

What is our approach?

The old trick of ...

reducing the problem to a known one.

Executive summary

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

What is our approach?

The old trick of ...

reducing the problem to a known one.

Executive summary

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

What is our approach?

The old trick of ...

reducing the problem to a known one.

Executive summary

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

What is our approach?

The old trick of ...

reducing the problem to a known one.

Executive summary

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.

- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.

- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t\, e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t\, e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{2}$$

  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}(\Delta, \lambda_\alpha) = T_t\, e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. dynamical exponent, z.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha\left(\mathbf{x}\right) \sigma_\alpha\left(\mathbf{x}\right), \tag{2}$$

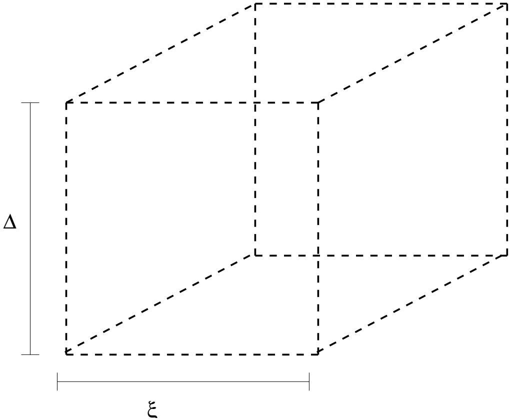
  where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.
- Hence, the evolution operator is

$$\hat{U}\left(\Delta,\lambda_\alpha\right) \quad = \quad T_t\, e^{-i\int_0^\Delta dt \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x},t)\sigma_\alpha(\mathbf{x})}, \tag{3}$$

Defining a coarse-grain grid

Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.



Defining a coarse-grain grid

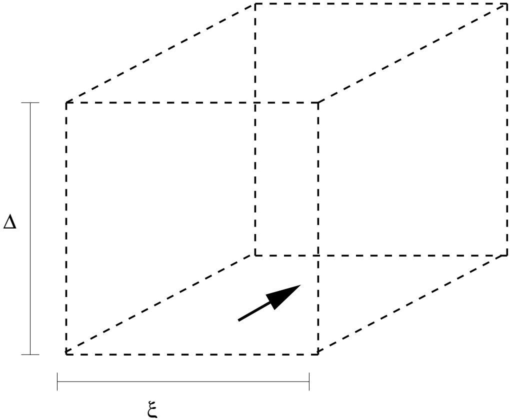
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

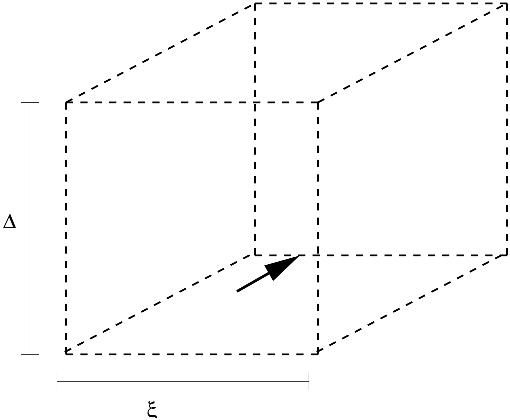
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

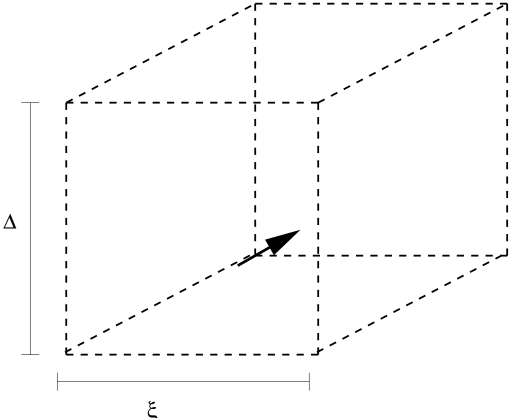
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

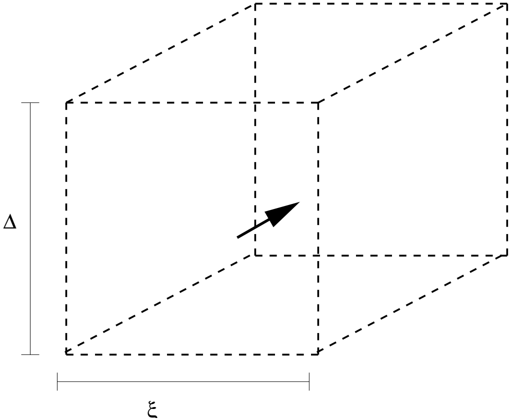
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

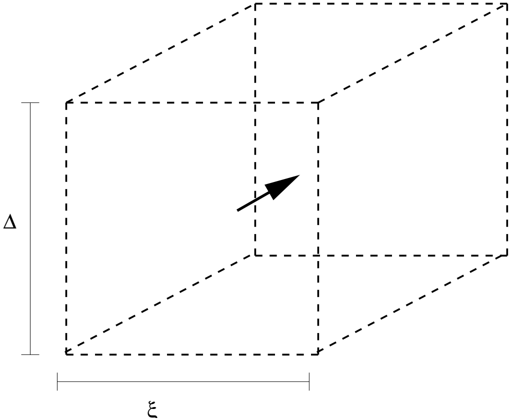
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

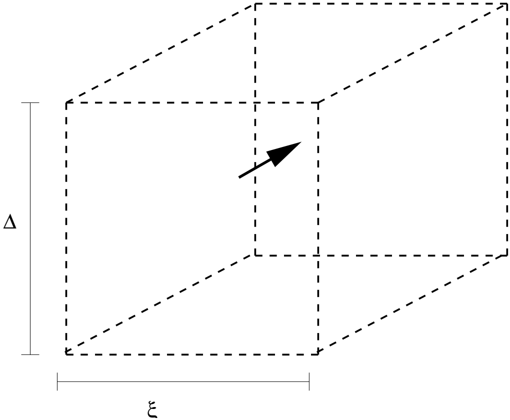
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

Hypothesis

There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



Defining a coarse-grain grid

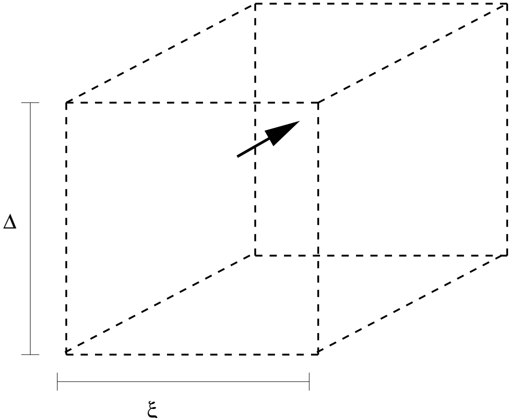
Hypercube

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.

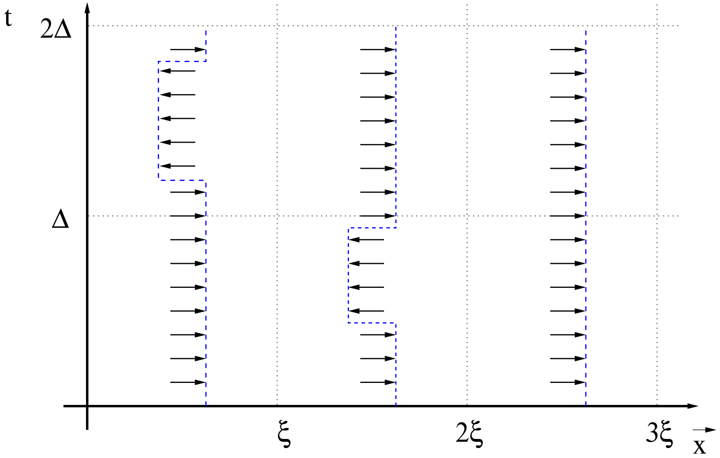
Hypothesis

There is only one qubit in each hypercube.

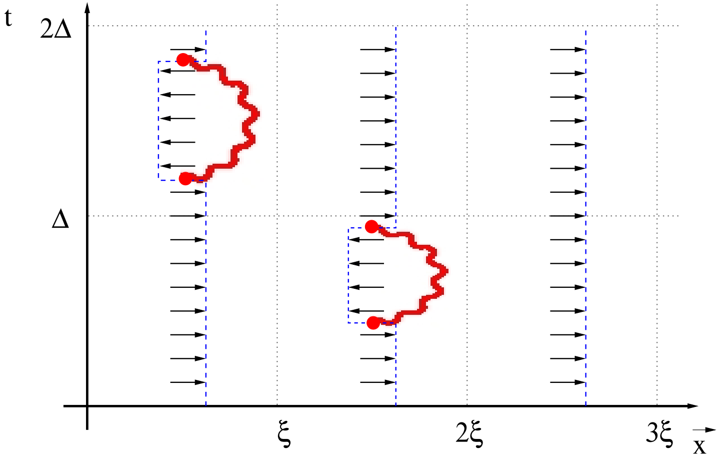
- allows to define the probability of an error in a qubit.
- for "short times" it is an impurity problem.



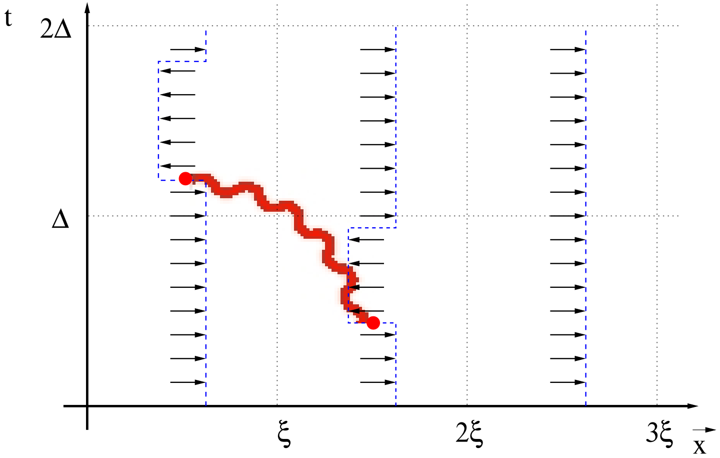
Defining a coarse-grain grid



Defining a coarse-grain grid



Defining a coarse-grain grid



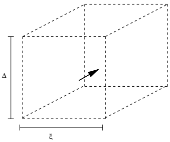
Qubit evolution in a QEC cycle

Lowest order perturbation theory for an error of type α.

$$\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha\right)\approx -i\lambda_\alpha\int_0^\Delta dt\, f_\alpha\left(\mathbf{x}_1,t\right),$$

Perturbation theory improved by RG

$$\begin{aligned}
\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha\right) &\approx -i\lambda_\alpha\int_0^\Delta dt\, f_\alpha\left(\mathbf{x}_1,t\right)\\
&- \frac{1}{2}\left|\varepsilon_{\alpha\beta\gamma}\right|\lambda_\beta\lambda_\gamma\boldsymbol{\sigma}_\alpha\left(\Delta\right)T_t\int_0^\Delta dt_1\,dt_2 f_\beta\left(\mathbf{x}_1,t_1\right)f_\gamma\left(\mathbf{x}_1,t_2\right)\boldsymbol{\sigma}_\beta\left(t_1\right)\boldsymbol{\sigma}_\gamma\left(t_2\right)\\
&+ \frac{i}{6}\sum_\beta\lambda_\alpha\lambda_\beta^2\boldsymbol{\sigma}_\alpha\left(\Delta\right)\\
&\times T_t\int_0^\Delta dt_1\,dt_2\,dt_3 f_\alpha\left(\mathbf{x}_1,t_1\right)f_\beta\left(\mathbf{x}_1,t_2\right)f_\beta\left(\mathbf{x}_1,t_3\right)\boldsymbol{\sigma}_\alpha\left(t_1\right)\boldsymbol{\sigma}_\beta\left(t_2\right)\boldsymbol{\sigma}_\beta\left(t_3\right),
\end{aligned}$$



Qubit evolution in a QEC cycle

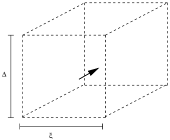
Lowest order perturbation theory for an error of type α.

$$\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha\right) \approx -i\lambda_\alpha \int_0^\Delta dt f_\alpha\left(\mathbf{x_1},t\right),$$

Perturbation theory improved by RG

$$\frac{d\lambda_\alpha}{d\ell} = g_{\beta\gamma}\left(\ell\right)\lambda_\beta\lambda_\gamma + \sum_\beta h_{\alpha\beta}\left(\ell\right)\lambda_\alpha\lambda_\beta^2,$$

integrate from the ultraviolet cut-off to Δ^{-1}, defines λ^*.



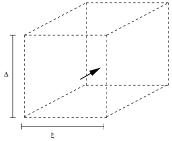
Qubit evolution in a QEC cycle

Lowest order perturbation theory for an error of type α.

$$\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha\right) \approx -i\lambda_\alpha \int_0^\Delta dt f_\alpha\left(\mathbf{x_1},t\right),$$

Perturbation theory improved by RG

$$\hat{\upsilon}_\alpha\left(\mathbf{x}_1,{\color{red}\lambda_\alpha^*}\right) \approx -i{\color{red}\lambda_\alpha^*} \int_0^\Delta dt f_\alpha\left(\mathbf{x_1},t\right),$$



Separating intra- and inter-hypercube components

Calculating a probability

$$P\left(\ldots;\alpha,\mathbf{x}_1;\ldots\right)\approx\left\langle\ldots\hat{\upsilon}_\alpha^\dagger\left(\mathbf{x}_1,\lambda_\alpha^*\right)\ldots\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha^*\right)\ldots\right\rangle.$$

$$\upsilon_\alpha^2\left(\mathbf{x}_1,\lambda_\alpha^*\right)\approx\varepsilon_\alpha+\left(\lambda_\alpha^*\Delta\right)^2:\left|f_\alpha\left(\mathbf{x_1},0\right)\right|^2:,$$

$$\varepsilon_\alpha=\left(\lambda_\alpha^*\right)^2\int_0^\Delta dt_1\int_0^\Delta dt_2\,\left\langle f_\alpha^\dagger\left(\mathbf{x_1},t_1\right)f_\alpha\left(\mathbf{x_1},t_2\right)\right\rangle$$

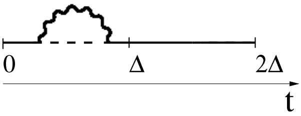
Separating intra- and inter-hypercube components

Calculating a probability

$$P(\ldots;\alpha,\mathbf{x}_1;\ldots) \approx \left\langle \ldots \hat{\upsilon}^{\dagger}_{\alpha}(\mathbf{x}_1,\lambda^*_{\alpha}) \ldots \hat{\upsilon}_{\alpha}(\mathbf{x}_1,\lambda^*_{\alpha}) \ldots \right\rangle .$$

$$\upsilon^2_{\alpha}(\mathbf{x}_1,\lambda^*_{\alpha}) \approx \varepsilon_{\alpha} + (\lambda^*_{\alpha}\Delta)^2 :\left|f_{\alpha}(\mathbf{x_1},0)\right|^2: ,$$

$$\varepsilon_{\alpha} = (\lambda^*_{\alpha})^2 \int_0^{\Delta} dt_1 \int_0^{\Delta} dt_2 \left\langle f^{\dagger}_{\alpha}(\mathbf{x_1},t_1) f_{\alpha}(\mathbf{x_1},t_2) \right\rangle$$



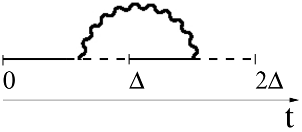
Separating intra- and inter-hypercube components

Calculating a probability

$$P\left(\ldots;\alpha,\mathbf{x}_1;\ldots\right)\approx\left\langle\ldots\hat{\upsilon}^{\dagger}_{\alpha}\left(\mathbf{x}_1,\lambda^{*}_{\alpha}\right)\ldots\hat{\upsilon}_{\alpha}\left(\mathbf{x}_1,\lambda^{*}_{\alpha}\right)\ldots\right\rangle.$$

$$\upsilon^{2}_{\alpha}\left(\mathbf{x}_1,\lambda^{*}_{\alpha}\right)\approx\varepsilon_{\alpha}+\left(\lambda^{*}_{\alpha}\Delta\right)^{2}:\left|f_{\alpha}\left(\mathbf{x}_1,0\right)\right|^{2}:,$$

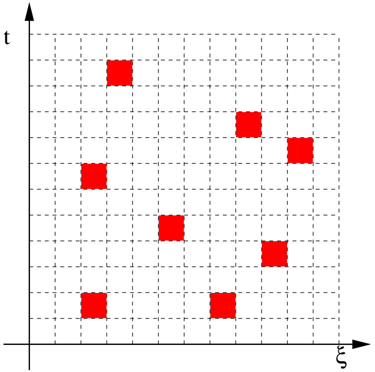
$$\varepsilon_{\alpha}=\left(\lambda^{*}_{\alpha}\right)^{2}\int_{0}^{\Delta}dt_1\int_{0}^{\Delta}dt_2\left\langle f^{\dagger}_{\alpha}\left(\mathbf{x}_1,t_1\right)f_{\alpha}\left(\mathbf{x}_1,t_2\right)\right\rangle$$



Probability of having m errors after N steps in R qubits

$$
\begin{aligned}
P_m^{\alpha} &= p_m \int \frac{d\mathbf{x}_1}{(v\Delta)^{D/z}} \cdots \frac{d\mathbf{x}_m}{(v\Delta)^{D/z}} \int_0^{N\Delta} \frac{dt_1}{\Delta} \cdots \int_0^{t_{m-1}} \frac{dt_m}{\Delta} \\
&\times \left\langle \left[\prod_{\zeta} F_0(\mathbf{x}_\zeta, t_\zeta) \right] \left[1 + F_\alpha(\mathbf{x_1}, t_1) \right] \ldots \left[1 + F_\alpha(\mathbf{x_m}, t_m) \right] \right\rangle
\end{aligned}
$$

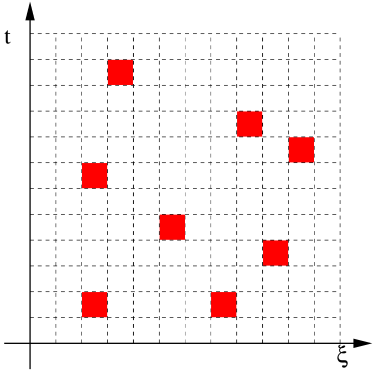
$$
\begin{aligned}
F_0(\mathbf{x_1}, 0) &= 1 - \frac{\sum_\beta \left(\lambda_\beta^* \Delta\right)^2 : |f_\beta(\mathbf{x_1}, 0)|^2 :}{1 - \sum_{\beta = x,y,z} \varepsilon_\beta}, \\
F_\alpha(\mathbf{x_1}, 0) &= \frac{(\lambda_\alpha^* \Delta)^2}{\varepsilon_\alpha} : |f_\alpha(\mathbf{x_1}, 0)|^2 : \; .
\end{aligned}
$$



"New" perturbation theory
in the coarse grain grid.

Probability of having m errors after N steps in R qubits

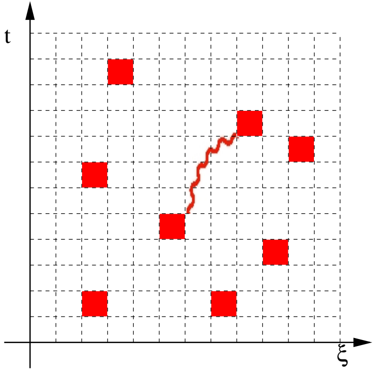
- zeroth order terms is just the stochastic probability:



$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} = p_m \binom{NR}{m}.$$

Probability of having m errors after N steps in R qubits

- second order term is the first correction due to long range correlations



$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} \left\langle F_\alpha\left(\mathbf{x_i}, t_i\right) F_\alpha\left(\mathbf{x_j}, t_j\right)\right\rangle .$$

Probability of having m errors after N steps in R qubits

correlation function

$$\left\langle F_\alpha\left(\mathbf{x_i},t_i\right)F_\alpha\left(\mathbf{x_j},t_j\right)\right\rangle \sim \mathcal{F}\left(\left|\mathbf{x}_i-\mathbf{x}_j\right|^{-4\delta_\alpha},\left|t_i-t_j\right|^{-4\delta_\alpha/z}\right)$$

where δ_α is the scaling dimension of f_α

using scaling again

$$\frac{d\lambda_\alpha^*}{d\ell}=\left(D+z-2\delta_\alpha\right)\lambda_\alpha^*.$$

defines the stability of the perturbation theory:

1. $D+z-2\delta_\alpha<0$ corrections are small,
2. $D+z-2\delta_\alpha>0$ new derivation needed.

Probability of having m errors after N steps in R qubits

correlation function

$$\left\langle F_\alpha\left(\mathbf{x_i},t_i\right)F_\alpha\left(\mathbf{x_j},t_j\right)\right\rangle \sim \mathcal{F}\left(\left|\mathbf{x}_i-\mathbf{x}_j\right|^{-4\delta_\alpha},\left|t_i-t_j\right|^{-4\delta_\alpha/z}\right)$$

where δ_α is the scaling dimension of f_α

using scaling again

$$\frac{d\lambda_\alpha^*}{d\ell}=\left(D+z-2\delta_\alpha\right)\lambda_\alpha^*.$$

defines the stability of the perturbation theory:

1. $D+z-2\delta_\alpha<0$ corrections are small,
2. $D+z-2\delta_\alpha>0$ new derivation needed.

Probability of having m errors after N steps in R qubits

correlation function

$$\left\langle F_\alpha\left(\mathbf{x_i},t_i\right)F_\alpha\left(\mathbf{x_j},t_j\right)\right\rangle \sim \mathcal{F}\left(\left|\mathbf{x}_i-\mathbf{x}_j\right|^{-4\delta_\alpha},\left|t_i-t_j\right|^{-4\delta_\alpha/z}\right)$$

where δ_α is the scaling dimension of f_α

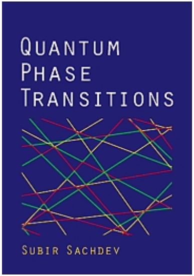
using scaling again

$$\frac{d\lambda_\alpha^*}{d\ell}=\left(D+z-2\delta_\alpha\right)\lambda_\alpha^*.$$

defines the stability of the perturbation theory:

1. $D+z-2\delta_\alpha<0$ corrections are small,
2. $D+z-2\delta_\alpha>0$ new derivation needed.

Threshold theorem as a quantum phase transition



Quantum Phase Transitions

We shall identify any point of non-analyticity in the ground state energy of the infinite lattice system as a quantum phase transition: the non-analyticity could be either the limiting case of an avoided level crossing, or an actual level crossing. ...

The phase transition is usually accompanied by a qualitative change in the nature of the correlations in the ground state, and describing this change shall clearly be one of our major interests. ...

It is important to notice that the discussion above refers to singularities in the ground state of the system. So strictly speaking, quantum phase transitions occur only at zero temperature, $T = 0$.

Threshold theorem as a quantum phase transition

high temperature - entangled phase

1. qubits and environment are strongly entangled,
2. the qubits are weakly entangle among themselves,
3. strong decoherence.

low temperature - entangled phase

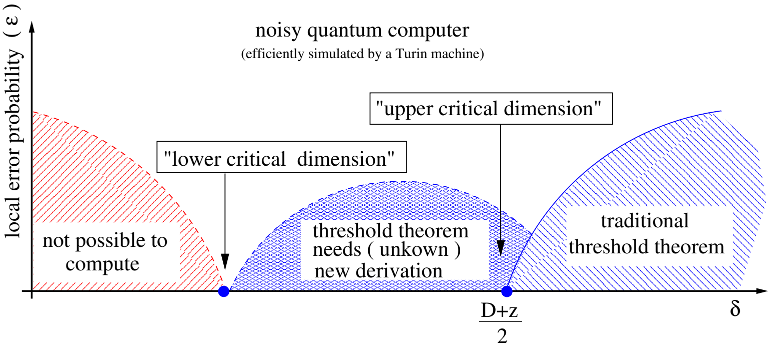
1. computer and qubits are strongly entangle,
2. strong decoherence.

low temperature - disentangled phase

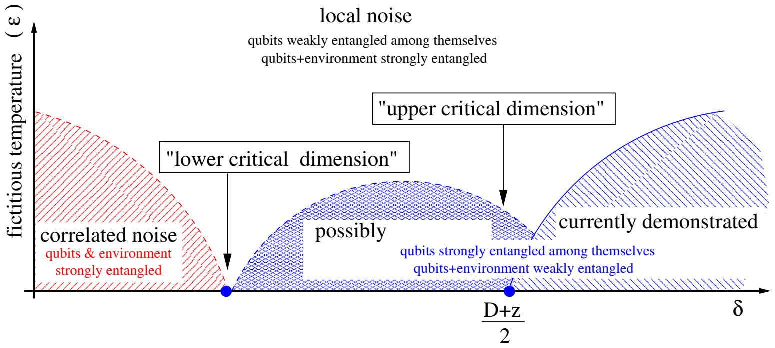
$$\left|\Psi_{total}\right\rangle = \left|\Psi_{computer}\right\rangle \otimes \left|\Psi_{environment}\right\rangle$$

1. qubits and environment are weakly entangled,
2. the qubits are strongly entangle among themselves,
3. low decoherence.

Schematic phase diagram



Schematic phase diagram



Conclusions

We...

1. ... studied decoherence in quantum computers in a correlated environment,
2. ... used an Hamiltonian description,
3. ... identify "software" methods to reduce the effects of correlations,
4. ... derive when is possible to use the "threshold theorem",
5. ... produce a parallel with the theorem of quantum phase transitions.

references

1. E. Novais and Harold U. Baranger.
   Decoherence by Correlated Noise and Quantum Error Correction
   Phys. Rev. Lett. 97, 040501 (2006).
2. E. Novais, Eduardo R. Mucciolo and Harold U. Baranger.
   Resilient Quantum Computation in Correlated Environments: A Quantum Phase Transition Perspective
   arXiv.org: quant-ph/0607155.