Seven Pines Symposium XIX
"General Relativity; a hundred years after its birth"
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From the Schwarzschild Singularity to the Black Hole Horizon.

Jean Eisenstaedt (Observatoire de Paris)

From the Schwarzschild singularity to the black hole horizon

- I. The early interpretation of the Schwarzschild solution.
  - General relativity before the sixties.
  - The solution: Schwarzschild, Droste.
  - - The “singularity": coordinates, covariance, junction conditions, singularities… And the proper time?
  - The "singularity": Schwarzschild, Droste, Hilbert, Laue, Eddington…
  - The trajectories: Droste, Hilbert, von Laue, de Jans, Hagihara, Rabe… And the free fall?

- II. The early way to a new interpretation.
  - Lanczos: is Schwarzschild's "radius" singular? (1921)
  - Synge (1934)
  - Lemaître on Schwarzschild's "singularity" (1932).
  - Lemaître, Robertson, Einstein, Synge, Tolman, Oppenheimer, Kruskal…

III. Cosmology, a space for thought in general relativity.

I. The early interpretation of the Schwarzschild solution.

The solution: Schwarzschild, Droste.

Karl Schwarzschild

Hochgeehrter Herr Kollege!

Ihre Arbeit habe ich mit grösstem Interesse durchgesehen. Ich hätte nicht erwartet, dass man so einfach die strenge Lösung der Aufgabe formulieren könne. Die rechnerische Behandlung des Gegenstandes gefällt mir ausgezeichnet. Nächsten Donnerstag werde ich die Arbeit mit einigen erläuternden Worten der Akademie übergeben.

« I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution of the problem in such a simple way. I liked very much your mathematical treatment of the subject. Next Thursday I shall present the work at the Academy with a few words of explanation. » (Einstein to Schwarzschild, January 9, 1916).

Einstein to Schwarzschild, 1916

Schwarzschild's original solution (1916)

$$ds^2 = (1-\frac{\alpha}{r})c^2\,dt^2 - \frac{dr^2}{1-\frac{\alpha}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

With $$\alpha = \frac{2Gm}{c^2}$$

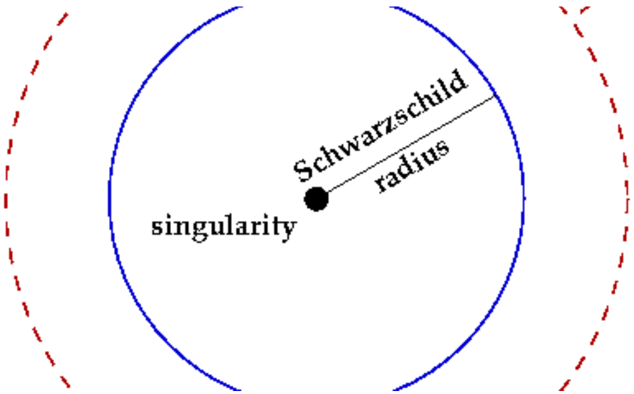
and $$r = \left(R^3 + \alpha^3\right)^{1/3}.$$

$$ds^2 = (1 - \frac{\alpha}{r})c^2 dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Schwarzschild's solution in Droste's coordinates (1916)

The early Schwarzschild structure

$$r_s = \frac{2GM}{c^2}$$



$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2\, dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

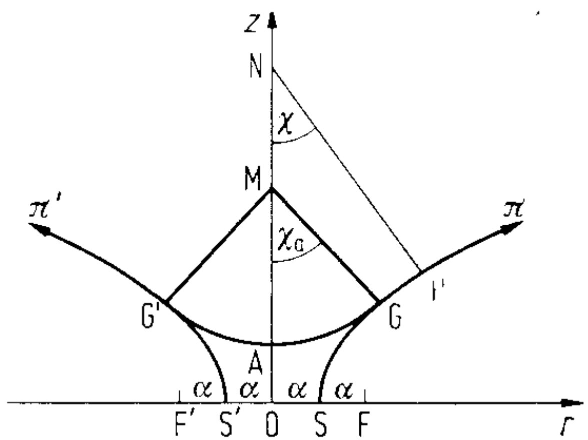
The "Schwarzschild singularity": coordinates, covariance, junction conditions, singularities...
And the proper time?

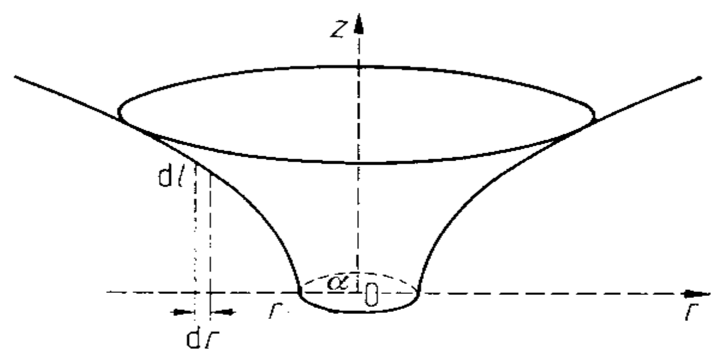
Eisenstaedt, Jean (1982). "Histoire et singularités de la solution de Schwarzschild (1915-1923)." Archive for History of Exact Sciences 27 : 157-198.

Embedding the spatial part of the Schwarzschild solution in an Euclidean space of four dimensions:

$$ds^2 = \frac{dr^2}{1-\frac{\alpha}{r}} + r^2\, d\theta^2 = dx^2 + dy^2 + dz^2$$

where: $x = r\sin\theta$, $y = r\cos\theta$ and $z = \int_0^r \frac{dr}{\sqrt{\frac{r}{\alpha} - 1}} = 2\sqrt{\alpha(r-\alpha)}$.





Embedding the solution, Flamm 1916, Becquerel 1922.

Line-elements' collection…

1. Schwarzschild's solution in standard polar coordinates:

$$ds^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)$$

with $r = \left(R^3 + \alpha^3\right)^{1/3}$.

2. Schwarzschild's solution in Droste's coordinates:

$$ds^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)$$

(with $r > \frac{2Gm}{rc^2}$).

3. Isotropic system (Droste 1916):

$$ds^2 = \left[\frac{\rho - \frac{mG}{2c^2}}{\rho + \frac{mG}{2c^2}}\right]^2 c^2\, dt^2 - \left(1 + \frac{mG}{2\rho c^2}\right)^4 \left\{d\rho^2 + \rho^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)\right\}$$

.

4. (Droste 1916) by means of the transformation $r = \tilde{r} + \alpha$:

$$ds^2 = \frac{c^2\, dt^2}{(1 + \frac{\alpha}{\tilde{r}})} - (1 + \frac{\alpha}{\tilde{r}})d\tilde{r}^2 - \left(\tilde{r} + \alpha\right)^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right).$$

Droste's comments on covariance

"To what formula shall we give the preference to that of Schwarzschild [2], to [3] or to [5]? It is in fact a matter of personal convenience. But we must remember that the **r coordinate doesn't represent the measured interval**. We are however free to choose any coordinate (provided that **all points may be reached**) but some choice of coordinates may appear more appropriate than another." (Droste 1916)

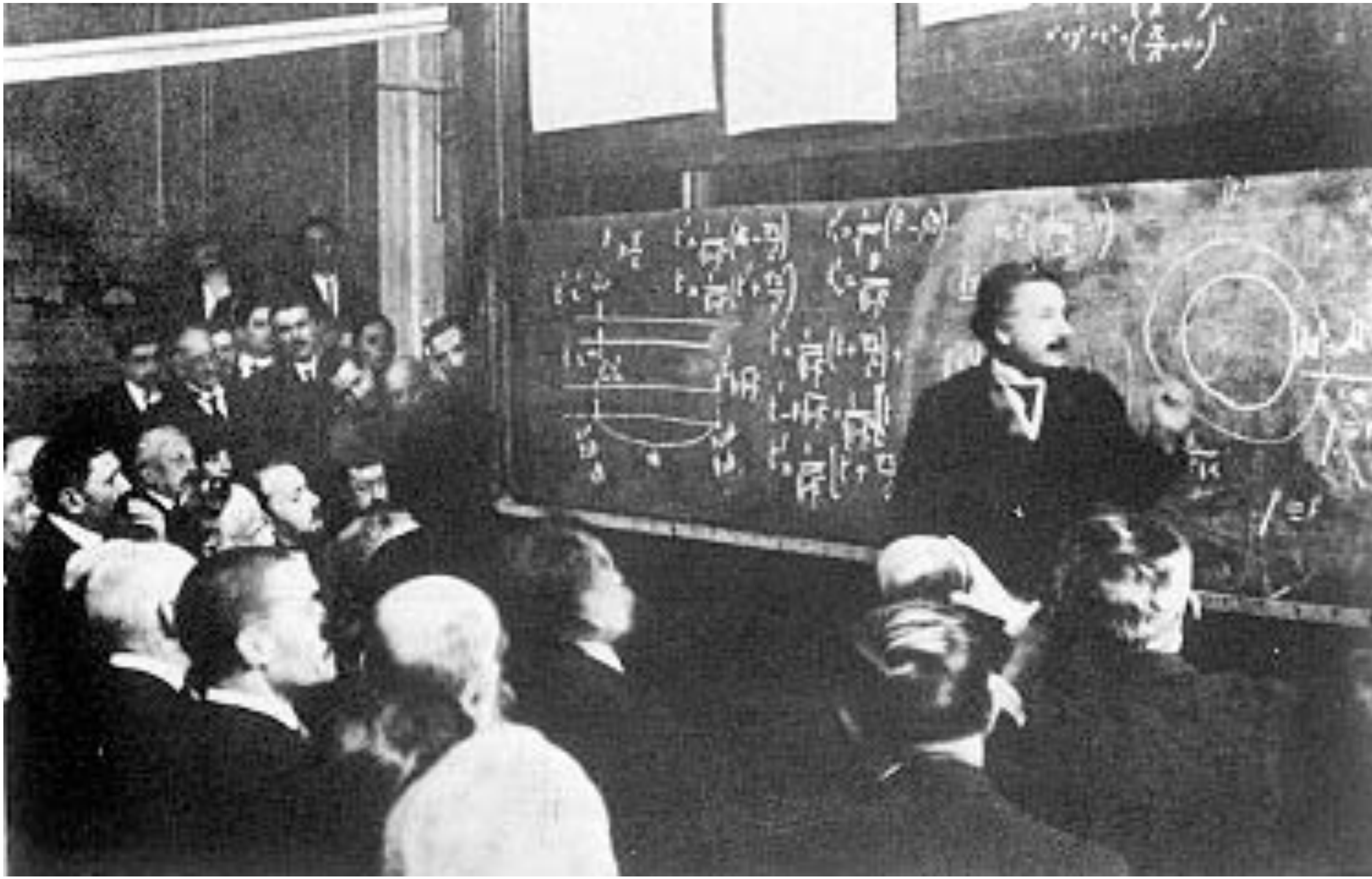
Weyl on the Schwarzschild solution (1917).

$$ds^2 = \left(\frac{\rho - m/2}{\rho + m/2} \right)^2 dt^2 - \left(1 + \frac{m}{2\rho} \right)^4 \left[d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]$$

- "the domain ρ > m/2 will correspond to the exterior and ρ > m/2 to the interior of the massif point. Through analytical extension

$$\frac{\rho - m/2}{\rho + m/2}$$

will be negative in the interior so that for a point at rest, the cosmic time and proper time are in opposition. »



Einstein at Collège de France, Paris 1922

Jacques Hadamard

Paris 1922: Einstein and Painlevé

Painlevé, Schwarzschild's solution and covariance.

$$ds^2 = (1 - \frac{\alpha}{r})c^2\,dt^2 + 2\sqrt{\frac{\alpha}{r}}drcdt - \left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

"It seems to me that the existence of [this] formula and the possibility of an infinity of others give a clear indication of the hazardous character of such predictions [...] it's pure imagination to claim that such consequences can be derived from the ds^2" (Painlevé 1921).

Einstein, the Schwarzschild solution and covariance.

$$ds^2 = (1-\frac{\alpha}{r})c^2\,dt^2 + 2\sqrt{\frac{\alpha}{r}}drcdt - \left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

"When, in the ds^2 of the static solution with central symmetry, you introduce any function of r instead of r, you do not obtain a new solution because the quantity r in itself has no physical meaning. [...] You must always keep in mind that **coordinates do not have any physical signification**; which means that they **do not represent the result of a measurement**; only conclusions, reached after the elimination of coordinates may pretend to an objective significance. Furthermore, the metrical interpretation of the quantity ds is not "pure imagination" but the deep core of the theory itself." (Einstein to Painlevé, December 7, 1921) (EA 19-004).

ALLVAR GULLSTRAND

Allvar Gullstrand (1862-1930).

Eddington 1932

Eddington on the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}\right)c^2 dt^2 - \left(1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

"We can go on shifting the measuring-rod through its own length time after time, but *dr* is zero; that is to say, we do not reduce *r*. There is a **magic circle** which no measurement can bring us inside. It is not unnatural that we should picture something obstructing our closer approach, and say that a particle of matter is filling up the interior." (Eddington 1920, 98)

"At a place where g_{44} vanishes there is an **impassable barrier**, since any change dr corresponds to an infinite distance ds surveyed by measuring-rods. [...] The first root would represent the boundary of the particle - if a genuine particle could exist - and give it the appearance of impenetrability. The second barrier is at a very great distance and may be described as the horizon of the world." (Eddington, 1923)

Eddington on the Schwarzschild singularity

"A singularity of ds^2 does not necessarily indicate material particles, for we can introduce or remove such singularities by making transformations of coordinates. It is impossible to know whether to blame the world-structure or the inappropriateness of the coordinate-system." (Eddington 1923, 165)

The Eddington-Finkelstein line-element (1924)

$$ds^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + c^2 dt^2 - \frac{2Gm}{rc^2}\left(cdt - dr\right)^2$$

This line element is actually due to Gullstrand. Eddington never claimed that the r = 2m surface was regular.

The trajectories: Droste, Hilbert, von Laue, de Jans, Hagihara, Rabe...
And the free fall

Droste thesis, December 1916

$$c^2\left(1-\frac{\alpha}{r}\right)\left(\frac{dt}{ds}\right)^2-\left(1-\frac{\alpha}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2-r^2\left(\frac{d\phi}{ds}\right)^2=1$$

$$r^2\left(\frac{d\phi}{ds}\right)=L$$

$$\left(1-\frac{\alpha}{r}\right)c\left(\frac{dt}{ds}\right)=E$$

$$\left(\frac{du}{d\phi}\right)^2=\frac{E^2-1}{L^2}+\frac{\alpha}{L^2}u-u^2+\alpha u^3$$

(where: u = 1/r)

Trajectories of the Schwarzschild field

Defining "a physical distance" δ as:

$$\delta = \int_{\frac{2Gm}{c^2}}^{r} \frac{dr}{\left(1 - \frac{2Gm}{rc^2}\right)^{1/2}}$$

and using t, the Newtonian coordinate time; from his equation of motion:

$$\dot{\delta}^2 = \left(\frac{d\delta}{dt}\right)^2 = \left(1 - \frac{2Gm}{rc^2}\right)\left[1 - \frac{1}{E^2}\left(1 - \frac{2Gm}{rc^2}\right)\right]c^2$$

$$\ddot{\delta} = \frac{d^2\delta}{dt^2} = \frac{Gm}{r^2}\left[1 - \frac{2}{E^2}\left(1 - \frac{2Gm}{rc^2}\right)\right]\left(1 - \frac{2Gm}{rc^2}\right)^{1/2}$$

$\dot{\delta}$ and $\ddot{\delta}$ go to zero at $r = 2Gm/c^2$ "where the motion stops infinitely smoothly". He will even get a repulsion around the "singularity".

- "a moving particule out of the sphere $r = \alpha$ would never get into this sphere"

- "the particule will never reach the sphere $r = \alpha$".

- He "will not consider the space $r < \alpha$"

He will come back to this question in his thesis (1917) but with his radial coordinate only and t, the time-coordinate.

Droste on the radial fall (1916, 1917)

Droste “for once, a different result“

"It might be said then that the material point **does not reach the center**. This result is, **for once, different** with any precision of the newtonian theory. We see here **to what an extent the movement is different near the center** with all what is said by the classical theory." (Droste 1916, thesis, 26).

De Jans, 1922-1924

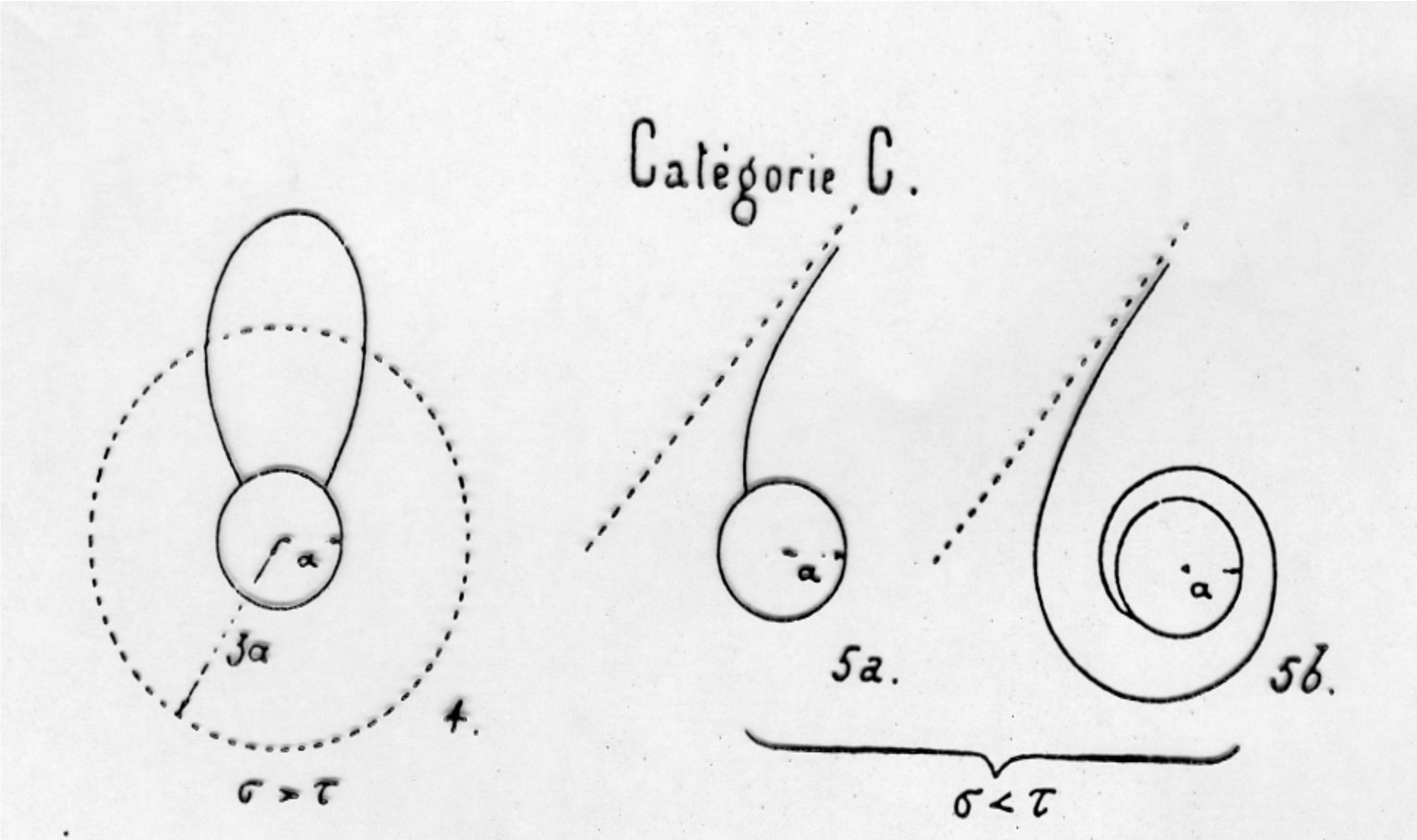
- As Droste, de Jans starts from:
- $$\left(\frac{du}{d\varphi}\right)^2 = \frac{E^2 - 1}{L^2} + \frac{2Gm}{c^2L^2}u - u^2 + \frac{2Gm}{c^2}u^3$$ [7]

- He calculates "the velocity of the particles on its orbit" as evaluated by an observer at infinity defined by $\frac{dl}{dt}$

the ratio of the spatial part of Schwarzschild's line-element to the time coordinate au temps coordonnée ; a non-covariant quantity:

$$v^2 = \left(\frac{dl}{dt}\right)^2 = c^2\left(1 - \frac{2Gm}{rc^2}\right)\left[1 - \frac{1}{E^2}\left(1 - \frac{2Gm}{rc^2}\right)\right]$$

« ν goes to zero for r = α » he wrote insisting on the fact that « r – 2m could not be negative ».



De Jans, 1923, trajectories

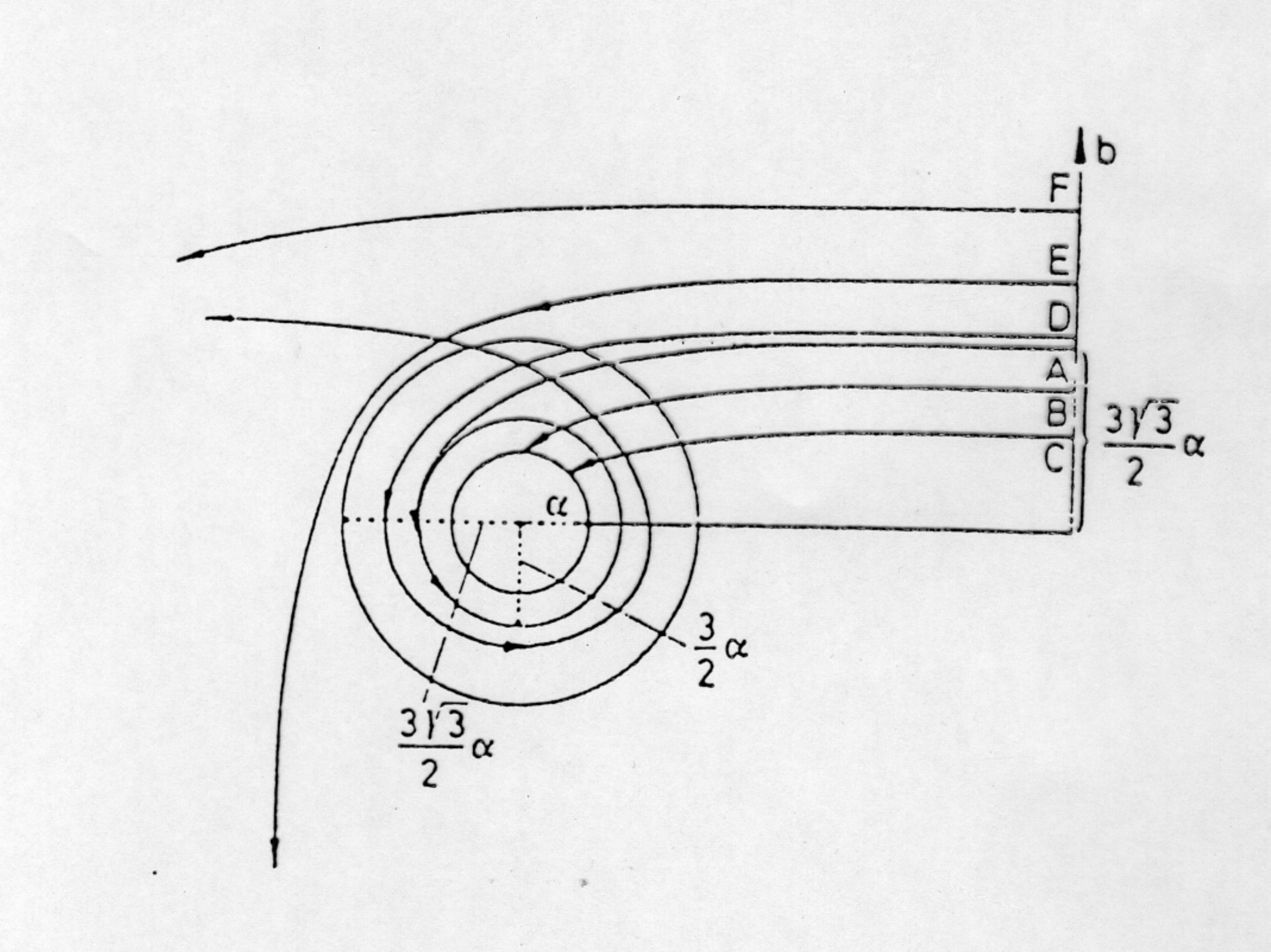
Light-rays in a Schwarzschild field

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1}{B^2} - \rho^2 + \alpha\rho^3$$

where $\rho = 1/r$ and $B = 3\sqrt{3}\,\frac{Gm}{c^2}$

Hilbert, 1917; Laue, 1921...

Max von Laue, 1879–1960



Laue 1921: light-rays

Newtonian equations in the radial case:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2Gm}{rc^2} + Cte$$

Relativistic equations in the radial case:

$$\left(\frac{dr}{ds}\right)^2 = E^2 - 1 + \frac{2Gm}{rc^2}$$

$$\left(\frac{dt}{ds}\right)\left(1 - \frac{2Gm}{rc^2}\right) = E$$

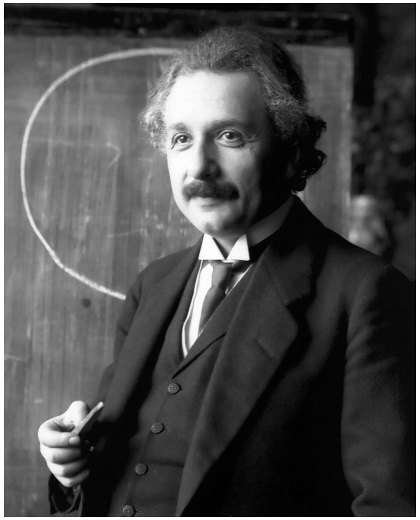
The free fall.

Regularity and jumping conditions

Hilbert on regularity (1917)

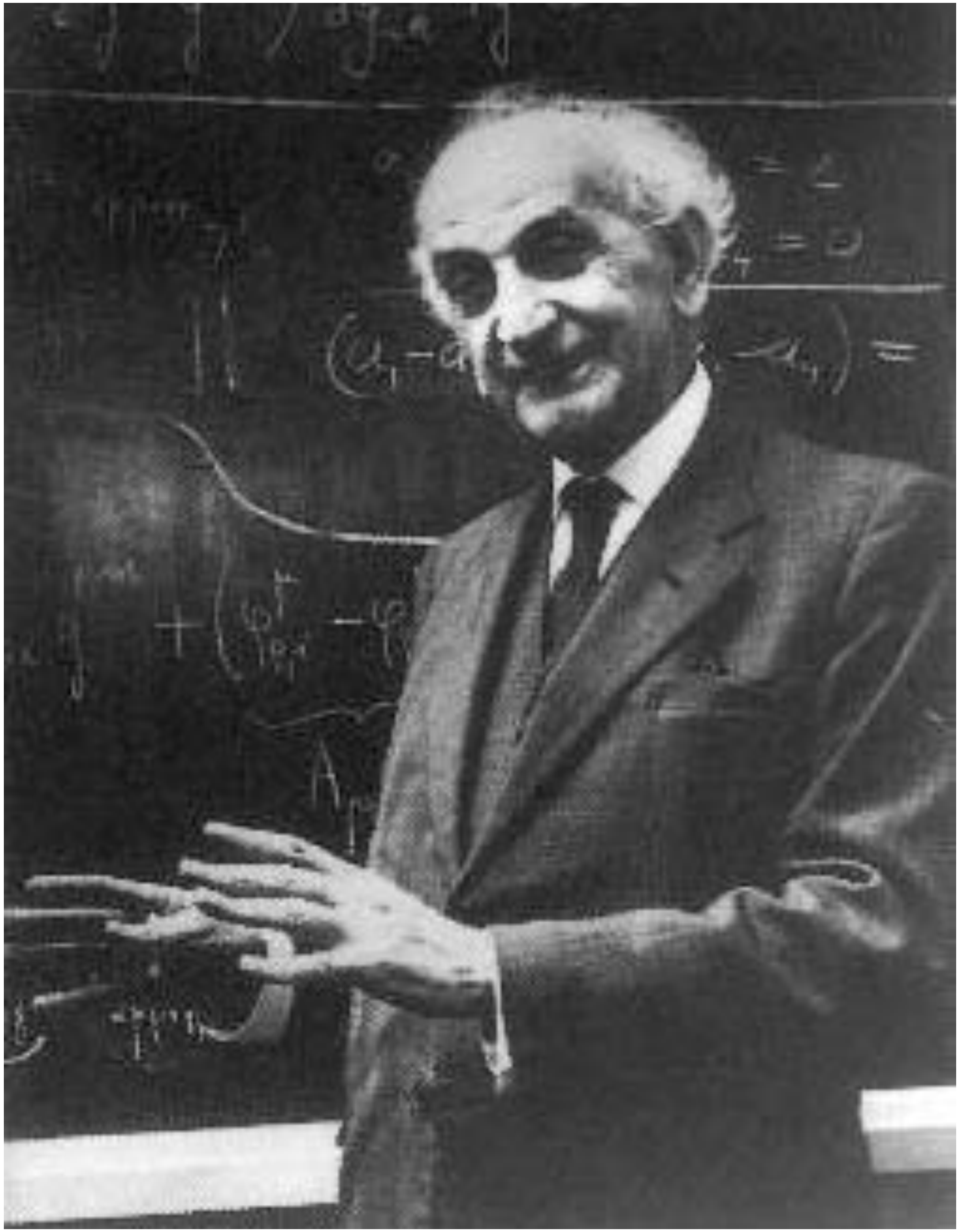
"a line element or a gravitational field gmn is *regular* at a point if it is possible by a **reversible one to one transformation** to introduce a coordinate system such that in this system the corresponding functions $g'_{\mu\nu}$ are regular at that point, i.e. **they are continuous and arbitrarily differentiable** at the point and in a neighborhood of the point, and the determinant g' is different from 0." (Hilbert 1917, p 70-71).

Einstein on regularity (1918)



"Moreover, the condition of continuity for the $g_{\mu\nu}$ and the $g^{\mu\nu}$ should not be taken as saying that there has to be a coordinate system such that continuity holds throughout space[time]. Clearly, one only has to require that in the neighborhood of every point there *exists* a coordinate system such that continuity holds in this neighborhood; *such a restriction of the demand of continuity naturally results from the general covariance of the [field] equations*." (Einstein 1918, 271); [my emphasis].

II. The early way to a new interpretation.



Cornelius Lanczos

Lanczos' insight, 1922.

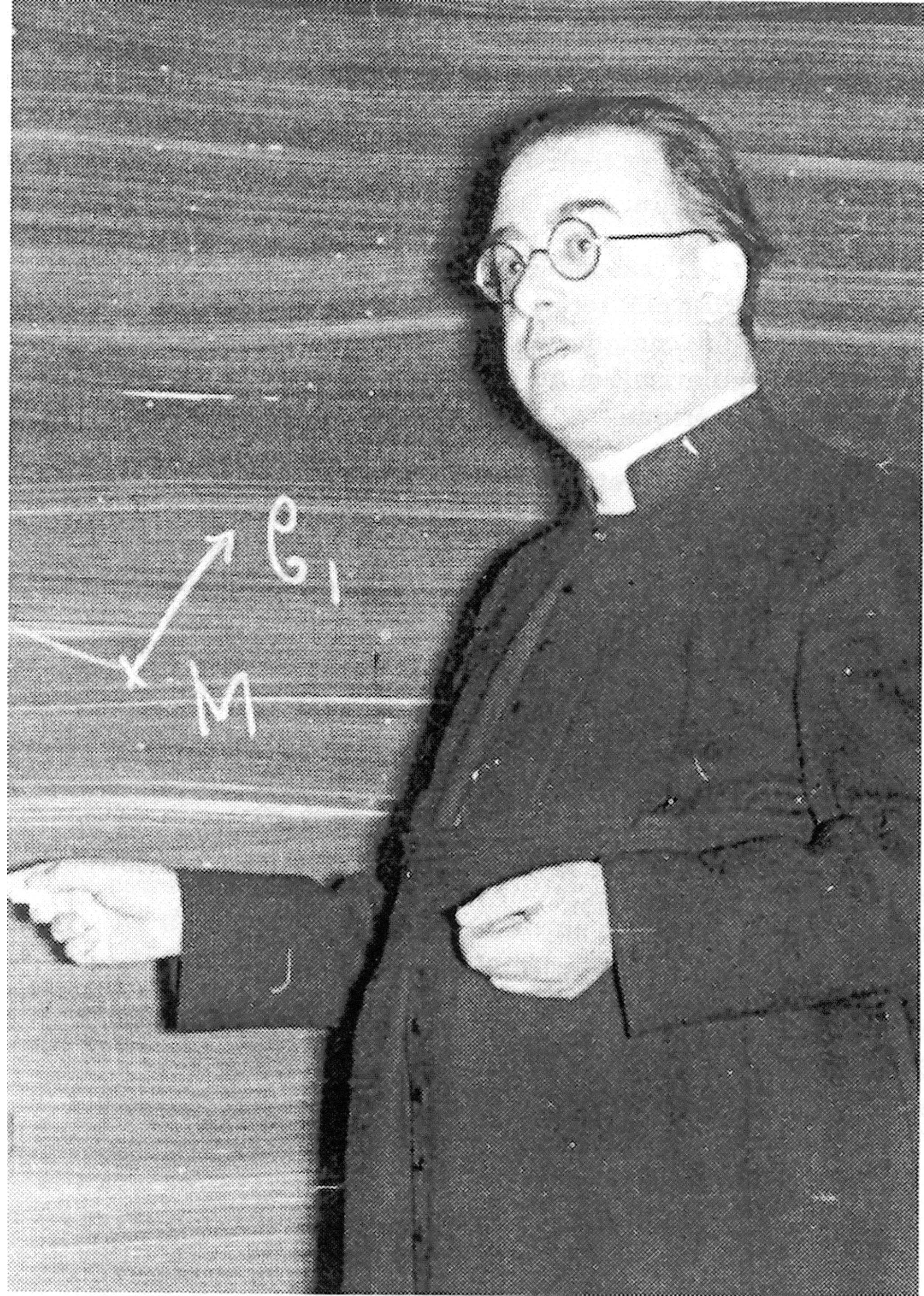
With $\bar{r} = r - \frac{\alpha}{2}$, Lanczos got:

$$ds^2 = \left[\frac{\bar{r} - \frac{\alpha}{2}}{\bar{r} + \frac{\alpha}{2}}\right] c^2\, dt^2 - \left[\frac{\bar{r} + \frac{\alpha}{2}}{\bar{r} - \frac{\alpha}{2}}\right] d\bar{r}^2 - \left(\bar{r} + \frac{\alpha}{2}\right)^2 \left(d\theta^2 + sin^2\,\theta d\phi^2\right).$$

Then, working out the expression for the determinant of the line-element corresponding to the euclidean systems of coordinates associated with the polar system $\bar{r}$, Lanczos obtains: $|g| = \left(1 + \frac{\alpha}{2\bar{r}}\right)^4$. Consequently, at $\bar{r} = 0$ the determinant is **singular**, which was **not** the case at the corresponding point $r = \frac{\alpha}{2}$ of the solution in Droste coordinates."

"This example shows how little one can infer an actual singularity of the field from the singular behavior of the functions g_{mn} since it may be **possible to remove** the latter **by a coordinate transformation**." (Lanczos 1922, 539)

Lanczos' conclusion, 1922.



Georges Lemaître (1894-1966)

Lemaître and the Schwarzschild « singularity », 1933.

Lemaître solved the field-equations with spherical symmetry, energy density $\rho(\chi, t)$ and **no pressure.** (Usually and wrongly called the Bondi-Tolman solution).

In the co-moving coordinate system he chose, the line element reads:

$$ds^2 = c^2\, d\tau^2 - a^2\, d\chi^2 - r^2\,(\, d\theta^2 + \sin^2\theta\;\; d\phi^2\,)\,,$$

where *c*, *a*, and r are functions of χ and t. Writing his dust solution in the exterior case and using the **non-static** transformation of coordinates:

$$r\,3/2 = \sqrt{\frac{Gm}{\lambda c^2}}\,\sinh\!\left(\sqrt{\frac{3\lambda c^2}{4}}(\tau - \chi)\right)$$

Lemaître obtains the Schwarzschild line element in his own coordinates:

$$ds^2 = c^2 d\tau^2 - \left(\frac{\lambda c^2}{3} r^2 + \frac{2Gm}{r}\right) d\chi^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

By the transformation

$$cdt = cd\tau + \frac{\sqrt{\dfrac{2Gm}{rc^2} + \dfrac{\lambda r^2}{3}}}{1 - \dfrac{2Gm}{rc^2} - \dfrac{\lambda r^2}{3}}\, dr,,$$

The “classical” Schwarzschild solution

$$ds^2 = \left(1 - \frac{2Gm}{rc^2} - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \dfrac{2Gm}{rc^2} - \dfrac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta . d\phi^2\right).$$

Lemaître has shown that:

"the singularity of the field is not real but the result of using a coordinate-system in which the field is static.”

Lemaître and the Schwarzschild « singularity », 1933.

"The equations of the Friedman universe admit [...] solutions in which the radius of the universe goes to zero. This contradicts the generally accepted result that a given mass cannot have a radius smaller than $2Gm/c^2$" (Lemaître 1932, 80).

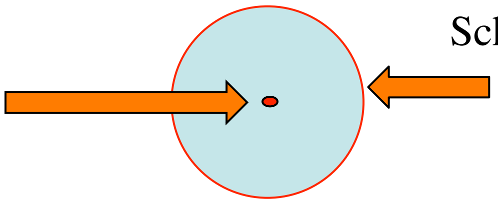
Friedmann’s space

«true» singularity

Schwarzschild’s exterior space

Schwarzschild’s singularity

«true» singularity: r = 0

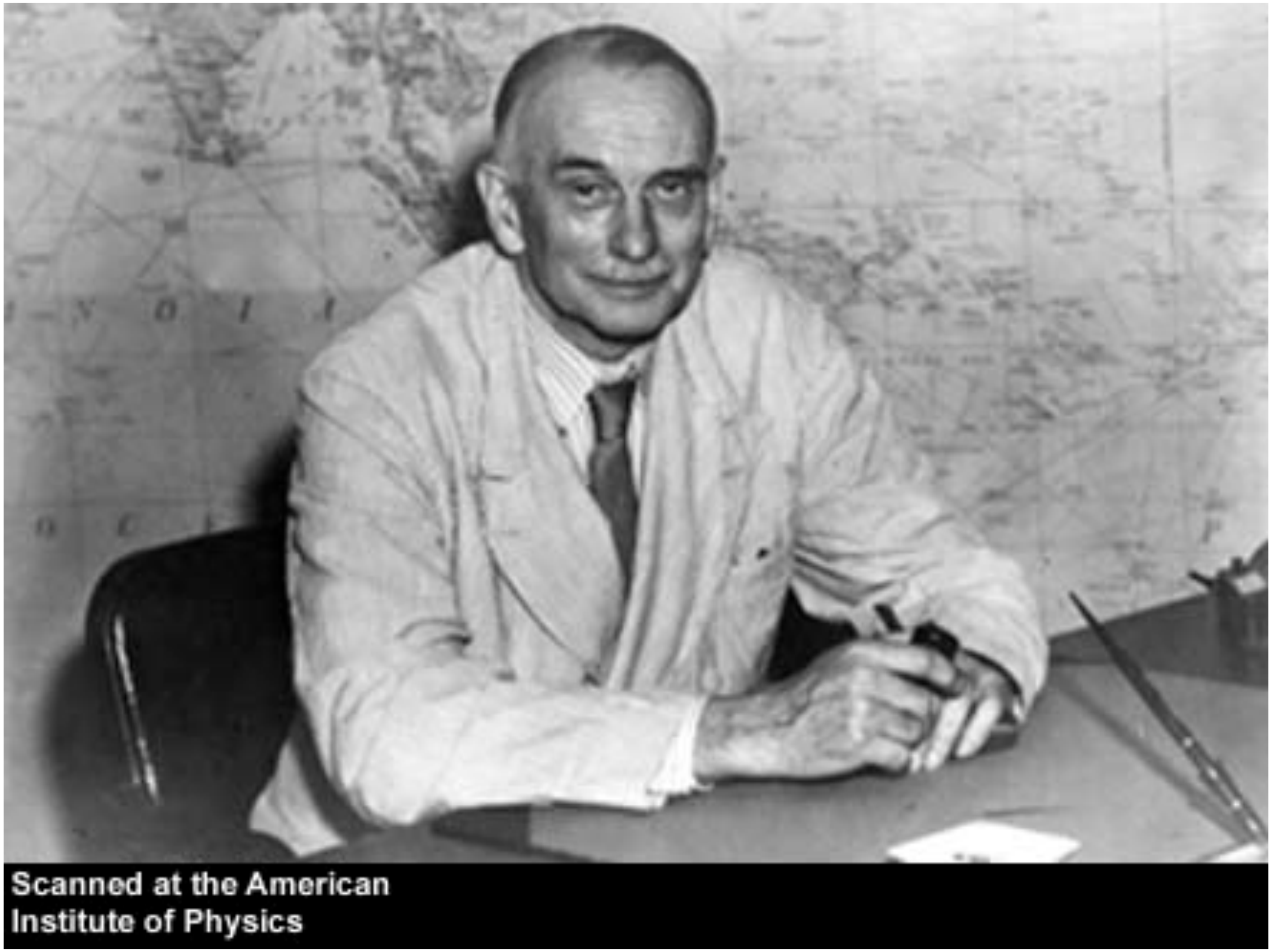


Schwarzschild’s solution before the sixties

Structure of Lemaître solution

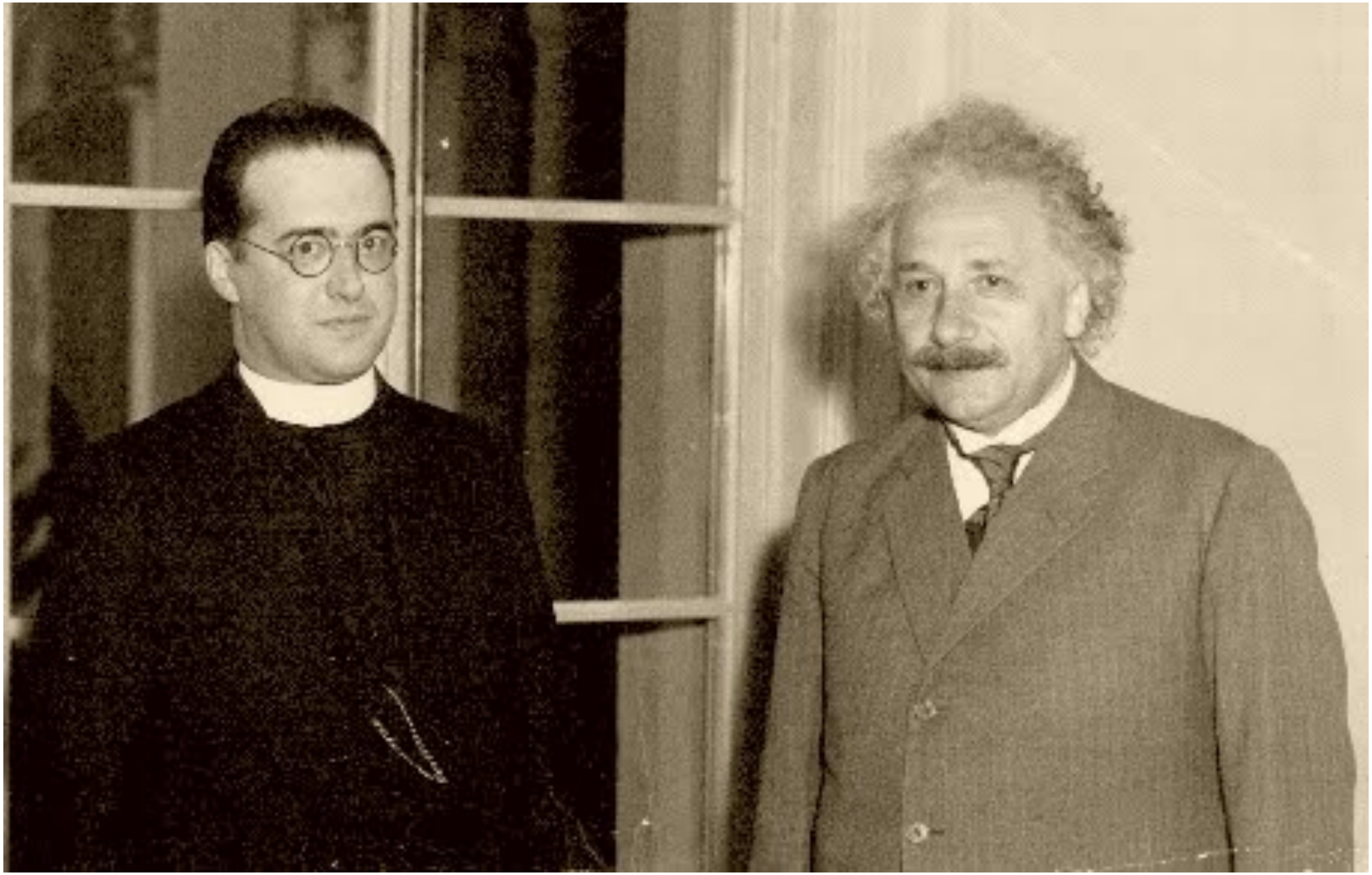
Lemaître on the Schwarzschild "singularity"

"The singularity of the Schwarzschild field then is a fictitious singularity, analogous to the one appearing on the horizon of the center in the original form of the de Sitter universe." (Lemaître 1932, 82).

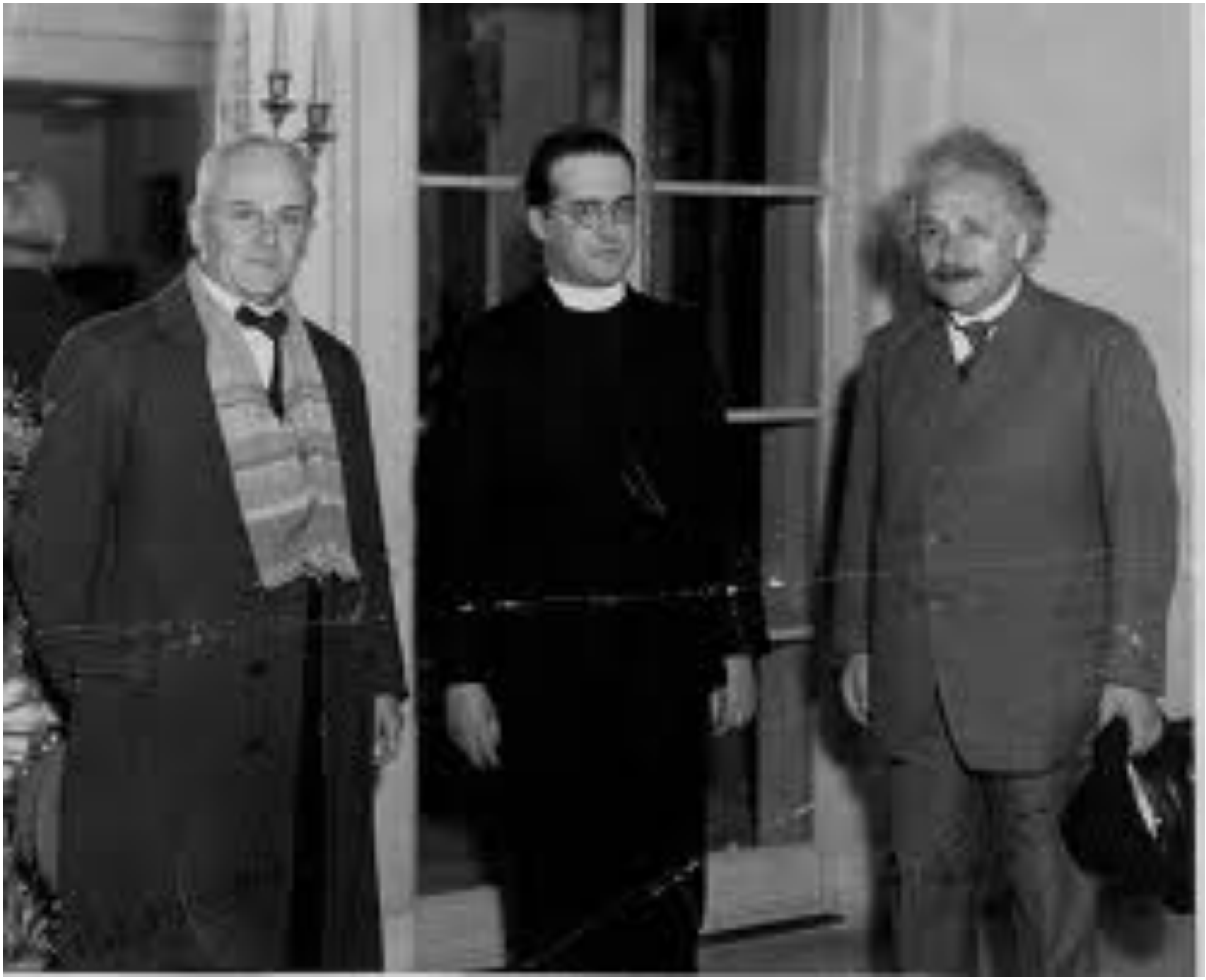


Richard Tolman (1881-1948)

John L. Synge (1897–1995).



Einstein and Lemaître



Millikan, Lemaître, Einstein.
(California Institute of Technology. Pasadena, 10 janvier 1933)

Howard Robertson (1903–1961)

Ratio of gravitational radius to radius

(from H. P. Robertson and T. W. Noonan, 1968).

Objet	mG/ac^2
Proton	$1.0 \quad 10^{-39}$
Metal sphere with radius 1 meter	$3 \quad 10^{-23}$
Earth	$6.95 \quad 10^{-10}$
Sun	$2.12 \quad 10^{-6}$
Certain white dwarfs stars	$2.5 \quad 10^{-4}$
Galactic nucleus	$3 \quad 10^{-7}$

"There are no known objects with so small a size » (Robertson and Noona n, 1968).

From Lemaître's line element:

$$ds^2 = c^2 d\tau^2 - \left(\frac{\lambda c^2}{3} r^2 + \frac{2Gm}{r}\right) d\chi^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

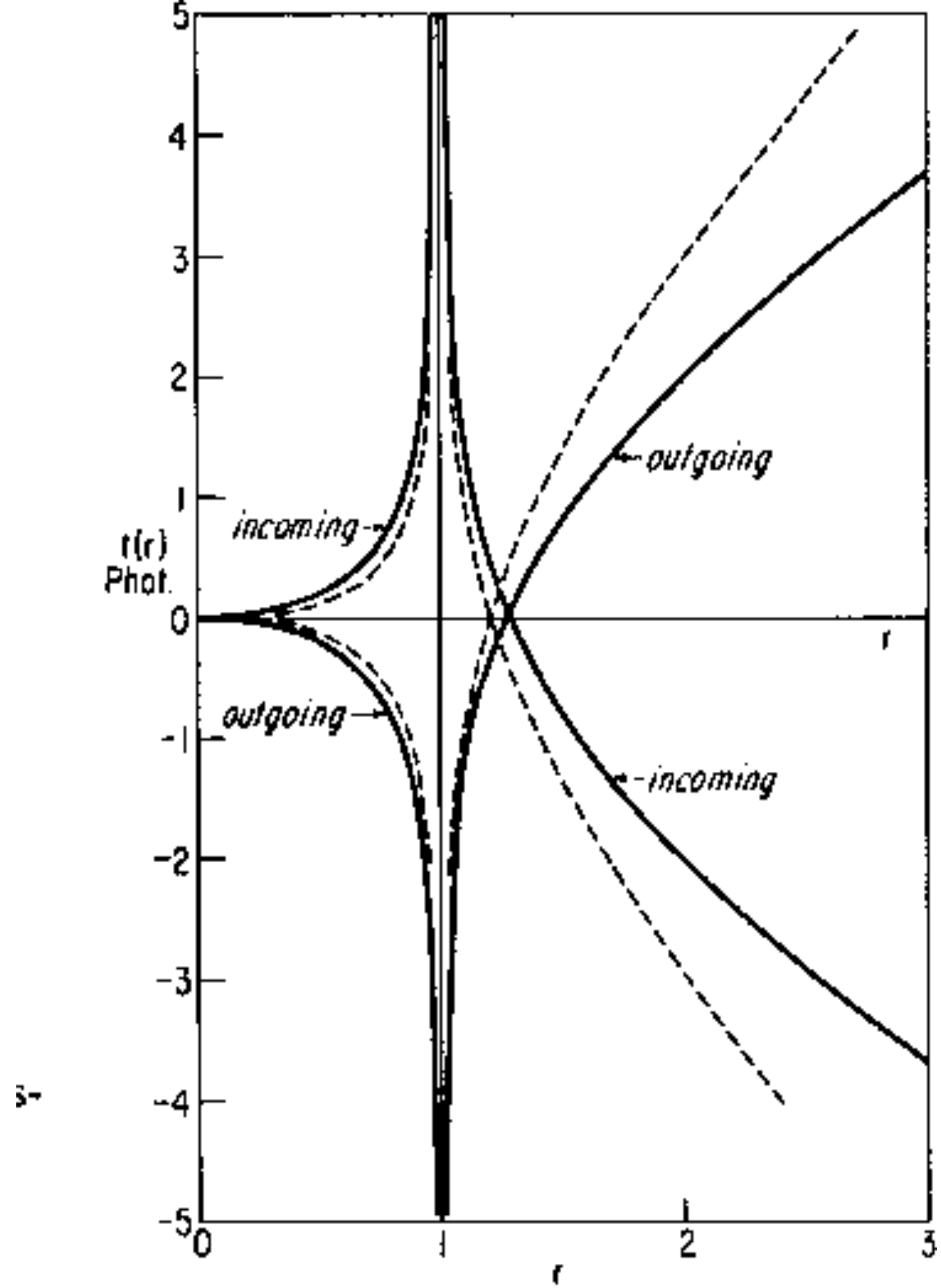
With $\lambda = 0$ and by using the coordinate transformation:

$$\chi = \frac{2}{3}\rho^{\frac{3}{2}}$$

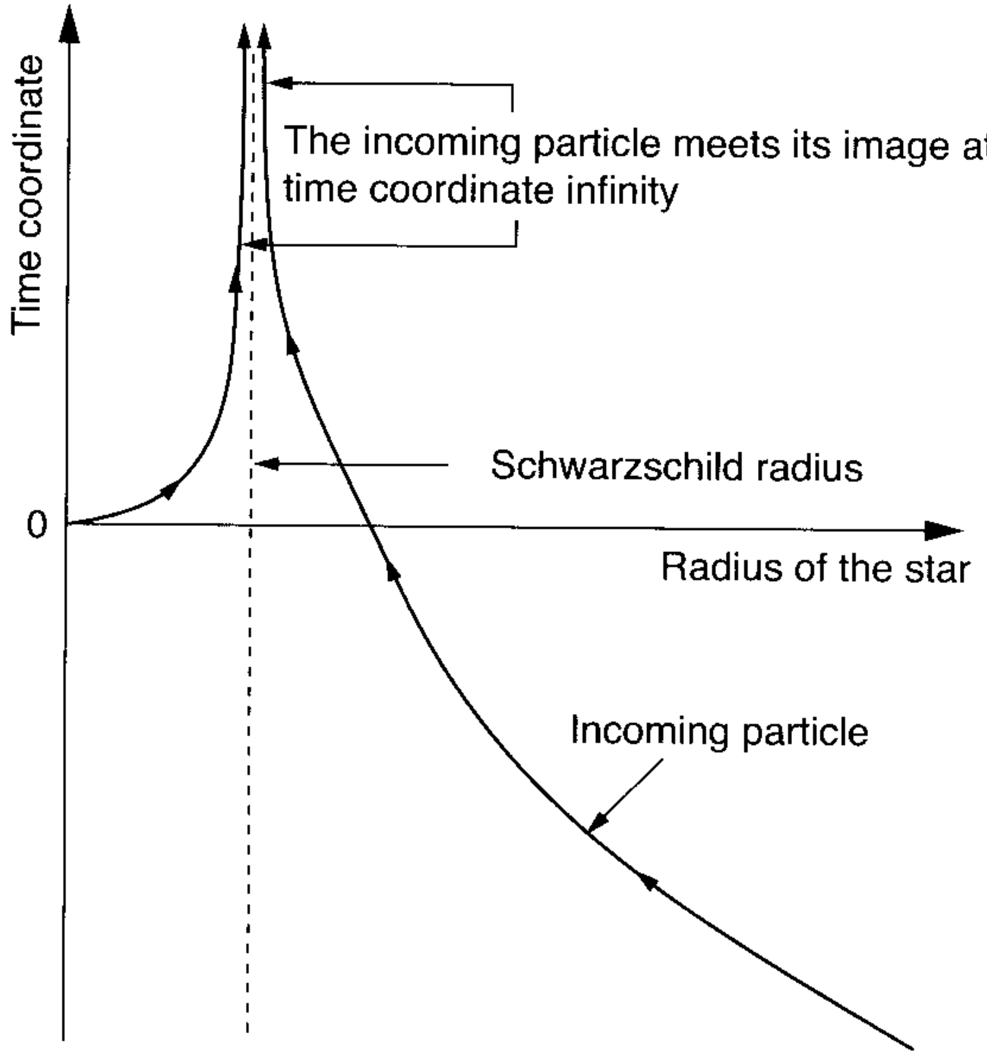
We get Robertson's form of the Schwarzschild line element,

$$ds^2 = c^2 d\tau^2 - \frac{\rho}{r} d\rho^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

From Lemaître's to Robertson's line-element



Robertson's trajectories of a Schwarzschild's field



A trajectory in Schwarzschild's space.
(from H. P. Robertson and T. W. Noonan, 1968).

Robertson approach to « r = 2m »

"The observer never sees the particle reach r = 2m, although the **particle passes r = 2m and reaches r = 0 in a finite proper time** ! [. . .] the light from the particle is redshited more and more; as the particle approaches r = 2m, z approaches ∞." (Robertson & Noonan 1968, 252).

Oppenheimer and Snyder on collapse

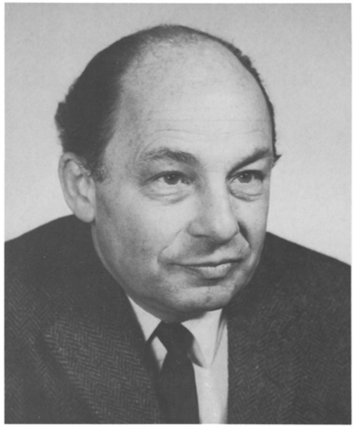
- "When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse…the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles…The total time of collapse for an observer comoving with the stellar matter is finite […] an external observer sees the star asymptotically shrinking to its gravitational radius." (Oppenheimer and Snyder 1939, 455)

Oppenheimer and Tolman, 1938

Einstein and the Schwarzschild “singularity”, 1939

"This investigation arose out of discussions the author conducted with Professor H.P. Robertson and with Drs. V. Bargmann and P. Bergmann on the mathematical and physical significance of the Schwarzschild singularity. The problem quite naturally leads to the question, answered by this paper in the negative, as to whether physical models are capable of exhibiting such a singularity."

Peter Bergmann and the Schwarzschild “singularity”, 1942



Peter G. Bergmann

Photo of Peter Bergmann (*anno* 1973) by the News Bureau of Syracuse University, furnished by courtesy of Ms. Ruth Newsholme.

Robertson came to Einstein's office and "told us that the Schwarzschild singularity (at r = 2M) might not be so bad. He used what is known as Finkelstein coordinates [...]. In these terms it takes only a finite 'time' to get inside, but 'forever' to get out. Or the converse. We thought this was important but puzzling." (Bergmann to J.E, 9 May 1986).

J. Robert Oppenheimer had to work closely with the military at Los Alamos

AMERICAN INST. PHYSICS/EMILIO SEGRE VISUAL ARCHIVES/UNITED PRESS INTERNATIONAL

Schwarzschild’s singularity becomes a black-hole horizon...

John L. Synge on singularities, 1950

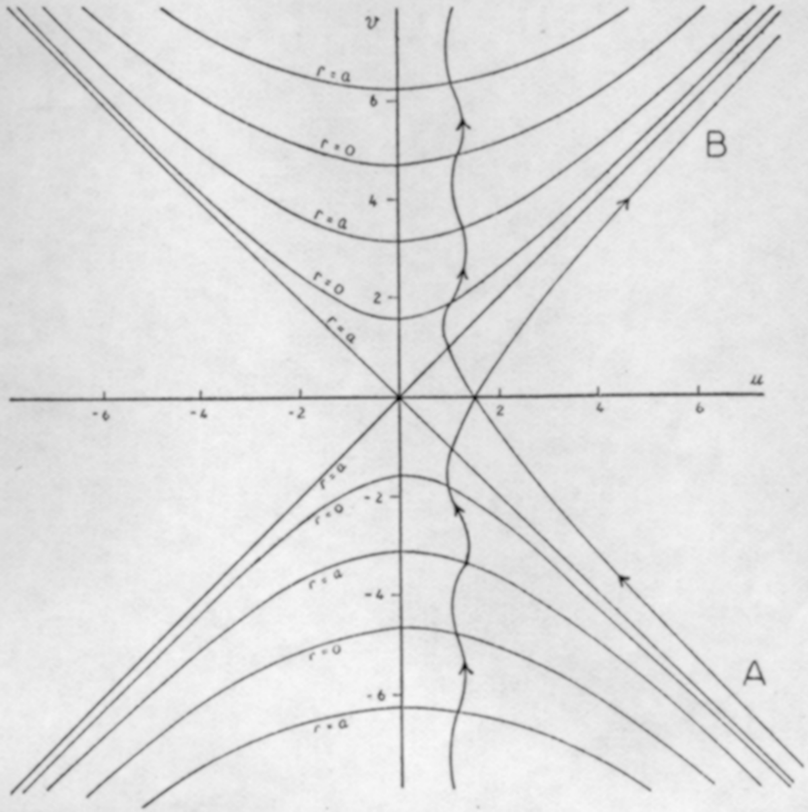


FIG. 14.—Histories of particles projected radially in from $r = \infty$ (A), and projected radially out to $r = \infty$ (B), with zero velocity at $r = \infty$ in each case.

Drawn for $a = 1$.

Synge's diagram 1950

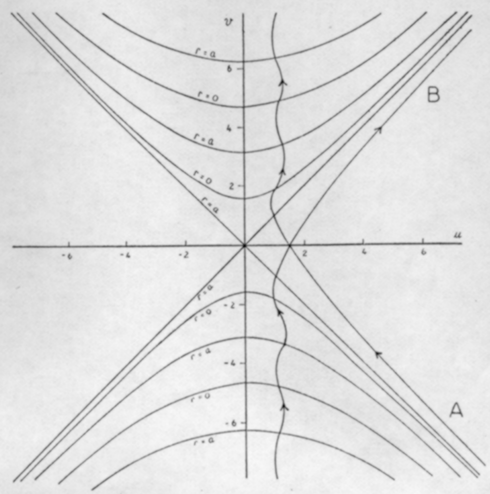


FIG. 14.—Histories of particles projected radially in from $r = \infty$ (A), and projected radially out to $r = \infty$ (B), with zero velocity at $r = \infty$ in each case.

Drawn for $a = 1$.

TABLE II. Relation of new coordinates to Schwarzschild coordinates.

New coordinates in terms of Schwarzschild coordinates	Schwarzschild coordinates in terms of new coordinates
$u = \left[\left(\frac{r}{2m^*}\right)-1\right]^{\frac{1}{2}} \exp\left(\frac{r}{4m^*}\right) \cosh\left(\frac{T}{4m^*}\right)$	$\left[\left(\frac{r}{2m^*}\right)-1\right] \exp\left(\frac{r}{2m^*}\right) = u^2 - v^2$
$v = \left[\left(\frac{r}{2m^*}\right)-1\right]^{\frac{1}{2}} \exp\left(\frac{r}{4m^*}\right) \sinh\left(\frac{T}{4m^*}\right)$	$T/4m^* = \text{arctanh}\,(v/u)$ $= \frac{1}{2}\,\text{arctanh}[2uv/(u^2+v^2)]$

$f^2 = (32m^{*3}/r)\exp(-r/2m^*) =$ a transcendental function of $(u^2 - v^2)$

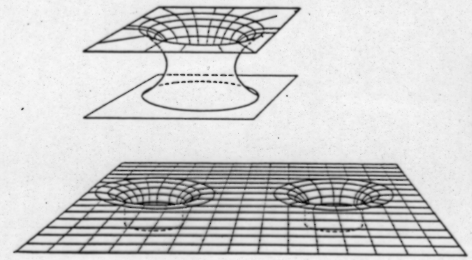


FIG. 1. Two interpretations of the 3-dimensional "maximally extended Schwarzschild metric" at the time $T=0$. Above: A connection or bridge in the sense of Einstein and Rosen between *two* otherwise Euclidean spaces. Below: A wormhole in the sense of Wheeler connecting two regions in *one* Euclidean space, in the limiting case where these regions are extremely far apart compared to the dimensions of the throat of the wormhole.

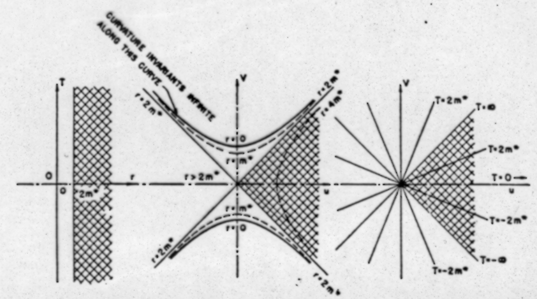


FIG. 2. Corresponding regions of the (r,T) and (u,v) planes. In the latter, curves of constant r are hyperbolas asymptotic to the lines $r=2m^*$ while T is constant on straight lines through the origin. The exterior of the singular sphere, $r>2m^*$, corresponds to the region $|v|<u$ (the hatched areas). The whole line $r=2m^*$ in the (r,T) plane corresponds to the origin $u=v=0$, while two one-dimensional families of ideal limit points with $r\to 2m^*$ and $T\to\pm\infty$ correspond to the remaining boundary points $u=|v|>0$.

In the (u,v) plane the metric is entirely regular not only in the hatched area but in the entire area between the two branches of the hyperbola $r=0$. This comprises two images of the exterior of the spherical singularity and two of its interior. (The expressions in Table II are valid in the right-hand quadrant $u>|v|$. To obtain formulas valid in the left-hand quadrant replace u and v by their negatives everywhere. To obtain formulas valid in the upper or lower quadrant replace u by $\pm v$, v by $\pm u$, and $r/2m^*-1$ by its negative everywhere. Note that the formula for r and the final formula for T remain invariant under these substitutions.) The purely radial ($d\theta=d\varphi=0$) null geodesics are lines inclined at 45°. The points with $r=2m^*$ have no local topological distinction, but still a global one: if a test particle crosses $r=2m^*$ into the interior (where r is time-like and T space-like), it can never get back out but must inevitably hit the irremovable singularity $r=0$ (curvature invariants infinite). This circumstance guarantees that one cannot violate ordinary causality in the "main universe" by sending signals via the wormhole effectively faster than light.

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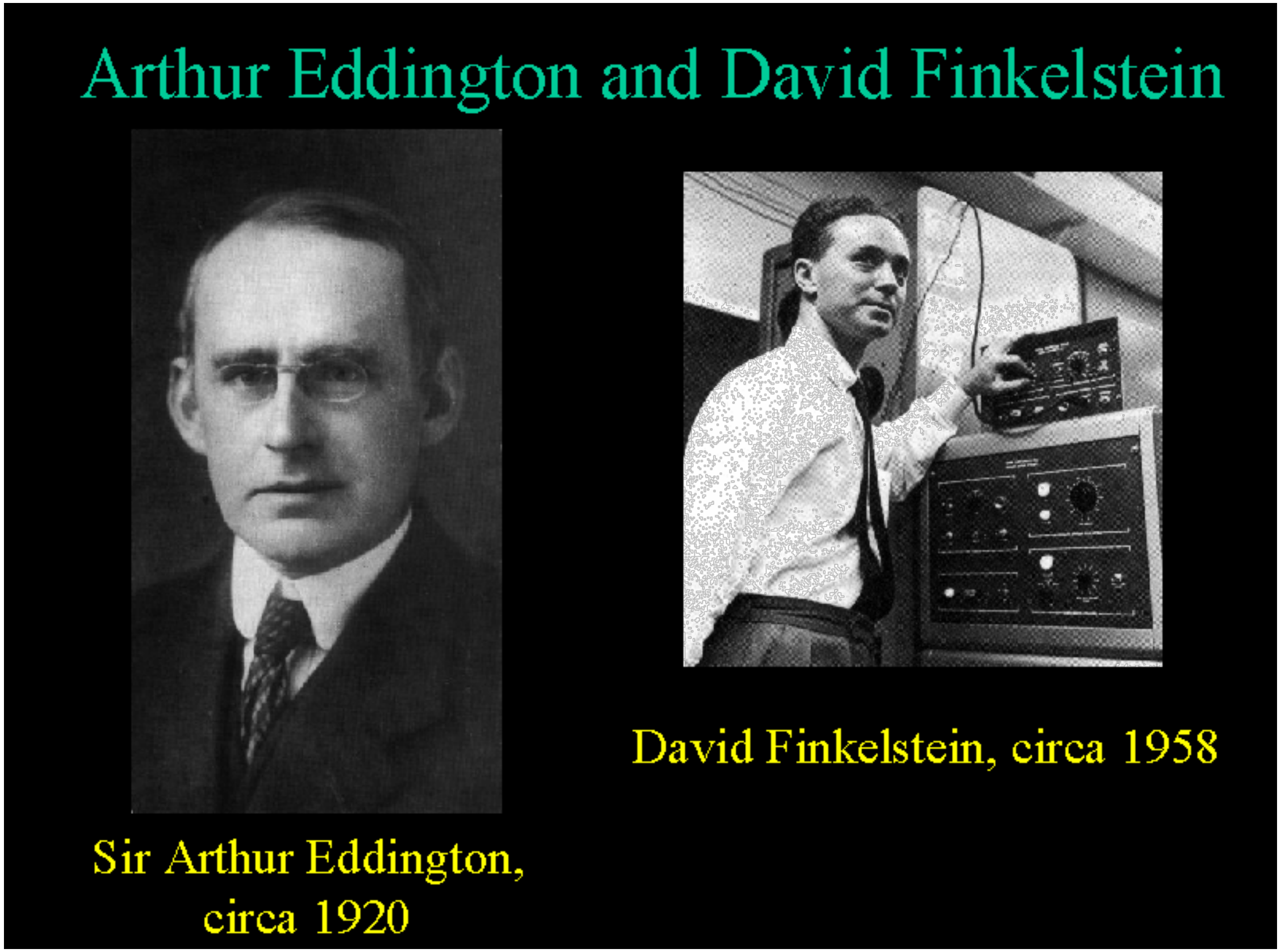
Maximal Extension of Schwarzschild Metric*

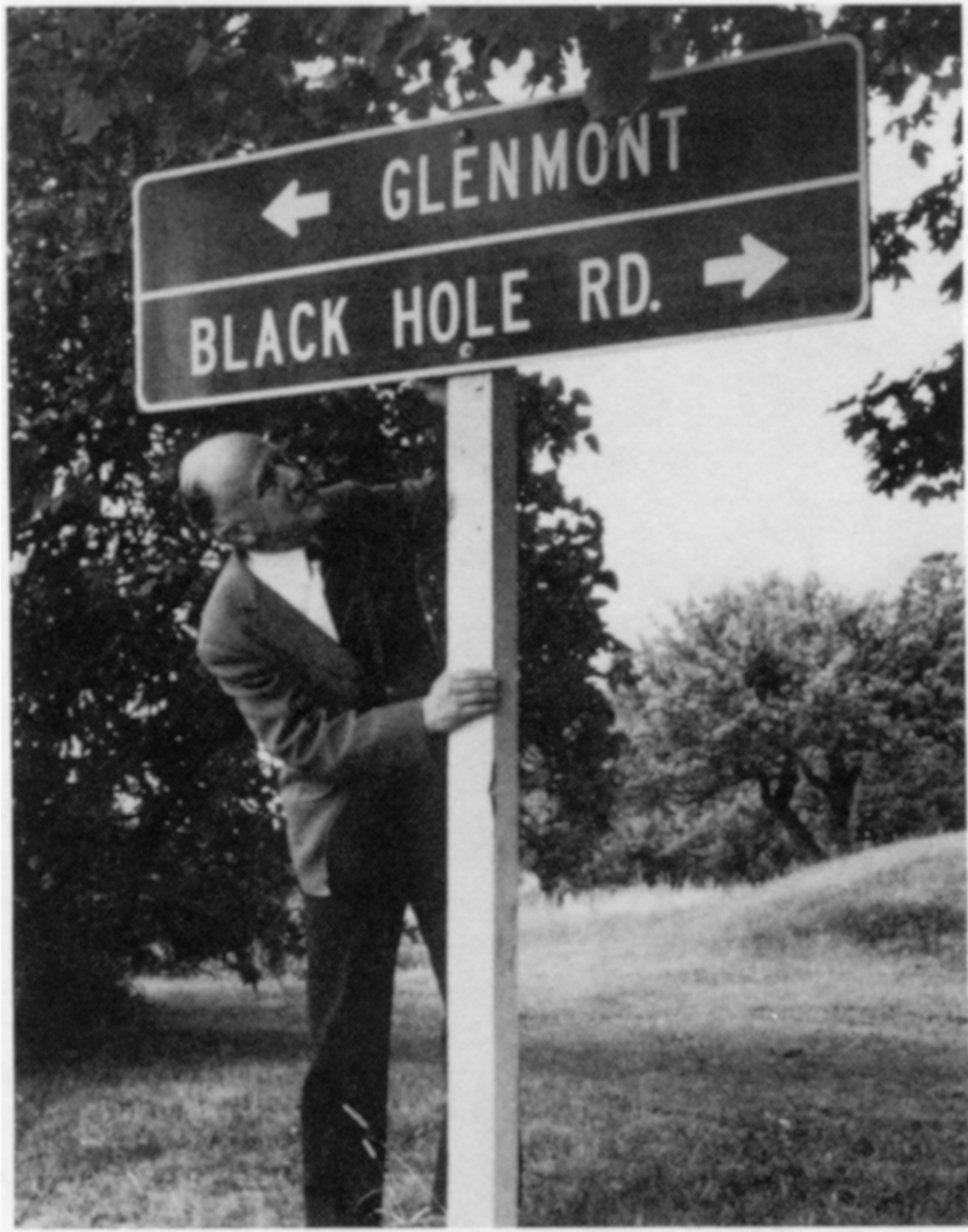
M. D. KRUSKAL†
Project Matterhorn, Princeton University, Princeton, New Jersey

(Received December 21, 1959)

Topology: Synge 1950 versus Kruskal 1960



Arthur Eddington and David Finkelstein



John A. Wheeler

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$v=\left[\left(\frac{r}{2m^*}\right)-1\right]^{\frac{1}{2}}\exp\left(\frac{r}{4m^*}\right)\sinh\left(\frac{T}{4m^*}\right)$	$T/4m^*=\text{arctanh}\,(v/u)$ $=\frac{1}{2}\,\text{arctanh}[2uv/(u^2+v^2)]$
$f^2=(32m^{*3}/r)\exp(-r/2m^*)=$ a transcendental function of (u^2-v^2)	

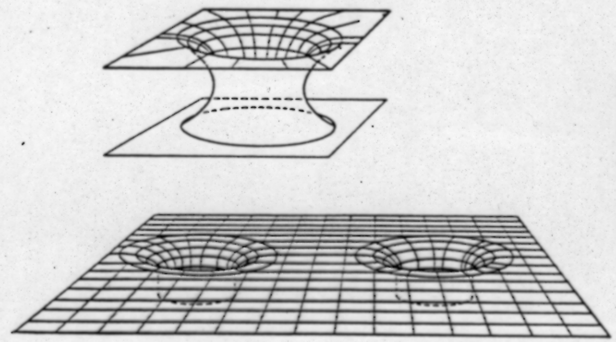

FIG. 1. Two interpretations of the 3-dimensional "maximally extended Schwarzschild metric" at the time $T=0$. Above: A connection or bridge in the sense of Einstein and Rosen between *two* otherwise Euclidean spaces. Below: A wormhole in the sense of Wheeler connecting two regions in *one* Euclidean space, in the limiting case where these regions are extremely far apart compared to the dimensions of the throat of the wormhole.

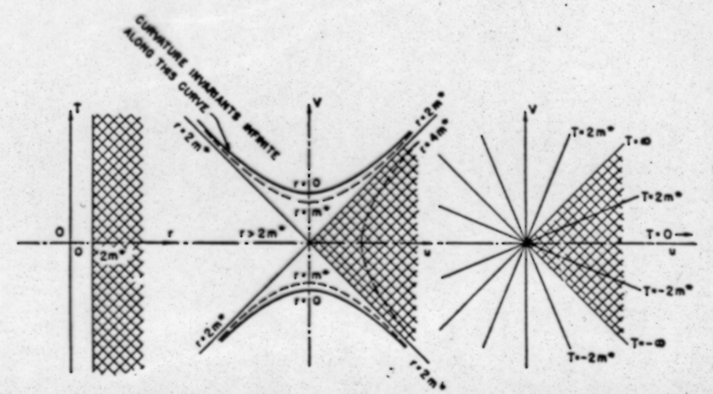

FIG. 2. Corresponding regions of the (r,T) and (u,v) planes. In the latter, curves of constant r are hyperbolas asymptotic to the lines $r=2m^*$ while T is constant on straight lines through the origin. The exterior of the singular sphere, $r>2m^*$, corresponds to the region $|v|<u$ (the hatched areas). The whole line $r=2m^*$ in the (r,T) plane corresponds to the origin $u=v=0$, while two one-dimensional families of ideal limit points with $r\rightarrow 2m^*$ and $T\rightarrow\pm\infty$ correspond to the remaining boundary points $u=|v|>0$.

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Kruskal's Extension

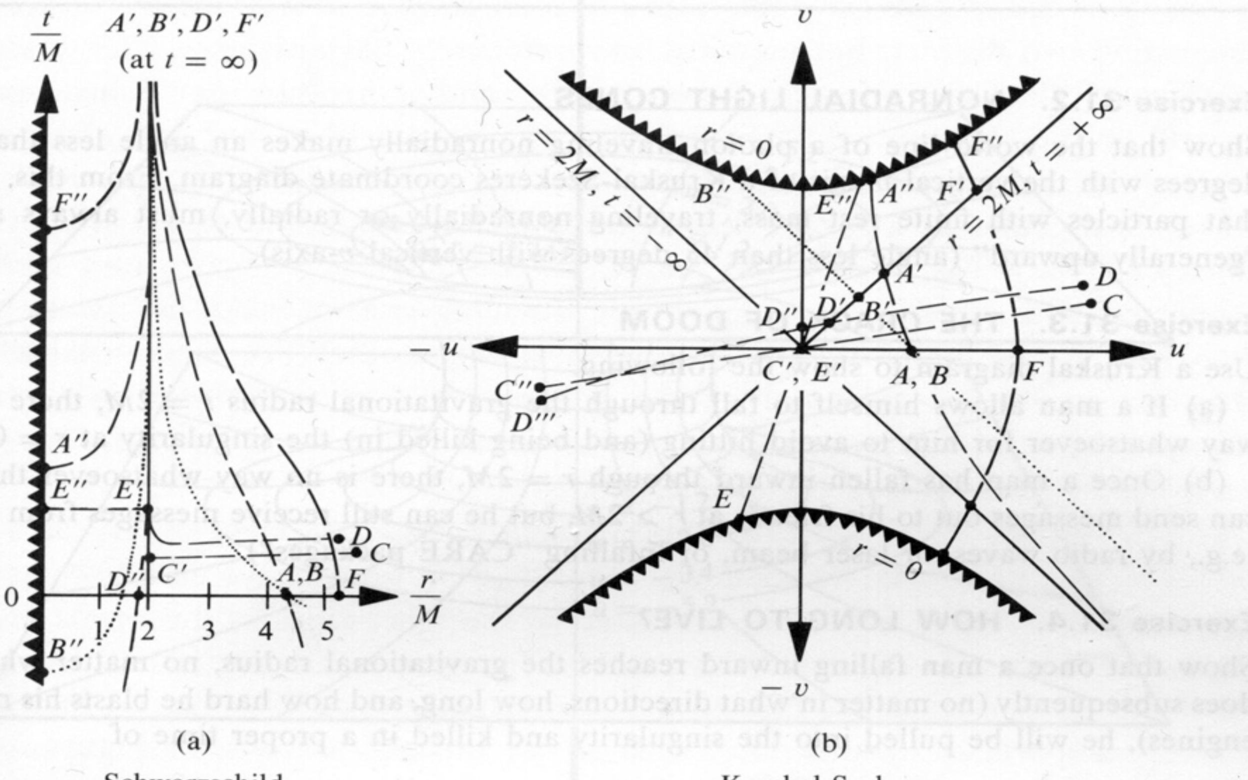
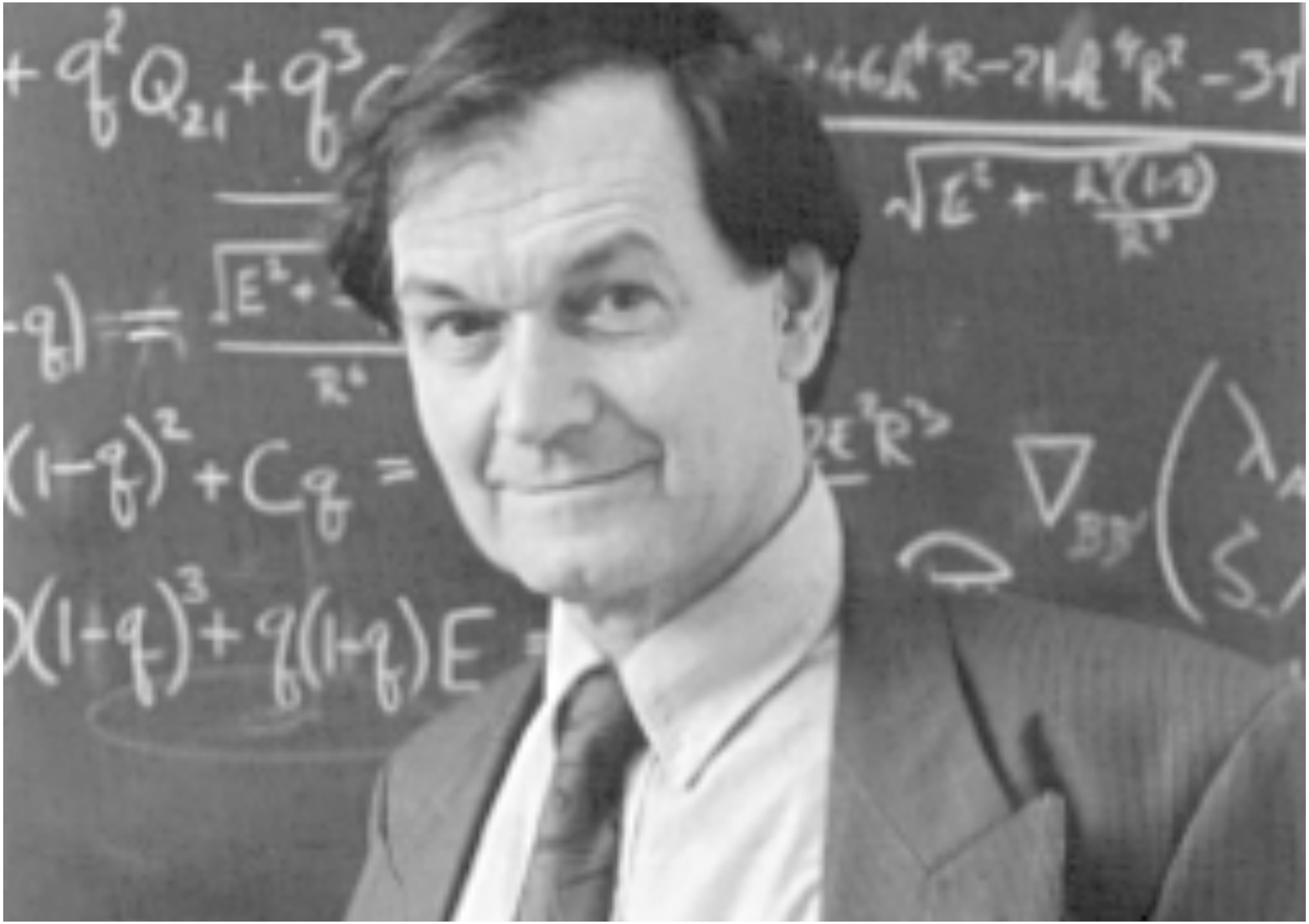


Figure 31.4.

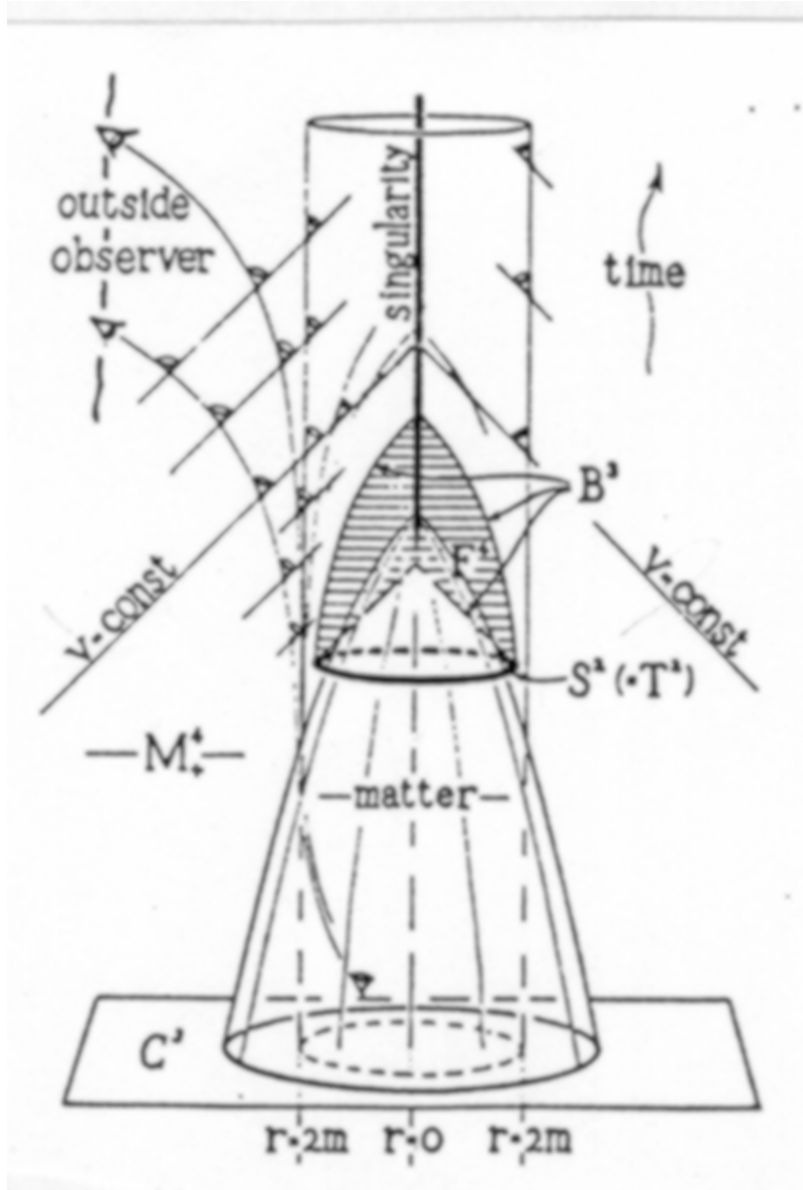
(a) Typical radial timelike (A, E, F), lightlike (B), and spacelike (C, D) geodesics of the Schwarzschild geometry, as seen in the Schwarzschild coordinate system (schematic only). This is a reproduction of Figure 31.1.

(b) The same geodesics, as seen in the Kruskal-Szekeres coordinate system, and as extended either to infinite length or to the singularity of infinite curvature at $r = 0$ (schematic only).

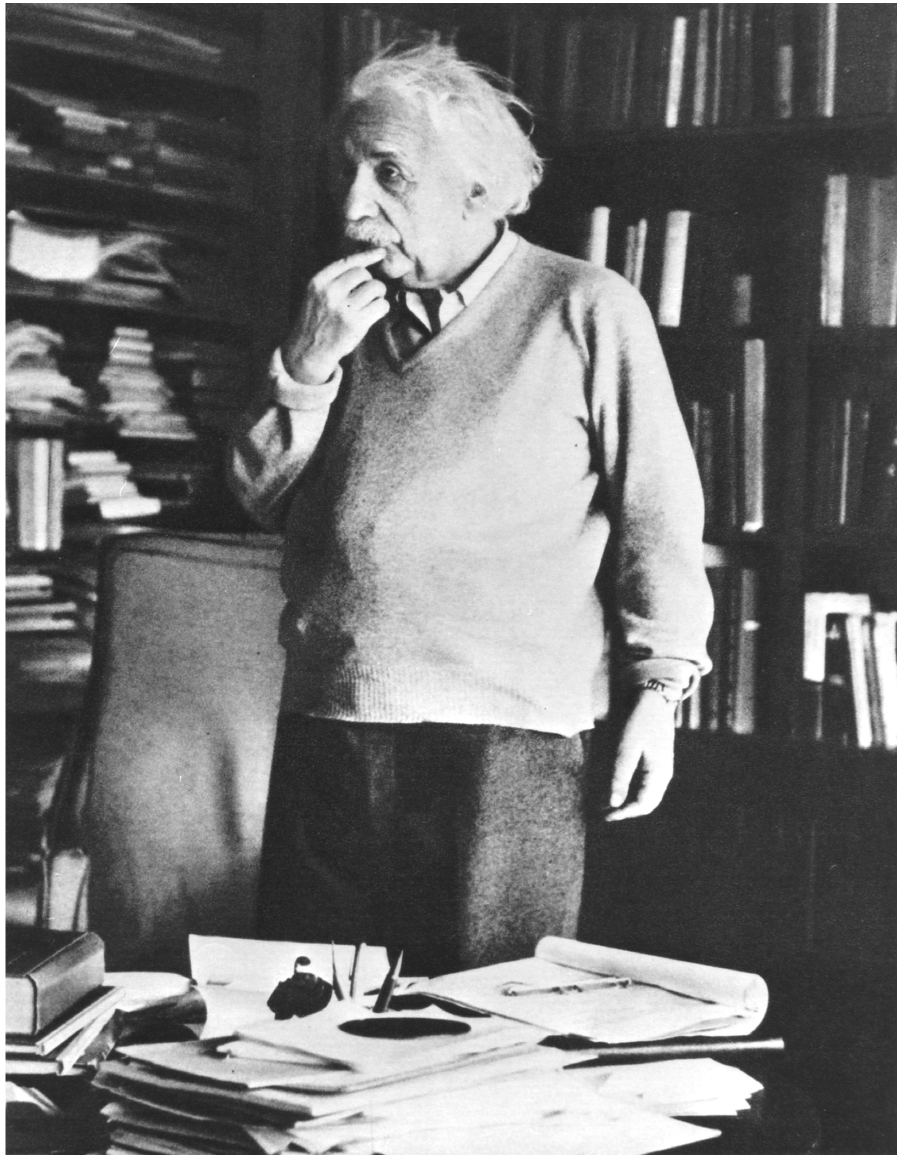
Schwarzschild, Kruskal- Szekeres diagrams (Wheeler et al, 1973)



Roger Penrose



Spherical collapse, Penrose, 1965



Albert Einstein 1954

EXPOSITION

DU SYSTÈME DU MONDE,

PAR PIERRE-SIMON LAPLACE,
de l'Institut National de France, et du Bureau des Longitudes.

TOME SECOND.

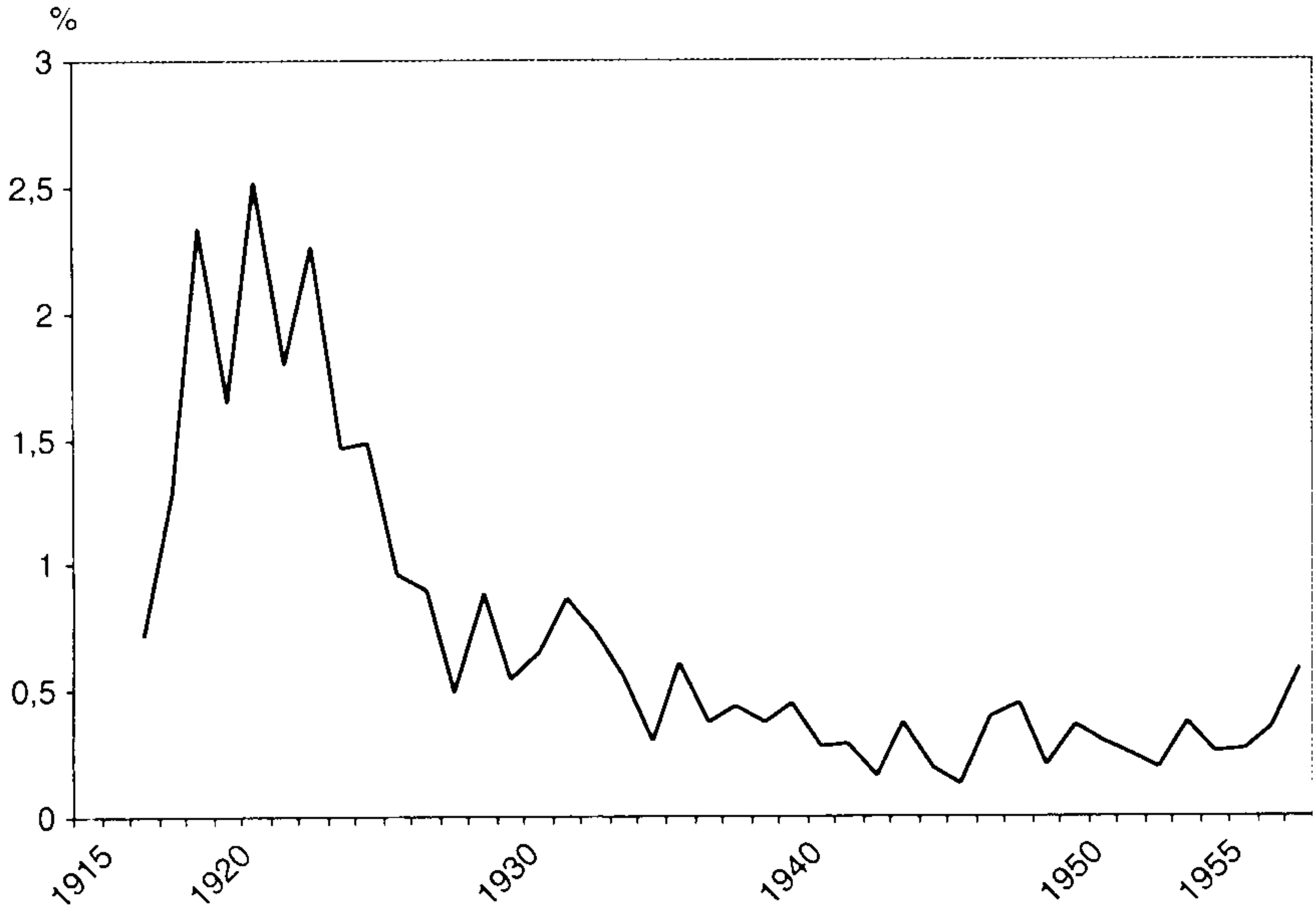
A PARIS,

De l'Imprimerie du CERCLE-SOCIAL, rue du Théâtre Français, N°. 4.

L'AN IV DE LA RÉPUBLIQUE FRANÇAISE.

Tous ces corps devenus invisibles, sont à la même place où ils ont été observés, puisqu'ils n'en ont point changé, durant leur apparition; il existe donc dans les espaces célestes, des corps obscurs aussi considérables, et peut être en aussi grand nombre, que les étoiles. Un astre lumineux de même densité que la terre, et dont le diamètre serait deux cents cinquante fois plus grand que celui du soleil, ne laisserait en vertu de son attraction, parvenir aucun de ses rayons jusqu'à nous; il est donc possible que les plus grands corps lumineux de l'univers, soient par cela même, invisibles. Une étoile qui, sans être de cette grandeur, surpasserait considérablement le soleil; affaiblirait sensiblement la vîtesse de la lumière, et augmenterait ainsi l'étendue de son aberration.

Laplace never quoted at John Michell who laid the foundations of the Newtonian theory of the propagation of light, of the action of gravitation on light and invented the "dark bodies" in 1784; all cousins of general relativity.



Number of publications in general relativity
as a percentage of the total number of publication in physics (from Science Abstract:1915-1955).

N.B. The absolute number of pubications goes from 10 in 1916 to 42 (greatest) in 1920; 4 in 1945 (lowest).

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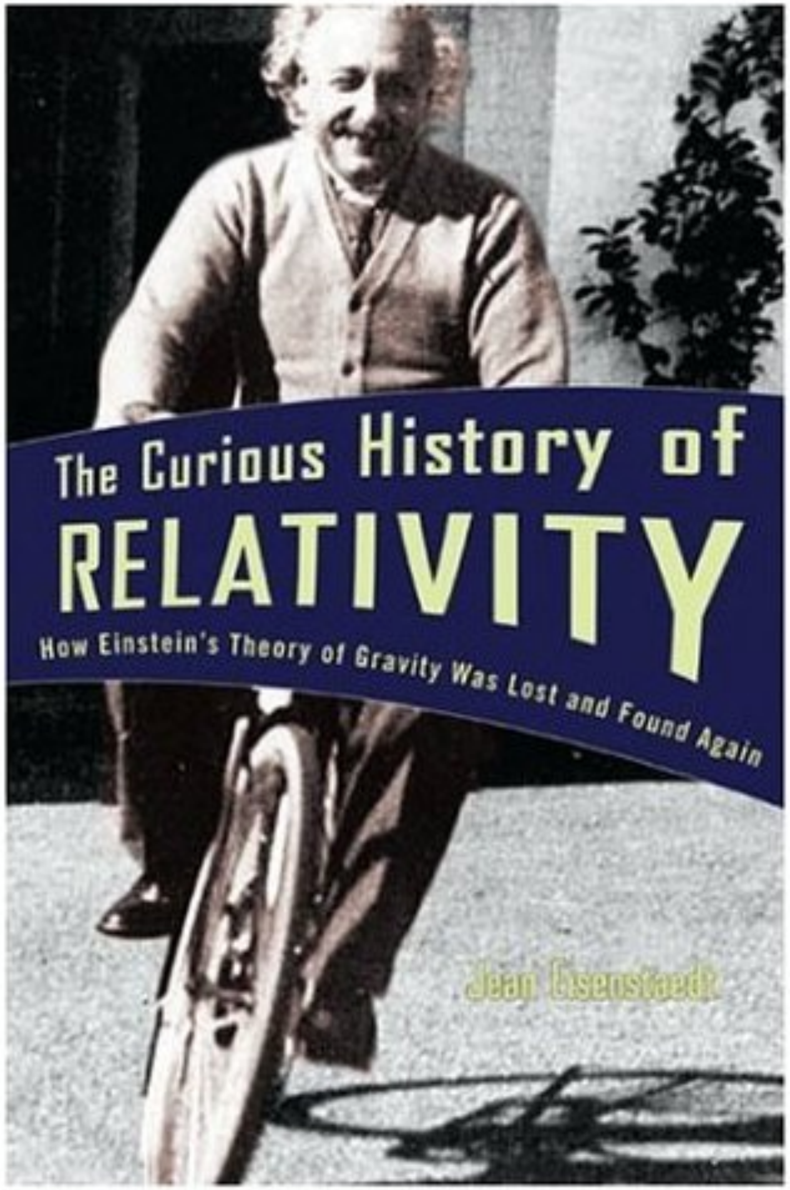

English Edition