A history of entanglement

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Basic mathematics for entanglement of pure states

- Let a compound system consists of two subsystems, with Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
- A state of the compound system is characterized is a unit vector in the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$,

  $$|\Psi\rangle = \sum_{ij} c_{ij} |\psi_i\rangle |\phi_j\rangle$$

  For arbitrary orthonormal bases $\{|\psi_i\rangle\}$ in $\mathcal{H}_1$ and $\{|\phi_j\rangle\}$ in $\mathcal{H}_2$,
- Except for the very special case that $c_{ij} = a_i b_j$, $|\Psi\rangle$ is not factorizable: there are no states $|\psi\rangle \in \mathcal{H}_1$, $|\phi\rangle \in \mathcal{H}_2$ such that

  $$|\Psi\rangle = |\psi\rangle |\phi\rangle$$

- In that case, the state $|\Psi\rangle$ is an entangled state.

- This formula, $|\Psi\rangle = |\psi\rangle|\phi\rangle$ invites a simple interpretation: the state of te compound system is such that each of the components are in states $|\psi\rangle$ and $|\phi\rangle$ respectively. For entangled states $|\Psi\rangle$ that simple interpretation is blocked.
- For any given o.n. basis $\{|\psi_i\rangle\}$ in $\mathcal{H}_1$, one can always write

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle|\phi_i\rangle$$

  but the $\{|\phi_j\rangle\}$ in $\mathcal{H}_2$ will not generally be an orthonormal basis.
- Thanks to Schmidt's biorthogonal decomposition theorem one can always find two special orthonormal bases (depending on $|\Psi\rangle$, such that $|\Psi\rangle$ takes a simpler form:

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle|\phi_i\rangle$$

- But: the choice of such bases is not unique iff from some $i, i' : |c_i| = |c_{i'}|$.
- In general there is no corresponding "triorthogonal decomposition" for systems with three or more components (but if there is one, it is always unique). (Elby & Bub, 1994)

why is the question of orthonormal bases relevant?

Only orthonormal bases are associated with values of physical quantities (observables). (eigenstate eigenvalue link)
It matters to what we want to associate to the unobserved particle: "the value of an observable of that particle?" or"just some quantum state? "

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Schrödinger 1935/6

- the term "entanglement" (*Verschränkung*) was coined by Schr̈odinger in 1935.
- within a year, he wrote three papers on the topic:

  “Die gegenwartige Situation in der Quantenmechanik” *Die Naturwissenschaften* 23 (1935) 807–812, 823–828, 844–849.

  “Discussion of probability relations between separated systems . *Proc. Cambr. Phil Soc.* 31 (1935) 555-563.

  “Probability relations between separated systems.” *Proc. Cambr. Phil Soc.* 31 (1936) 446–452.

"When two systems of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described the same way as before, viz., by endowing each of them with a representative state of its own. I would not call that one but the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or ψ-functions) have become entangled."

Entanglement before Schrödinger : EPR

- Schrödinger, of course, responded to the EPR paper of 1935.
- EPR exhibited a special entangled state with multiple biorthonormal expansions:

$$|\Psi\rangle = \int dp e^{ipa} \,|-p\rangle|p\rangle = \int dq \,|q-a\rangle|q\rangle$$

- EPR used to example to argue that the theory was incomplete.
- Einstein said that the EPR paper was "smothered in the formalism (*Gelehrsahmkeit*).

Prehistory of entanglement II

- How did previous authors look at entangled states?
- Born "Zur Quantenmechanik der Stossvorgänge" (1926)

> *If one wishes to calculate quantum mechanically the interaction of two systems, then, as is well known, one cannot, as in classical mechanics pick out a state of the one system and determine how this is influenced by a state of the other system, since all states of both systems are coupled in a complicated way. This is true also [...] in a collision [...]. Yet, there is no escape from the conclusion that before, as well as after the collision a definite state must be specifiable for the atom and likewise [...] for the electron. The problem is to formulate mathematically this asymptotic behavior of the interacting particles. I did not succeed in doing this with the matrix form of quantum mechanics but did with the Schrödinger formulation.*

Born proceeds by assuming that, initially, the two systems are in a product state

$$|\Psi\rangle_i = |\psi\rangle|\phi\rangle$$

where the incident particle state $|\phi\rangle$ is a a momentum eigenstate (i.e. a plane wave incident form the z-direction upon the target atom in energy eigenstate $|\psi\rangle$.
After the interaction, he gives the result in the form

$$|\Psi\rangle_f = \sum_i c_i|\psi_i\rangle|\phi_i\rangle$$

where the states $|\phi_i\rangle$ refer to momentum eigenstates scattered in different directions, and the target atom states $|\psi_i\rangle$ are energy eigenstates that have been (de)excited by the interaction.

He concludes:

> *"If one translates this result into terms of particles, only one interpretation is possible: c_i gives the probability* for the electron, arriving from the z-direction, to be thrown out into the direction designated by $|\phi_i\rangle$. Schrödinger's quantum mechanics therefore gives quite a definite answer to the question of the effects of the collision; but there is no answer to the question "what is the state after the collision?" but only to the question "how probable is a specified outcome of the collision?" [...].*
>
> * *Addition in proof: more careful consideration shows that the probability is proportional to $|c_i|^2$.*

Apparently, Born is well aware of the non-classical nature of entangled states. But he immediately gives an interpretation in terms of properties possessed by the components in an entangled state. Clearly he did not realize that the decomposition of entangled states could be non-unique.

1935- 1964

- This period shows very little development of the concept of entanglement.
- With one important exception: Bohm's (1951) book on QM contains a chapter on the EPR paradox that replaces the original EPR state with a singlet.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$$

- which is much simpler, and the starting point of all subsequent authors.

  It also evades some problems of EPR. we consider two particles that have interacted and are then supposed to separate to distant locations. We then contemplate making a momentum measurement on one of them, and predict the outcome of a momentum measurement on the other to assign momentum eigenstates to both. But, momentum eigenstates (plane waves) overlap. What sense would it then have to say the systems are separated?

1964 -1985 The Bell inequalities

- Bell showed that any local hidden variables theory had to obey certain experimentally accessible inequalities, which were violated by the singlet state.
- Bell's result brought a significant change: it showed that incompleteness of QM was not the core issue.
- Moreover, the experimental testability of the inequalities meant that one did not even have to assume the validity of QM.

  Most theoretical effort after Bell's initial presentation was directed at cutting away superfluous assumptions.

  Bell's original inequalities assumed that the HV theory was deterministic. The Clauser-Horne-Shimony-Holt version of his inequality avoided that assumption.

$$|\langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle| \leq 2$$

- They are violated by many entangled states, not only the singlet. But singlet does give maximal violations.

$\pm$1990-present Entanglement as a resource rather than a problem.

- A broader interest into developing the theoretical understanding of entanglement for many particles, mixed states, and higher-dimensional state spaces.
- Bell inequality violations, nonseparability and entanglement turn out as different issues
- (But Bell inequality violation $\Longleftrightarrow$ existence of local hidden variable model.

Generalization to more particles & mixed states

- Greenberger, Horne and Zeilinger (1989) were the first to discuss a three-particle entangled state.

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$$

  They showed that for such a state one could get a contradiction with local realism that did not require testing a statistical correlation inequality.

- Werner (1989) first provided a general definition mixed states: they are entangled iff

$$\rho \neq \sum_i p_i \rho_{1i} \otimes \cdots \rho_{Ni}$$

  and separable otherwise.

- He studied a class of mixed states (Werner states)

$$W_p = (1-p)\frac{1}{N}\mathbb{1} + p|GHZ\rangle\langle GHZ|$$

  These are entangled if $p > 1/3$, and violate Bell inequalities only if $p > 1/\sqrt{2}$
  Hence entanglement $\not\Longrightarrow$ BI violation.

- This is also true for 2 particles: Seevinck & J.U. obtained inequality that is satisfied by all separable quantum states (for orthogonal spin-directions)

$$\langle AB' + A'B\rangle^2 + \langle AB - A'B'\rangle^2 \leq 1$$

multipartite entanglement

- Mermin 1990 derived generalizations of the BI for N-particle states and showed that GHZ states could violate them with exponentially increasing factors.

$$|\langle M\rangle_{LHV}| \leq 2 \qquad |\langle M\rangle_{GHZ}| = 2^{(N+1)/2}$$

- Svetlichny found a similar but inequivalent set of inequalities.
- The many-particle case also brought a distinction in types of entanglement, e.g.for $N = 3$:

$$(123), \quad (12)3, \quad 1(23) \quad 2(13), \quad (1)(2)(3)$$

The Svetlichny inequalities provided a hierarchy of bound for such partial entanglement: If a state N-particle is at most k-particle entanglement:

$$\langle S\rangle \leq 2^{(k+1)2}$$

Non-separability without entanglement

Popescu-Rohrlich (1994) showed that some non-classical and non-quantum model for 2 particles with 2-dimensional state space could violate the Tsirelson bound they achieved (without violating the non-signaling condition!)

$$\langle CHSH \rangle = 4$$

Apparently, a violation of separability could have been much weirder than quantum entanglement permits.

- Bennett et al. provide example of an observable for 2 spin-1 particles for which all eigenstates are separable (non-entangled) but that is nevertheless not a product-observable.
- Hence, it is impossible to determine a value of this observable by local measurements. Again: violation of separability principle without invoking entanglement.