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# Hamiltonian Methods in QEC and Fault Tolerance

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**Abstract**

In this Chapter, we look at how memory effects induced by a correlated environment can alter the long-time dynamics of a quantum computer running a quantum error correction protocol. This is done through a Hamiltonian formulation which allows for a physical, microscopic modeling of the interaction between the computer and the environment. Assuming that single-qubit error probabilities are well defined, we use a perturbative expansion to find expressions for the probability of faulty paths in the evolution of the quantum computer. We obtain a dimensional criterion for the stability of the perturbation theory and the applicability of the error threshold theorem of fault-tolerant quantum computation. When this criterion is not satisfied, perturbation theory breaks down and no definitive statement can be made about the existence of an error threshold. We discuss the parallels between this situation and the theory of quantum phase transitions in condensed matter systems.

## 1.1 Introduction

The proof that efficient quantum error correction (QEC) codes exist, combined with the concept of error threshold [16, 20], brought confidence that reliable quantum computation is achievable in practice. However, it is fundamental to understand if there are *physical* limitations to resilient quantum computation within this framework. In this chapter, we discuss one of the few situations that still poses some doubts [7, 12, 18, 6, 9] about the effectiveness of QEC codes: Critical environments.

The term "critical environment" originates from condensed matter physics. It refers to physical systems where quantum correlations decay

as power laws. In this context, the Born-Markov approximation used to evaluate decoherence rates cannot be formally justified. For quantum computation, this fact translates into the appearance of errors that can depend on previous events in the computer history. The ultimate nightmare is that this memory effects may eventually lead to error probabilities above the threshold value and therefore to the breakdown of resilient quantum computation.

The first step in our quantitative study of critical environments is to formulate the dynamics of the computer and the environment with a Hamiltonian formulation. This allows for a systematic treatment of the interaction between the computer and the environment. Crucial elements such as space dimensionality, correlation function exponents, and coupling constants can be incorporated into calculations. The first step is to separate the total Hamiltonian into two distinct parts. The first part, $H_0$, we call "free". We will assume that this operator has a simple enough form as to allow for an explicit evaluation of the corresponding quantum evolution operator. The second part we define as the "interaction", $V$, which includes everything that preclude us from writing explicitly the complete evolution operator for the problem at hand. In other words, $H_0$ contains the ideal evolution of the computer and the dynamics of the environment while isolated from each other, while $V$ represents the coupling between the two.

Since $V$ destroys our ability to write down explicitly the quantum evolution of the entire system, we have to resort to an expansion of the evolution operator in powers of the "interaction". The result is usually called a Dyson series [22],

$$U\left(T,0\right)=T_t\, e^{-i\int_0^T dt'\, V\left(t'\right)}, \tag{1.1}$$

where $T_t$ denotes the time ordering operator, $T$ is the duration of the computation, and $V(t) = e^{iH_0t}Ve^{-iH_0t}$. In this series, every insertion of $V$ is a deviation of the computer evolution from the path that we envisaged in $H_0$. Of course, some of these deviations are harmless, since these "good" paths bear no effects on the result of the computation. Our problem is to evaluate the likelihood of "bad" paths [28]. A reasonable idea is to consider the evolution of the system with at least one insertion of $V$ [19, 28, 10, 4],

$$\begin{aligned}\mathcal{E}\left(T\right) &= U\left(T,0\right)-1\\ &= -i\int_0^T dt'\, V\left(t'\right)\, U\left(t',0\right).\end{aligned} \tag{1.2}$$

It has been shown that using the operator norm, one can derive an upper bound to the error probability of "bad" paths. The result is that

$$||\mathcal{E}(T)||_\infty \leq \int_0^T dt' \; ||V(t')||_\infty \leq \Lambda T, \tag{1.3}$$

where $\Lambda$ is the largest eigenvalue of $V$ [19, 28, 10].

Without quantum error correction, Eq. (1.3) does not help much. That is because the computational time $T$ must be regarded as an unbounded parameter. Therefore, after a time $\propto 1/\Lambda$, the computation would certainly fail. Nevertheless, the situation changes when QEC is added to the discussion. Error correction introduces another time scale into the problem, namely, the periodicity $\Delta$ in which error correction operations are repeatedly carried out. In this case, the relevant time for Eq. (1.3) is not the total computation time $T$, but rather $\Delta$. The remaining issue to be tackle is the dependence on $\Lambda$.

In many relevant physical situations, $\Lambda$ can be extremely large since it usually grows with the number of degrees of freedom of the environment [28]. An illustrative example is that of a single qubit, $\sigma$, interacting with $N$ two-level systems, $\{\tau^j\}$, through

$$V = \lambda \sum_{j=1}^{N} \sigma_z \tau_z^{(j)}. \tag{1.4}$$

This is a simplified version of the central spin problem [15] that has been studied in the context of decoherence [5, 13]. It is straightforward to see that $\Lambda = \lambda N$, thus diverging with the number of degrees of freedom of the spin bath.

Physical interactions are ultimately mediated by gauge fields. Hence, a very natural assumption for $V$ is to consider the minimum coupling model

$$V = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{1.5}$$

where $\mathbf{f}$ is some (vector) function of the environmental variables and $\sigma_{x,y,z}$ are Pauli matrices representing the qubit degrees of freedom. The bath are the gauge fields. In this case, it is clear that $\Lambda$ diverges with the number of modes in the field.

A way around was introduced by Aharonov, Kitaev, and Preskill [4]. In their discussion, they bypassed the divergence due to the gauge fields by integrating them out of the problem. In this case, an effective many-body interaction among the qubits of the computer is generated. They

simplified the problem by considering these extra interaction only up to second-order perturbation theory in $V$ and by assuming that the velocity of the gauge field modes is infinite. The net result was an instantaneous interaction between any two qubits which decayed spatially as a power law,

$$\frac{\lambda_\alpha \lambda_\beta}{4} \sum_{\mathbf{x},\mathbf{y}} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2\delta}} \sigma_\alpha(\mathbf{x}) \sigma_\beta(\mathbf{y}), \tag{1.6}$$

where $\delta$ is a pure number that depends on the specific form of $\mathbf{f}$. Their analysis then proved that if the volume integral

$$\int d^D y \, \frac{1}{|\mathbf{x}-\mathbf{y}|^{2\delta}} \tag{1.7}$$

converges, then the error probability can be bounded from above by the resulting number (here, $D$ denotes the spatial dimension).

This is a very elegant and general approach. However it precludes the possibility of self-interaction and retardation effects. In order to deal with these issues, we will follow a different path, namely we will study the stability of the Dyson series in $V$ [24, 26, 25]. For this purpose, we assume that the environment is described by a free-field theory, with the relevant two-point correlation function given by

$$\langle \Psi_{\text{env}} | f_\alpha(\mathbf{x}_1, t_1) \, f_\beta(\mathbf{x}_2, t_2) | \, \Psi_{\text{env}} \rangle \sim \mathcal{F}_{\alpha\beta} \left( \frac{1}{(\Delta x)^{2\delta}}, \frac{1}{(\Delta t)^{2\delta/z}} \right), \tag{1.8}$$

and that we can use Wick's theorem to calculate the higher-order correlation functions. The parameters $\delta$ and $z$ are usually called scaling dimension and dynamical exponent, respectively.

Another critical assumption we make is that the qubits are separated by a minimum distance in such a way that an entire error correction cycle can be performed before correlations between any neighboring qubits develop. We call this the "hypercube" assumption and adopted it in order to allow for a connection to the usual derivation of the threshold theorem. This assumption allows for the introduction of a well-defined error probability for a single qubit during a QEC cycle.

The basic strategy that we will follow is to use the Dyson series to find a reasonable way to calculate the probability of an error in a single qubit. We will then determine the conditions necessary to reduce the problem to a stochastic error problem.

### 1.2 Microscopic Hamiltonian models

There is a variety of physical realizations of qubits where the interaction between the environmental degrees of freedom and the qubits can be cast in the form of Eq. (1.5). For instance, the spin of an electron confined in a GaAs lateral quantum dot couples to nuclear spin through a hyperfine interaction, in which case the field $f_\alpha(\mathbf{x})$ represents a component of the local nuclear magnetization, also known as Overhauser field [11]. The minimum coupling model also appears in superconducting (Josephson) qubits, where the field $f_\alpha(\mathbf{x})$ accounts for the coupling to electromagnetic fluctuations. These fluctuations typically arise from Johnson-Nyquist noise in currents and voltages and can be described by a bath of harmonic oscillators, i.e., a bosonic field [23]. Another common situation where Eq. (1.5) applies is in qubits based on charge motion (e.g., double-dot charge qubits or impurities embedded in a semiconductor matrix) [29, 17]. In this case the field $f_\alpha(\mathbf{x})$ accounts for the coupling to acoustic phonons and can also be represented by bosonic degree of freedoms. Finally, if the qubit is a localized magnetic moment embedded in a conducting medium, Eq. (1.5) can be used to represent the coupling between the spin of itinerant electrons and the local moment (the so-called Kondo problem [1]).

For many qubit systems, particularly in solid-state, the most common environment is a bosonic one. A very ubiquitous interaction is

$$V = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \sigma_\alpha(\mathbf{x}) \sum_{\mathbf{q}} \lambda_{\alpha,\mathbf{q}}\, e^{i\mathbf{q}\cdot\mathbf{x}} \left( a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}} \right), \tag{1.9}$$

where the bosonic field $a_{\mathbf{q}}$ is usually assumed to have a free dynamics described by a quadratic Hamiltonian,

$$H_{\text{bath}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}}\, a^\dagger_{\mathbf{q}} a_{\mathbf{q}}. \tag{1.10}$$

(Generalizations where multiple bosonic baths couple to the qubits are straightforward.) Equations (1.9) and (1.10) define the so-called spin-boson model [21]. It has been intensively studied in the contexts of dissipative quantum mechanics and condensed matter physics (for a detailed discussion, see Ref. [30]). The spin-boson model is very representative of the kind of physical constraints faced by qubits implementations and should be regarded as a paradigmatic model.

Before proceeding to the discussion of QEC, we still need to highlight an important aspect of the Hamiltonian formulation: What is the form of $V$ in the interaction picture that enters into the Dyson series? As we

argue below, this issue is related to the duration of the quantum gates used in the computation.

The "free Hamitonian" is composed by two parts: The Hamiltonian that dictates the time evolution of the environment and the control Hamiltonian that implements quantum gates:

$$H_0\,(t) \quad = \quad H_{\rm bath} + H_{\rm QC}(t). \tag{1.11}$$

These terms act on different Hilbert spaces, hence $[H_{\rm bath}, H_{QC}] = 0$. In this case, it is straightforward to write an interaction picture which takes into account not only the environment but also the free evolution of the computer,

$$\begin{aligned} V(t) \quad &= \quad \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} \left[e^{iH_0 t} f_\alpha(\mathbf{x})\, e^{-iH_0 t}\right] W^\dagger(t)\, \sigma_\alpha\,(\mathbf{x}) W(t), \\ &= \quad \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x},t)\, G_\alpha(\mathbf{x},t), \end{aligned} \tag{1.12}$$

where $W(t,0) = T_t\, e^{-i\int_0^t dt' H_{QC}(t')}$. The operator $G_\alpha$ is a $SU(2^N)$ matrix that depends on the particular sequence of quantum gate which is being performed. In order to keep the discussion general, it is necessary to introduce some simplification. There are two possible paths:

(i) We can assume that the quantum gates are performed much faster that the environment's response time. In this limit, we can prove that the gate operations implicit in Eq. (1.12) lead to the same microscopic form of Eq. (1.5), namely

$$V(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x},t)\, \sigma_\alpha(\mathbf{x}). \tag{1.13}$$

(ii) The other possibility is to derive an effective Hamiltonian that provides an upper estimate to the effect of errors. The point here is to realize that the information encoded in the qubits is exposed to different components of the environment depending on the particular gate being performed. For instance, single-qubit gates can be written as

$$G_{\alpha,1}(\mathbf{x},t) = \sum_{\beta=\{x,y,z\}} g_{\alpha\beta}(\mathbf{x},t)\, \sigma_\beta(\mathbf{x}), \tag{1.14}$$

where $g_{\alpha\beta}(\mathbf{x},t)$ are ordinary functions. Consequently, the "interaction" Hamiltonian can be written as

$$V_1(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \sum_{\beta=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x},t)\, g_{\alpha\beta}(\mathbf{x},t)\, \sigma_\beta(\mathbf{x}), \tag{1.15}$$

which tell us that all components of the qubits mix with all components of the environment. In order to define a suitable upper bound estimate, we can make all functions $g_{\alpha\beta}(\mathbf{x},t)$ constant and set them to unity. This obviously breaks the unitarity of the gates but has the virtue of simplicity. More accurate upper bounds could in principle be obtained by making use of the functional form of $g_{\alpha\beta}(\mathbf{x},t)$, but this would also make any calculation considerably more difficult.

A similar argument can also be made for two-qubit gates. In this case, a two-qubit rotation can be written as

$$\begin{aligned} W\left(\mathbf{x}_1,\mathbf{x}_2,t\right) &= \cos\left[\theta(\mathbf{x}_1,\mathbf{x}_2,t)\right] + i\sin\left[\theta(\mathbf{x}_1,\mathbf{x}_2,t)\right] \\ &\quad \times \sigma_a(\mathbf{x}_1)\,\sigma_b(\mathbf{x}_2), \end{aligned} \tag{1.16}$$

where $\theta(\mathbf{x}_1,\mathbf{x}_2,t)$ is an ordinary function, $\mathbf{x}_1$ and $\mathbf{x}_2$ tag the position of the qubits involved in the gate, and $a$ and $b$ denote two arbitrary qubit components. This implies that

$$\begin{aligned} G_{\alpha,2}(\mathbf{x}_1,t) &= \sin\left[2\theta(\mathbf{x}_!,\mathbf{x}_2,t)\right]\epsilon_{a\alpha\gamma}\sigma_\gamma(\mathbf{x}_1)\,\sigma_b(\mathbf{x}_2) \\ &\quad + \cos\left[2\theta(\mathbf{x}_1,\mathbf{x}_2,t)\right](1-\delta_{a,\alpha})\;\sigma_\alpha(\mathbf{x}_1) \\ &\quad + \delta_{a,\alpha}\,\sigma_\alpha(\mathbf{x}_1), \end{aligned} \tag{1.17}$$

where $\epsilon_{\alpha\beta\gamma}$ is the usual anti-symmetric tensor. Once again, we can define a suitable upper bound by setting all the functions in the prefactors to unity, yielding

$$G_{\alpha,2}(\mathbf{x},t) = \epsilon_{a\alpha\gamma}\sigma_\gamma(\mathbf{x})\,\sigma_b(\mathbf{y}) + \sigma_\alpha(\mathbf{x}). \tag{1.18}$$

The corresponding "interaction" Hamiltonian is then

$$V_2(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x},t)\,G_{\alpha,2}(\mathbf{x},t). \tag{1.19}$$

The second term on the r.h.s. of Eq. (1.18) is just a local noise. The first term is more worrisome since it leads to errors propagating between target and control qubits. Obviously it is not unique to have the error occurring during the two-qubit gate. In fact, we can re-interpret Eq. (1.19) as an error on the qubit $\mathbf{x}$ before the gate is performed. This error is then propagated by a perfect two-qubit gate. Propagation of errors is in general unavoidable in a quantum circuit. Nevertheless, we can assume that it can be handle by fault tolerant procedures. As a result, we conclude that an upper bound estimate to the action of gates in the

microscopic model is given by the interaction Hamiltonian

$$V_{\rm eff}(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda}{2} f_{\rm eff}(\mathbf{x},t)\, \sigma_\alpha(\mathbf{x},t), \tag{1.20}$$

where

$$f_{\rm eff}\left(\mathbf{x},t\right) = \frac{1}{\lambda^2}\left[\sum_{\beta=\{x,y,z\}} \lambda_\beta f_\beta\left(\mathbf{x},t\right)\right] \tag{1.21}$$

and $\lambda = \sqrt{\sum_{\beta=\{x,y,z\}} \lambda_\beta^2}$ is the new coupling parameter.

Because Eqs. (1.13) and (1.20) have the same functional form, we hereafter drop the subscript "eff".

### 1.3 Time evolution with quantum error correction

In this chapter we will mostly consider the case where the initial state of the computer, $\psi_0$, and the environment, $\varphi_0$, are pure states. However, the generalization of our discussion to the case of mixed initial states is straightforward.

A particularly simple case occurs when the environment is at a finite temperature, T. In many situations, this simply introduces a thermal coherence length $\xi_{\rm T}$ for the bath modes. For distances smaller than $\xi_{\rm T}$, the correlation functions have a power-law behavior that is potentially troublesome. However, for distances larger than $\xi_{\rm T}$, the correlation functions will decay exponentially. This raises the obvious question: Why don't we simply operate the computer at a finite temperature and use the results from Aharonov and Ben-Or [3]? The answer is that the same mechanism that is setting the temperature T for the environment is also affecting the qubits. Thus, this is likely to introduce an exponentially short coherence time for the qubits as well. Furthermore, since one of the basic assumptions of quantum computation is the ability to prepare the initial state of the computer, it is reasonable to consider that the computer and the environment are not entangled at beginning of the calculation. Hence, we assume that the initial state vector of the system is

$$\left|\Psi\left(t=0\right)\right\rangle = \left|\psi_0\right\rangle \otimes \left|\varphi_0\right\rangle. \tag{1.22}$$

The state $\Psi$ will evolve according to the unitary operator $U(\Delta,0)$. At a time $\Delta$, the syndrome is extracted and the computer wave function is projected,

$$P_{\rm m}\, U\left(\Delta,0\right)\left|\Psi\left(0\right)\right\rangle,$$

where m corresponds to a particular syndrome, with $\sum_{\mathrm{m}} P_{\mathrm{m}} = I$ and $P_{\mathrm{m}}^2 = P_{\mathrm{m}}$. In the case of many logical qubits evolving together, the symbol m denotes the set of all the syndromes extracted at time $\Delta$. Finally, as required by QEC, an appropriate recovery operation $R_{\mathrm{m}}$ is performed:

$$|\Psi(\Delta)\rangle = R_{\mathrm{m}}(\Delta + \delta_r, \Delta)\, P_{\mathrm{m}}\, U(\Delta, 0)\, |\Psi(0)\rangle\,, \tag{1.23}$$

where $\delta_r$ denotes the duration of the recovery operation.

It is well known that quantum error correction could also be performed without the measurement of the syndrome. However, in that case a fresh supply of cold ancillas must be made available at each QEC step. Thus, we must consider two possible scenarios: i) If the ancillas are only briefly in contact with the computer and bath, then to our purposes this procedure is completely equivalent to the use a measurement; ii) if, however, the ancillas cannot be separated from the computer and the bath, then we must also follow their dynamics. Although the inclusion of the ancillas would not change our conclusion, it would introduce some non-essential elements to the discussion (such as where and how they are stored). Therefore, we will limit ourselves to the more usual prescription of QEC using syndrome extraction.

The generalization of Eq. (1.23) to a sequence of QEC cycles is straightforward [24],

$$\Upsilon_{\mathbf{w}} = \upsilon_{w_N}\big(N\Delta, (N-1)\Delta\big)...\upsilon_{w_1}(\Delta, 0), \tag{1.24}$$

where $\mathbf{w}$ is the particular history of syndromes for all the qubits and

$$\upsilon_{w_j}\big(j\Delta, (j-1)\Delta\big) = R_{w_j}\big(j(\Delta+\delta_r), j\Delta\big) P_{w_j} U\big(j\Delta, (j-1)\Delta\big) \tag{1.25}$$

is the quantum error correction evolution after each cycle.

There are two useful quantities that we can now calculate. The first one is the probability to have a particular history of syndromes,

$$\Pr(\Upsilon_{\mathbf{w}}) = \langle\varphi_0|\langle\psi_0|\,\Upsilon_{\mathbf{w}}^{\dagger}\Upsilon_{\mathbf{w}}\,|\psi_0\rangle\,|\varphi_0\rangle\,. \tag{1.26}$$

The second quantity is the residual decoherence, which can be read from the reduced density matrix

$$\rho_{\vec{r},\vec{s}}(\Upsilon_{\mathbf{w}}) = \frac{\langle\varphi_0|\left[\langle\psi_0|\,\Upsilon_{\mathbf{w}}^{\dagger}\,|\vec{s}\rangle\,\langle\vec{r}|\,\Upsilon_{\mathbf{w}}\,|\psi_0\rangle\right]|\varphi_0\rangle}{\langle\varphi_0|\,\langle\psi_0|\,\Upsilon_{\mathbf{w}}^{\dagger}\Upsilon_{\mathbf{w}}\,|\psi_0\rangle\,|\varphi_0\rangle}, \tag{1.27}$$

with $\vec{r}$ and $\vec{s}$ being elements of the logical subspace.

The presence of the "interacting" Hamiltonian $V$ precludes us from explicitly writing the exact quantum evolution. Therefore, the best that

we can do with Eqs. (1.26) and (1.27) is to write them as a double series in $V$, namely, one for each $\Upsilon_{\mathbf{w}}$ operator. These are usually represented graphically as a double contour in time [see Fig. (1.1)]. The upper leg stands for the time ordered series ($\Upsilon_{\mathbf{w}}$) , while the lower leg stands for the anti-time-ordered ($\Upsilon^{\dagger}_{\mathbf{w}}$). In the out-of-equilibrium literature, this sort of diagram is sometimes referred as Keldysh's contour. There are six (non independent) Green functions in such representation: The usual advanced and retarded functions for the time-ordered series; the advanced and retarded functions for the anti-time-ordered; and the $<$ and $>$ functions, corresponding to contracting a term from the time-order series with a term from the anti-time-ordered one. There are very good reviews on the Keldysh formalism, but unfortunately the diagrammatic rules can be very cumbersome. Hence, before we start a general discussion, it is instructive to consider a simple case. Also, for clarity, we will focus our discussion on the simplest quantity to calculate, $\Pr(\Upsilon_{\mathbf{w}})$.

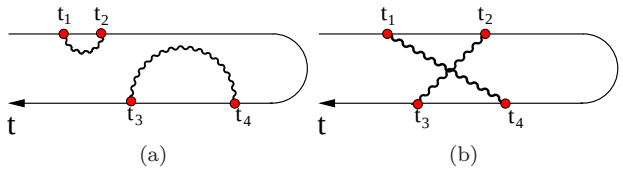



Fig. 1.1. Graphical representation of two fourth-order terms in a "time-loop" expansion for either the probability of a given evolution or the reduced density matrix (spatial dimensions are suppressed for clarity). Points of interaction with the bath (circles) are connected by propagation of the environmental modes (wiggly lines).

### *1.3.1 Qualitative discussion*

In this section we develop some intuitive understanding of how QEC works in a critical environment. Hence, we assume (for the moment) two simplifications: i) For quantum error correction periods where an error was diagnosed, we expand the evolution to lowest order in $U$; ii) for QEC periods where a "non-error" was diagnosed, we approximate $U \approx I$. Neither of these assumptions is rigorously valid in general, but they strip the discussion of many unimportant details.

The simplest case to discuss is a single error and a single quantum error correction step. For example, let us consider that the syndrome $m_1$ tell us that a $Z$ error occurred at qubit 1 in the first quantum error

correction cycle. Thus, using Eqs. (1.25) and (1.26), it is straightforward to write the probability of this history,

$$\Pr\left(\upsilon_{m_1}(\Delta,0)\right) = \left(\frac{\lambda_z}{2}\right)^2 \int_0^\Delta dt_2 \int_0^\Delta dt_1 \, \left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z(\mathbf{x_1},t_1)\right\rangle + \mathcal{O}(\lambda_z^4). \tag{1.28}$$

If the theory is properly regularized in the ultraviolet, Eq. (1.28) is a well-defined object, $\mathcal{P}\left(\upsilon_{m_1}(\Delta,0)\right) = \epsilon$.

The next case is to consider two $Z$ errors at different periods. For instance, let us assume the history points to an error in the first period at qubit 1 and another error in period $j+1$ at qubit 3. The quantum evolution now reads

$$\upsilon_{m_3}\upsilon_{m_1} = \left(\frac{\lambda_z}{2}\right)^2 \int_{j\Delta}^{(j+1)\Delta} dt_3 \int_0^\Delta dt_1 \, f_z(\mathbf{x_3},t_3) f_z(\mathbf{x_1},t_1) + \mathcal{O}(\lambda_z^4), \tag{1.29}$$

which implies a probability

$$\begin{aligned} \Pr\left(\upsilon_{m_3}\upsilon_{m_1}\right) \;=\; & \left(\frac{\lambda_z}{2}\right)^4 \int_{j\Delta}^{(j+1)\Delta} dt_4\, dt_3 \int_0^\Delta dt_1\, dt_2 \\ & \times \left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z^\dagger(\mathbf{x_3},t_4) f_z(\mathbf{x_3},t_3) f_z(\mathbf{x_1},t_1)\right\rangle \\ & + \mathcal{O}(\lambda_z^6). \end{aligned} \tag{1.30}$$

Using Wick's theorem, we can write the four-point correlation function in terms of products of two-point correlation functions, Eq. (1.8). Hence, Eq. (1.30) can be simplified to

$$\left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z^\dagger(\mathbf{x_3},t_4) f_z(\mathbf{x_3},t_3) f_z(\mathbf{x_1},t_1)\right\rangle$$

$$\begin{aligned} &= \overbrace{\left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z^\dagger(\underbrace{\mathbf{x_3},t_4}) f_z(\underbrace{\mathbf{x_3},t_3}) f_z(\mathbf{x_1},t_1)\right\rangle} \\ &+ \overbrace{\left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z^\dagger(\mathbf{x_3},t_4) f_z(\mathbf{x_3},t_3) f_z(\underbrace{\mathbf{x_1},t_1})\right\rangle} \\ &+ \overbrace{\left\langle f_z^\dagger(\mathbf{x_1},t_2) f_z^\dagger(\mathbf{x_3},t_4) f_z(\mathbf{x_3},t_3) f_z(\mathbf{x_1},t_1)\right\rangle} \end{aligned} \tag{1.31}$$

where the bars indicate the correspondent two-point functions. The first term in this equation is the simplest to understand. The domains of integration are disjoint and we simply obtain a result proportional to $\epsilon^2$,

i.e., the probability of having two "uncorrelated" errors. The other terms are corrections to $\epsilon^2$ due to "correlations" between errors. It is expected that these terms should produce small corrections to the "uncorrelated" value. Hence, the strategy that we shall follow is to derive a perturbative expansion for these corrections.

If errors occur far away in space-time, the precise position of each one is not very relevant to the calculation. Hence, we can coarse grain the space-time to a volume $\Delta \times (\mathtt{v}\Delta)^{D/z}$. Probably the simplest method to perform this step is to use the operator product expansion on $\upsilon_{m_3}$ and $\upsilon_{m_1}$,

$$f_z(\mathbf{x},t) \quad \sim \quad f_z(\mathbf{x},0) + \partial_t f_z(\mathbf{x},0)\, t + \text{l.r.t.}$$

where $\sim$ stands for equal up to non-singular terms. Thus, the *leading* corrections to the "uncorrelated" probability are

$$\begin{aligned}
\Pr\left(\upsilon_{m_3}\upsilon_{m_1}\right) \quad &\approx \quad \epsilon^2 \\
&+ \quad \left(\frac{\lambda_z}{2}\right)^4 \overbrace{\left\langle f_z^\dagger(\mathbf{x_1},0) f_z^\dagger(\underbrace{\mathbf{x_3},j\Delta) f_z(\mathbf{x_3},j\Delta}) f_z(\mathbf{x_1},0)\right\rangle}\, \Delta^2 \\
&+ \quad \left(\frac{\lambda_z}{2}\right)^4 \overbrace{\left\langle f_z^\dagger(\mathbf{x_1},0) f_z^\dagger(\underbrace{\mathbf{x_3},j\Delta) f_z(\mathbf{x_3},j\Delta}) f_z(\mathbf{x_1},0)\right\rangle}\, \Delta^2
\end{aligned}$$

Of course there are many more terms to be calculated, but these leading terms already reveal a very important pattern. Since we known from the syndrome that a particular event (error or no-error) has happened, both branches of the Keldysh contour (the time-ordered and anti-time ordered series) must have insertions of $V$ occurring at the same coarse-grained times (see Fig. 1.2). This reduces the number of time integrals to be taken from four to two. Thus, the effective scaling dimension for the infrared component of the probability is doubled in comparison to the naive expectation. Most of our discussion will now turn to formalize this result and to properly calculate $\epsilon$.

### *1.3.2 Quantitative discussion*

The key step in the discussion of the previous section was the coarse graining of space-time. But this crucial step introduces a conceptual problem. If two qubits are separated by a distance smaller than $(\mathtt{v}\Delta)^{1/z}$, then we cannot define neither a unique probability for an event nor

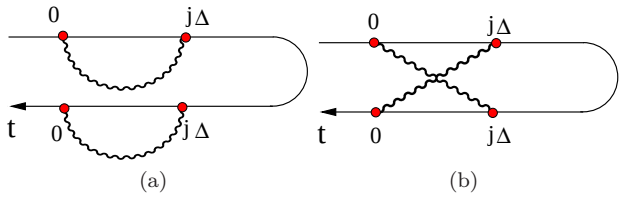



Fig. 1.2. Graphical representation of two fourth-order terms in the coarse-grained scale.

a unique "long-range" operator, such as the one we wrote using the operator product expansion. For instance, imagine that two errors were diagnosed in two physical qubit belonging to different logical qubits. If the two errors are inside the fundamental volume $\Delta \times (\mathrm{v}\Delta)^{D/z}$, then the probability of this two events, to lowest order in $V$, is

$$\begin{aligned} \Pr(\upsilon_{\mathbf{m}_1}) &= \epsilon^2 + \left(\frac{\lambda_\alpha}{2}\right)^4 \int_0^\Delta dt_1\, dt_2\, dt_3\, dt_4 \\ &\quad \big\langle f_\alpha^\dagger(\mathbf{x_1}, t_4) f_\alpha^\dagger(\mathbf{x_2}, t_2) f_\alpha(\mathbf{x_2}, t_3) f_\alpha(\mathbf{x_1}, t_1) \big\rangle \\ &\quad + \big\langle f_\alpha^\dagger(\mathbf{x_1}, t_4) f_\alpha^\dagger(\mathbf{x_2}, t_2) f_z(\mathbf{x_2}, t_3) f_\alpha(\mathbf{x_1}, t_1) \big\rangle \end{aligned} \tag{1.32}$$

The last two terms are corrections to the probability $\epsilon$ of an error in a qubit that is conditional to the events in the other qubits inside the fundamental volume. Similarly, if we try to coarse grain space-time, then we would find a different operator for each possible set of events in $\Delta \times (\mathrm{v}\Delta)^{D/z}$.

Since we do not want to deal with such conditional probabilities, we assume the *single most important simplifying hypothesis* of our discussion: We assume *hereafter* that the qubits are separated by a minimum distance

$$\xi = (\mathrm{v}\Delta)^{1/z}, \tag{1.33}$$

where $\mathrm{v}$ is the velocity of the bath modes and $z$ is the dynamical exponent of the theory describing the environment. This implies that for all qubits separated by this minimum distance, $\mathbf{x_1} \neq \mathbf{x_2}$ and $|t_1 - t_1| < \Delta$, we have $\langle f_\alpha(\mathbf{x_2}, t_2) f_\alpha(\mathbf{x_1}, t_1)\rangle \approx 0$. Thus, if we impose this "hypercube" assumption in the previous example, we see that Eq (1.32) is reduced to $\epsilon^2$. In conclusion, the hypercube assumption allow us to assign an independent probability for an error to each qubit.

Until this point, all our discussion was based on using an expansion to the evolution operator to lowest nontrivial order. However, in order to obtain a better quantitative result, it is important to try to improve the expansion by taking into account higher order contributions.

We start by writing the evolution operator at the end of a quantum error correction cycle. Using the hypercube assumption, it is straightforward to see that if an error $\alpha$ was diagnosed on qubit 1, the lowest-order terms in the evolution are

$$\begin{aligned} \upsilon_{\alpha}\left(\mathbf{x}_1,\lambda_{\alpha}\right) \approx & -i\lambda_{\alpha}\int_0^{\Delta} dt\, f_{\alpha}\left(\mathbf{x_1},t\right) \\ & -\frac{1}{2}\left|\epsilon_{\alpha\beta\gamma}\right|\lambda_{\beta}\lambda_{\gamma}\,\sigma_{\alpha}\left(\Delta\right)T_t\int_0^{\Delta} dt_1\,dt_2 \\ & \times f_{\beta}\left(\mathbf{x_1},t_1\right)f_{\gamma}\left(\mathbf{x_1},t_2\right)\sigma_{\beta}\left(t_1\right)\sigma_{\gamma}\left(t_2\right) \\ & +\frac{i}{6}\sum_{\beta}\lambda_{\alpha}\lambda_{\beta}^2\,\sigma_{\alpha}\left(\Delta\right)T_t\int_0^{\Delta} dt_1\,dt_2\,dt_3 \\ & \times f_{\alpha}\left(\mathbf{x_1},t_1\right)f_{\beta}\left(\mathbf{x_1},t_2\right)f_{\beta}\left(\mathbf{x_1},t_3\right) \\ & \times\sigma_{\alpha}\left(t_1\right)\sigma_{\beta}\left(t_2\right)\sigma_{\beta}\left(t_3\right)+\ldots, \end{aligned} \tag{1.34}$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor. In Refs. [26, 25], we employed a perturbation theory improved by the renormalization group to take into account higher-order terms. This is certainly a very compact and elegant way to proceed, but a conceptually simpler approach is also possible in many cases by summing the so-called bubble diagrams.

By time-ordering the probability, $T_t\mathcal{P}\left(\upsilon_{\mathbf{m}_1}\right)$, we obtain a term at $m$-th order in the perturbation theory as

$$\int_0^{\Delta} dt_1 ... \int_0^{t_{m-1}} dt_m\left\langle f_{\alpha}\left(t_1\right)f_{\alpha}\left(t_2\right)\right\rangle ...\left\langle f_{\alpha}\left(t_{m-1}\right)f_{\alpha}\left(t_m\right)\right\rangle . \tag{1.35}$$

In the case where $\langle f_{\alpha}f_{\beta}\rangle\propto\delta_{\alpha\beta}$, the series of diagrams with this form is certainly the most divergent set.

From our qualitative discussion, we know that there are two very different frequency regimes: i) A high-frequency domain from $\Delta^{-1}$ to the cutoff frequency of $V$, $\omega_{\Lambda}$, which corresponds to summing up the most divergent diagrams inside a hypercube; ii) a low-frequency domain, for contractions between hypercubes. Hence, it is convenient to separate these two domains,

$$f_{\alpha}=f_{\alpha}^{(<)}+f_{\alpha}^{(>)}, \tag{1.36}$$

and integrate out of the problem the high-frequency components of the environment. This integration produces two effects.

First, the sum of diagrams containing only high-frequency terms usually produces a geometric progression. Assuming, for the sake of simplicity, bosonic commutation relations, the series adds to

$$\epsilon_\alpha \sim \frac{\lambda_\alpha^2 \left(\Omega\Delta\right)^{2(z-\delta)}}{1+\lambda_\alpha^2 \left(\Omega\Delta\right)^{2(z-\delta)}}, \tag{1.37}$$

where $\Omega$ is a function of $\omega_\Lambda$ and other microscopic parameters. The second effect is to dress the low-frequency part and create an effective coupling constant

$$\lambda_\alpha^* \sim \frac{\lambda_\alpha}{\sqrt{1+\lambda^2 \left(\Omega\Delta\right)^{2(z-\delta)}}}. \tag{1.38}$$

These two contributions can be put together under a new set of operators defined in the coarse-grained space-time grid,

$$v_0^2 \left(\mathbf{x}, \Delta, 0\right) \approx \left(1 - \sum_\alpha \epsilon_\alpha\right) F_0 \left(\mathbf{x}, 0\right) \tag{1.39}$$

and

$$v_\alpha^2 \left(\mathbf{x}, \Delta, 0\right) \approx \epsilon_\alpha \left[1 + F_\alpha \left(\mathbf{x}, 0\right)\right], \tag{1.40}$$

where

$$F_0 \left(\mathbf{x}, 0\right) = 1 - \frac{\sum_\alpha \left(\lambda_\alpha^* \Delta\right)^2}{1 - \sum_\alpha \epsilon_\alpha} : \left| f_\alpha^{(<)} \left(\mathbf{x}, 0\right)\right|^2 : \tag{1.41}$$

and

$$F_\alpha \left(\mathbf{x}, 0\right) = \frac{1}{\epsilon_\alpha} \left(\lambda_\alpha^* \Delta\right)^2 : \left| f_\alpha^{(<)} \left(\mathbf{x}, 0\right)\right|^2 :, \tag{1.42}$$

with $::$ denoting the normal ordering with respect to the environment state $\varphi_0$. If $\epsilon_\alpha$ and $\lambda_\alpha^*$ are small parameters, Eqs. (1.39) and (1.40) are a good approximation to the expectation value that we are trying to evaluate. They separate the local (stochastic) contribution to the probability of a particular syndrome from the long-range (correlated) part.

### 1.4 The threshold theorem in a critical environment

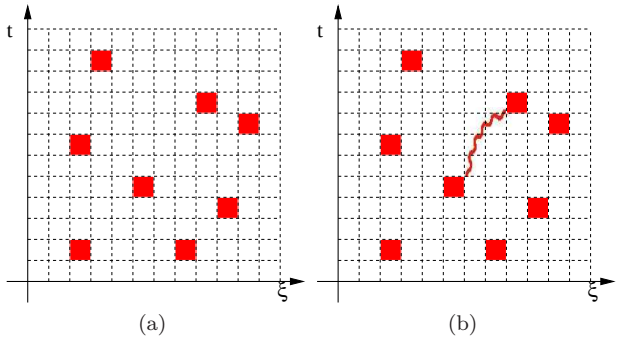
Using the results from the previous section, it is relatively simple to try to calculate the probability of a particular history of syndromes.

In this section, we use such calculation to address the issue of the resilience of quantum computation in a critical environment. The basic strategy is rather simple: If a correlated error model can be reasonably approximated by a stochastic error model, then we can make use of the traditional derivation of the threshold theorem [16, 3, 20].

We start by asking what is the probability of a computer with $R$ qubits to have $m$ errors of type $\alpha$ diagnosed after $N$ quantum error correction cycles. Recalling Eqs (1.39) and (1.40), this probability can be written as

$$\begin{aligned} \mathrm{Pr}_m^\alpha &= p_m \int \frac{d\mathbf{x}_1}{(v\Delta)^{D/z}} \cdots \frac{d\mathbf{x}_m}{(v\Delta)^{D/z}} \int_0^{N\Delta} \frac{dt_1}{\Delta} \cdots \int_0^{t_{m-1}} \frac{dt_m}{\Delta} \\ &\times \left\langle \Big[ \prod_\zeta F_0(\mathbf{x}_\zeta, t_\zeta) \Big] \big[1 + F_\alpha(\mathbf{x_1}, t_1)\big] \ldots \big[1 + F_\alpha(\mathbf{x_m}, t_m)\big] \right\rangle , \end{aligned} \tag{1.43}$$

where we integrated over all possible grid positions, $(\mathbf{x}_j, t_j)$, $\zeta$ denotes the set of remaining hypercubes, and $p_m = (1 - \sum_\alpha \epsilon_\alpha)^{RN-m} (\epsilon_\alpha)^m$.




(a) (b)

Fig. 1.3. Graphical representation of Eqs. (1.44) and (1.45). Wiggly line represents a pair contraction $\langle F_\alpha(\mathbf{x_i}, t_i) F_\alpha(\mathbf{x_j}, t_j)\rangle$.

We now organize the expectation value of Eq. (1.43) in powers of $(\lambda^*, \epsilon_\alpha)$ and invoke Wick's theorem again. The first term is just the stochastic contribution to the probability [see Fig. (1.3-a)],

$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} = p_m \begin{pmatrix} NR \\ m \end{pmatrix} \sim p_m (NR)^m . \tag{1.44}$$

The next term is typically of the form [see Fig. (1.3-b)]

$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} \left\langle F_\alpha \left(\mathbf{x_i}, t_i\right) F_\alpha \left(\mathbf{x_j}, t_j\right) \right\rangle . \tag{1.45}$$

All higher-order terms can be systematically taken into account and simply provide corrections to the stochastic probability shown in Eq. (1.44).

The last step to complete the discussion is to use Wick's theorem and Eq. (1.8) to show that the two-point correlation function for $F_\alpha$ has the general form

$$\left\langle F_\alpha \left(\mathbf{x_i}, t_i\right) F_\alpha \left(\mathbf{x_j}, t_j\right) \right\rangle \sim \mathcal{F}\left( \frac{1}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^{4\delta_\alpha}}, \frac{1}{\left|t_i - t_j\right|^{4\delta_\alpha / z}} \right). \tag{1.46}$$

Hence, a perturbative expansion in $\lambda_\alpha^*$ is guaranteed to be stable if the term in (1.45) remains finite and smaller than the leading term (1.44). This requires

$$D + z - 2\delta < 0. \tag{1.47}$$

In other words, whenever Eq.(1.47) is satisfied, correlations between hypercubes produce small corrections to Eq. (1.44). Therefore, the traditional proof of resilience holds whenever the error probability $\epsilon_\alpha$ is below the threshold value.

The opposite situation, $D + z - 2\delta > 0$, is much less clear. In this case, the expansion is not stable and no conclusion can be draw. It is important to emphasize that the calculation we just did does not precludes that QEC can still be effective. The instability is only telling us that a perturbative expansion in $\lambda_\alpha^*$ is not well define and that the threshold theorem, as we stated, does not hold. It is conceivable that some different derivation of the theorem could still exist in this case.

## 1.5 The threshold theorem and quantum phase transitions

It is possible to construct a very nice analogy between our discussion and the theory of quantum phase transitions [27]. In fact, the traditional threshold theorem can be thought as a quantum/classical phase transition: Ref. [2] tell us that the error probability $\epsilon$ plays to fault tolerance a role similar to the temperature in a physical system. This is how:

i) For $\epsilon < \epsilon_c$, the computer components can maintain a large entanglement through fault-tolerant procedures, which in turn means that the computer and the environment are weakly entangled. Hence, due to

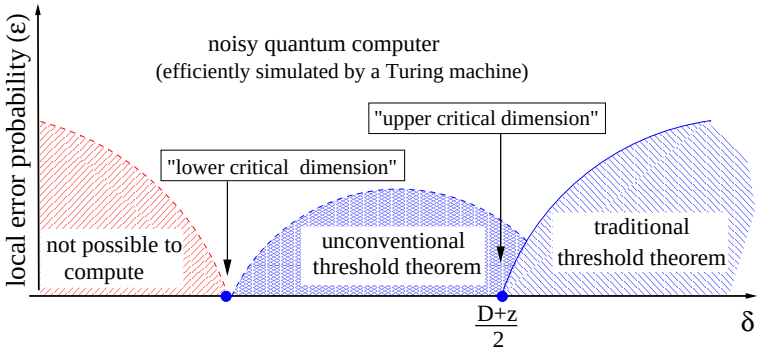



Fig. 1.4. Phase diagram of a quantum computer running QEC. The parameter $\delta$ denotes the scaling dimension of the environment operator in the system-environment interaction [see Eq. (1.8)].

this large internal entanglement, the quantum computer departs from the classical computer model and can not be efficiently simulated by a Turing machine.

ii) For $\epsilon > \epsilon_c$, the computer components are weakly entangled and, therefore, can be efficiently simulated by a Turing machine. In other words, the computer density matrix is no longer represents a pure state, but rather a statistical mixture, and the computer components are strongly entangled with the environment.

In this sense, the threshold theorem defines a "phase transition" from a high-temperature phase, where qubits are "independent" from each other, to a low-temperature phase, where quantum coherence and entanglement are possible.

Our analysis of a critical environment adds another parameter to this interpretation: The scaling dimension $\delta$. This quantum parameter defines another axis in the "phase diagram" of a quantum computer [see Fig. (1.4)].

In the theory of quantum phase transitions, a dimension criterion as $D+z = 2\delta$ defines what is usually referred as the upper critical dimension of the model. Above the upper critical dimension [when the inequality (1.47] is satisfied) the model is essentially free and perturbation theory works. Below the upper critical dimension, there are two possibilities: i) If the system is above its lower critical dimension, some infinite resummation of diagrams is necessary, but the theory is still preserve its weak coupling character (although no longer with mean-field exponents); ii) if, however, the theory is below its lower critical dimension, the theory

is in a strong coupling regime, and no re-summation of diagrams will help us to determine its physical characteristics.

If we take the analogy with the theory of quantum phase transitions literally, we could argue that below the upper critical dimension of the computer, $D + z > 2\delta$, there are two possible scenarios. It is possible that for a range of scaling dimensions $\delta$ another proof of resilience can be achieved. However, there may also be a range of $\delta$ where no proof of resilience exists (in other words, there is never resilience to errors). Although this issue is rather speculative, the similarity between the threshold theorem and quantum phase transitions is so striking and intuitive that the existence of these extra "phases" is rather tantalizing and likely worth exploring.

## 1.6 Conclusions

It is believed that some sort of Quantum Error Correction must be always implemented at last logical level[31]. Thus, it is undoubtedly one of the cornerstones of quantum computation. However, it has been argued that QEC relies on a set of unphysical assumptions [7, 8, 9], namely: (i) "fast" measurements, (ii) "fast" gates, and (iii) describing decoherence by error models. Our perspective is that all these problems are not of fundamental nature. They are legitimate concerns, but they also have been discussed extensively in the literature: First, in Ref. [14] DiVincenzo and Aliferis demonstrated that resilient circuits can be constructed with slow measurements. Second, we used some reasonable assumptions to treat "slow" gates and laid the groundwork for a theoretical framework that connects microscopic Hamiltonian's with error models in correlated environments. Finally, our results, in conjunction with those of Ref. [4], show that a large class of critical environments are already properly treated within the QEC framework.

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