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Readings

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Do we really understand quantum mechanics? Strange correlations, paradoxes and theorems.

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Abstract

This article presents a general discussion of several aspects of our present understanding of quantum mechanics. The emphasis is put on the very special correlations that this theory makes possible: they are forbidden by very general arguments based on realism and local causality. In fact, these correlations are completely impossible in any circumstance, except the very special situations designed by physicists especially to observe these purely quantum effects. Another general point that is emphasized is the necessity for the theory to predict the emergence of a single result in a single realization of an experiment. For this purpose, orthodox quantum mechanics introduces a special postulate: the reduction of the state vector, which comes in addition to the Schrödinger evolution postulate. Nevertheless, the presence in parallel of two evolution processes of the same object (the state vector) may be a potential source for conflicts; various attitudes that are possible to avoid this problem are discussed in this text. After a brief historical introduction, recalling how the very special status of the state vector has emerged in quantum mechanics, various conceptual difficulties are introduced and discussed. The Einstein Podolsky Rosen (EPR) theorem is presented with the help of a botanical parable, in a way that emphasizes how deeply the EPR reasoning is rooted into what is often called “scientific method”. In another section the GHZ argument, the Hardy impossibilities, as well as the BKS theorem are introduced in simple terms. The final two sections attempt to give a summary of the present situation: one section discusses non-locality and entanglement as we see it presently, with brief mention of recent experiments; the last section contains a (non-exhaustive) list of various attitudes that are found among physicists, and that are helpful to alleviate the conceptual difficulties of quantum mechanics.

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Quantum mechanics describes physical systems through a mathematical object, the state vector $|\Psi\rangle$, which replaces positions and velocities of classical mechanics. This is an enormous change, not only mathematically, but also conceptually. The relations between $|\Psi\rangle$ and physical properties are much less direct than in classical mechanics; the distance between the formalism and the experimental predictions leaves much more room for discussions about the interpretation of the theory. Actually, many difficulties encountered by those who tried (or are still trying) to “really understand” quantum mechanics are related to questions pertaining to the exact status of $|\Psi\rangle$: for instance, does it describe the physical reality itself, or only some partial knowledge that we might have of this reality? Does it fully describe ensemble of systems only (statistical description), or one single system as well (single events)? Assume that, indeed, $|\Psi\rangle$ is affected by an imperfect knowledge of the system; is it then not natural to expect that a better description should exist, at least in principle? If so, what would be this deeper and more precise description of the reality?

Another confusing feature of $|\Psi\rangle$ is that, for systems extended in space (for instance, a system made of two particles at very different locations), it gives an overall description of all its physical properties in a single block from which the notion of space seems to have disappeared; in some cases, the physical properties of the two remote particles seem to be completely “entangled” (the word was introduced by Schrödinger in the early days of quantum mechanics) in a way where the usual notions of space-time and local events seem to become dimmed. Of course, one could think

that this entanglement is just an innocent feature of the formalism with no special consequence: for instance, in classical electromagnetism, it is often convenient to introduce a choice of gauge for describing the fields in an intermediate step, but we know very well that gauge invariance is actually fully preserved at the end. But, and as we will see below, it turns out that the situation is different in quantum mechanics: in fact, a mathematical entanglement in $|\Psi\rangle$ can indeed have important physical consequences on the result of experiments, and even lead to predictions that are, in a sense, contradictory with locality (we will see below in what sense).

Without any doubt, the state vector is a rather curious object to describe reality; one purpose of this article is to describe some situations in which its use in quantum mechanics leads to predictions that are particularly unexpected. As an introduction, and in order to set the stage for this discussion, we will start with a brief historical introduction, which will remind us of the successive steps from which the present status of $|\Psi\rangle$ emerged. Paying attention to history is not inappropriate in a field where the same recurrent ideas are so often rediscovered; they appear again and again, sometimes almost identical over the years, sometimes remodelled or rephrased with new words, but in fact more or less unchanged. Therefore, a look at the past is not necessarily a waste of time!

1 Historical perspective

The founding fathers of quantum mechanics had already perceived the essence of many aspects of the discussions on quantum mechanics; today, after almost a century, the discussions are still lively and, if some very interesting new aspects have emerged, at a deeper level the questions have not changed so much. What is more recent, nevertheless, is a general change of attitude among physicists: until about 20 years ago, probably as a result of the famous discussions between Bohr, Einstein, Schrödinger, Heisenberg, Pauli, de Broglie and others (in particular at the famous Solvay meetings [1]), most physicists seemed to consider that “Bohr was right and proved his opponents to be wrong”, even if this was expressed with more nuance. In other words, the majority of physicists thought that the so called “Copenhagen interpretation” had clearly emerged from the infancy of quantum mechanics as the only sensible attitude for good scientists. As we all know, this interpretation introduced the idea that modern physics must contain indeterminacy as an essential ingredient: it is fundamentally impossible to predict the outcome of single microscopical events; it is impossible to go beyond the formalism of the wave function (or its equivalent, the state vector $|\Psi\rangle$ ¹) and complete it; for some physicists, the Copenhagen interpretation also includes the diffi-

¹In all this article, we will not make any distinction between the words “wave function” and “state vector”.

cult notion of “complementarity” even if it is true that, depending on the context, complementarity comes in many varieties and has been interpreted in many different ways! By and large, the impression of the vast majority was that Bohr had eventually won the debate with Einstein, so that discussing again the foundations of quantum mechanics after these giants was pretentious, useless, and maybe even bad taste.

Nowadays, the attitude of physicists is much more moderate concerning these matters, probably partly because the community has better realized the non-relevance of the “impossibility theorems” put forward by the defenders of the Copenhagen orthodoxy, in particular by Von Neumann [2] (see [3], [4] and [5], as well as the discussion given in [6]); another reason is, of course, the great impact of the discoveries and ideas of J. Bell [7]. At the turn of the century, it is probably fair to say that we are no longer sure that the Copenhagen interpretation is the only possible consistent attitude for physicists - see for instance the doubts expressed in [8]. Alternative points of view are considered as perfectly consistent: theories including additional variables (or “hidden variables”²) [9] [10]; modified dynamics of the state vector [4] [11] [12] [13] (non-linear and/or stochastic evolution) ; at the other extreme we have points of view such as the so called “many worlds interpretation” (or multibranching universe interpretation) [14], or more recently other interpretations such as that of “decoherent histories” [15] (the list is non-exhaustive). All these interpretations will be discussed in § 6. For a recent review containing many references, see [16], which emphasizes additional variables, but which is also characteristic of the variety of positions among contemporary scientists³, as well as an older but very interesting debate published in *Physics Today* [17]; another very useful source of older references is the 1971 *AJP* “Resource Letter” [18]. But recognizing this variety of positions should not be the source of misunderstandings! It should also be emphasized very clearly that, until now, no new fact whatsoever (or no new reasoning) has appeared that has made the Copenhagen interpretation obsolete in any sense.

1.1 Three periods

Three successive periods may be distinguished in the history of the elaboration of the fundamental quantum concepts; they have resulted in the point of view that we may call “the orthodox interpretation”, with all provisos that have just been made above. Here we give only a brief historical summary, but we refer the reader who would like to know more about the history of the

²As we discuss in more detail in § 6.2, we prefer to use the words “additional variables” since they are not hidden, but actually appear directly in the results of measurements; what is actually hidden in these theories is rather the wave function itself, since it evolves independently of these variables and can never be measured directly.

³It is amusing to contrast the titles of refs. [8] and [16].

conceptual development of quantum mechanics to the book of Jammer [19]; see also [20]; for detailed discussions of fundamental problems in quantum mechanics, one could also look for references such as [21] [22] [8] or those given in [18].

1.1.1 Prehistory

Planck's name is obviously the first that comes to mind when one thinks about the birth of quantum mechanics: he is the one who introduced the famous constant h , which now bears his name, even if his method was phenomenological. His motivation was actually to explain the properties of the radiation in thermal equilibrium (black body radiation) by introducing the notion of finite grains of energy in the calculation of the entropy, later interpreted by him as resulting from discontinuous exchange between radiation and matter. It is Einstein who, later, took the idea more seriously and really introduced the notion of quantum of light (which would be named "photon" much later), in order to explain the wavelength dependence of the photoelectric effect- for a general discussion of the many contributions of Einstein to quantum theory, see [23].

One should nevertheless realize that the most important and urgent question at the time was not so much to explain fine details of the properties of radiation-matter interaction, or the peculiarities of the blackbody radiation; it was, rather, to understand the origin of the stability of atoms, that is of all matter which surrounds us and of which we are made! Despite several attempts, explaining why atoms do not collapse almost instantaneously was still a complete challenge in physics. One had to wait a little bit more, until Bohr introduced his celebrated atomic model, to see the appearance of the first elements allowing to treat the question. He proposed the notion of "quantized permitted orbits" for electrons, as well as that of "quantum jumps" to describe how they would go from one orbit to another, during radiation emission processes for instance. To be fair, we must concede that these notions have now almost disappeared from modern physics, at least in their initial forms; quantum jumps are replaced by a much more precise theory of spontaneous emission in quantum electrodynamics. But, on the other hand, one may also see a resurgence of the old quantum jumps in the modern use of the postulate of the wave packet reduction. After Bohr, came Heisenberg who introduced the theory that is now known as "matrix mechanics", an abstract intellectual construction with a strong philosophical component, sometimes close to positivism; the classical physical quantities are replaced by "observables", mathematically matrices, defined by suitable postulates without much help of the intuition. Nevertheless, matrix mechanics contained many elements which turned out to be building blocks of modern quantum mechanics!

In retrospect, one can be struck by the very abstract and somewhat

mysterious character of atomic theory at this period of history; why should electrons obey such rules which forbid them to leave a given class of orbits, as if they were miraculously guided on simple trajectories? What was the origin of these quantum jumps, which were supposed to have no duration at all, so that it would make no sense to ask what were the intermediate states of the electrons during such a jump? Why should matrices appear in physics in such an abstract way, with no apparent relation with the classical description of the motion of a particle? One can guess how relieved many physicists felt when another point of view emerged, a point of view which looked at the same time much simpler and in the tradition of the physics of the 19th century: the undulatory (or wave) theory.

1.1.2 The undulatory period

It is well known that de Broglie was the first who introduced the idea of associating a wave with every material particle; this was soon proven to be correct by Davisson and Germer in their famous electron diffraction experiment. Nevertheless, for some reason, at that time de Broglie did not proceed much further in the mathematical study of this wave, so that only part of the veil of mystery was raised by him (see for instance the discussion in [24]). It is sometimes said that Debye was the first who, after hearing about de Broglie's ideas, remarked that in physics a wave generally has a wave equation: the next step would then be to try and propose an equation for this new wave. The story adds that the remark was made in the presence of Schrödinger, who soon started to work on this program; he successfully and rapidly completed it by proposing the equation which now bears his name, one of the most basic equations of all physics. Amusingly, Debye himself does not seem to have remembered the event. The anecdote may not be accurate; in fact, different reports about the discovery of this equation have been given and we will probably never know exactly what happened. What remains clear anyway is that the introduction of the Schrödinger equation is one of the essential milestones in the history of physics. Initially, it allowed one to understand the energy spectrum of the hydrogen atom, but we now know that it also gives successful predictions for all other atoms, molecules and ions, solids (the theory of bands for instance), etc. It is presently the major basic tool of many branches of modern physics and chemistry.

Conceptually, at the time of its introduction, the undulatory theory was welcomed as an enormous simplification of the new mechanics; this is particularly true because Schrödinger and others (Dirac, Heisenberg) promptly showed how it allowed one to recover the predictions of the complicated matrix mechanics from more intuitive considerations on the properties of the newly introduced "wave function" - the solution of the Schrödinger equation. The natural hope was then to be able to extend this success, and to simplify all problems raised by the mechanics of atomic particles: one would replace

it by a mechanics of waves, which would be analogous to electromagnetic or sound waves. For instance, Schrödinger thought initially that all particles in the universe looked to us like point particles just because we observe them at a scale which is too large; in fact, they are tiny “wave packets” which remain localized in small regions of space. He had even shown that these wave packets remain small (they do not spread in space) when the system under study is a harmonic oscillator... alas, we now know that this is only one of the very few special cases where this is true; in general, they do constantly spread in space!

1.1.3 Emergence of the Copenhagen interpretation

It did not take a long time before it became clear that the undulatory theory of matter also suffers from very serious difficulties, actually so serious that physicists were soon led to abandon it. A first example of difficulty is provided by a collision between particles, where the Schrödinger wave spreads in all directions, exactly as the water wave stirred in a pond by a stone thrown into it; but, in all collision experiments, particles are observed to follow well-defined trajectories which remain perfectly localized, going in some precise direction. For instance, every photograph taken in the collision chamber of a particle accelerator shows very clearly that particles never get “diluted” in all space! This remark stimulated the introduction, by Born, of the probabilistic interpretation of the wave function. Another difficulty, even more serious, arises as soon as one considers systems made of more than one single particle: then, the Schrödinger wave is no longer an ordinary wave since, instead of propagating in normal space, it propagates in the so called “configuration space” of the system, a space which has $3N$ dimensions for a system made of N particles! For instance, already for the simplest of all atoms, the hydrogen atom, the wave which propagates in 6 dimensions (if spins are taken into account, four such waves propagate in 6 dimensions); for a macroscopic collection of atoms, the dimension quickly becomes an astronomical number. Clearly the new wave was not at all similar to classical waves, which propagate in ordinary space; this deep difference will be a sort of Leitmotiv in this text⁴, reappearing under various aspects here and there⁵.

In passing, and as a side remark, it is amusing to notice that the recent observation of the phenomenon of Bose-Einstein condensation in dilute

⁴For instance, the non-locality effects occurring with two correlated particles can be seen as a consequence of the fact that the wave function propagates locally, but in a 6 dimension space, while the usual definition of locality refers to ordinary space which has 3 dimensions.

⁵One should probably mention at this point that quantum mechanics can indeed be formulated in a way which does not involve the configuration space, but just the ordinary space: the formalism of field operators (sometimes called second quantization for historical reasons). The price to pay, however, is that the wave function (a complex number) is then replaced by an operator, so that any analogy with a classical field is even less valid.

gases [25] can be seen, in a sense, as a sort of realization of the initial hope of Schrödinger: this condensation provides a case where the many-particle matter wave does propagate in ordinary space. Before condensation takes place, we have the usual situation: the atoms belong to a degenerate quantum gas, which has to be described by wave functions defined in a huge configuration space. But, when they are completely condensed, they are restricted to a much simpler many-particle state that can be described by the same wave function, exactly as a single particle. In other words, the matter wave becomes similar to a classical field with two components (the real part and the imaginary part of the wave function), resembling an ordinary sound wave for instance. This illustrates why, somewhat paradoxically, the “exciting new states of matter” provided by Bose-Einstein condensates are not an example of an extreme quantum situation; they are actually more classical than the gases from which they originate (in terms of quantum description, interparticle correlations, etc.). Conceptually, of course, this remains a very special case and does not solve the general problem associated with a naive view of the Schrödinger waves as real waves.

The purely undulatory description of particles has now disappeared from modern quantum mechanics. In addition to Born and Bohr, Heisenberg [26], Jordan, Dirac [27] and others played an essential role in the appearance of a new formulation of quantum mechanics [20], where probabilistic and undulatory notions are incorporated in a single complex logical edifice. The now classical Copenhagen interpretation of quantum mechanics (often also called “orthodox interpretation”) incorporates both a progressive, deterministic, evolution of the wave function/state vector according to the Schrödinger equation, as well as a second postulate of evolution that is often called the “wave packet reduction” (or also “wave function collapse”). The Schrödinger equation in itself does not select precise experimental results, but keeps all of them as potentialities in a coherent way; forcing the emergence of a single result in a single experiment is precisely the role of the postulate of the wave packet reduction. In this scheme, separate postulates and equations are therefore introduced, one for the “natural” evolution of the system, another for measurements performed on it.

1.2 The status of the state vector

With two kinds of evolution, it is no surprise if the state vector should get, in orthodox quantum theory, a non-trivial status - actually it has no equivalent in all the rest of physics.

1.2.1 Two extremes and the orthodox solution

Two opposite mistakes should be avoided, since both “miss the target” on different sides. The first is to endorse the initial hopes of Schrödinger and

to decide that the (many-dimension) wave function directly describes the physical properties of the system. In such a purely undulatory view, the position and velocities of particles are replaced by the amplitude of a complex wave, and the very notion of point particle becomes diluted; but the difficulties introduced by this view are now so well known - see discussion in the preceding section - that few physicists seem to be tempted to support it. Now, by contrast, it is surprising to hear relatively often colleagues falling to the other extreme, and endorsing the point of view where the wave function does not attempt to describe the physical properties of the system itself, but just the information that we have on it - in other words, the wave function should get a relative (or contextual) status, and become analogous to a classical probability distribution in usual probability theory. Of course, at first sight, this would bring a really elementary solution to all fundamental problems of quantum mechanics: we all know that classical probabilities undergo sudden jumps, and nobody considers this as a special problem. For instance, as soon as a new information becomes available on any system to us, the probability distribution that we associate with it changes suddenly; is this not the obvious way to explain the sudden wave packet reduction?

One first problem with this point of view is that it would naturally lead to a relative character of the wave function: if two observers had different information on the same system, should they use different wave functions to describe the same system⁶? In classical probability theory, there would be no problem at all with “observer-dependent” distribution probabilities, but standard quantum mechanics clearly rejects this possibility: it certainly does not attribute such a character to the wave function⁷. Moreover, when in ordinary probability theory a distribution undergoes a sudden “jump” to a more precise distribution, the reason is simply that more precise values of the variables already exist - they actually existed before the jump. In other words, the very fact that the probability distribution reflected our imperfect knowledge implies the possibility for a more precise description, closer to the reality of the system itself. But this is in complete opposition with orthodox quantum mechanics, which negates the very idea of a better description of the reality than the wave function. In fact, introducing the notion of pre-existing values is precisely the basis of unorthodox theories with additional variables (hidden variables)! So the advocates of this “information inter-

⁶Here we just give a simplified discussion; in a more elaborate context, one would introduce for instance the notion of intersubjectivity, etc. [8] [21].

⁷We implicitly assume that the two observers use the same space-time referential; otherwise, one should apply simple mathematical transformations to go from one state vector to the other. But this has no more conceptual impact than the transformations which allow us, in classical mechanics, to transform positions and conjugate momenta.

We should add that there is also room in quantum mechanics for classical uncertainties arising from an imperfect knowledge of the system; the formalism of the density operator is a convenient way to treat these uncertainties. Here, we intentionally limit ourselves to the discussion of wave functions (pure states).

pretation”⁸ are often advocates of additional variables (often called hidden variables - see § 6.2 and note 2), without being aware of it! It is therefore important to keep in mind that, in the classical interpretation of quantum mechanics, the wave function (or state vector) gives THE ultimate physical description of the system, with all its physical properties; it is neither contextual, nor observer dependent; if it gives probabilistic predictions on the result of future measurements, it nevertheless remains inherently completely different from an ordinary classical distribution of probabilities.

If none of these extremes is correct, how should we combine them? To what extent should we consider that the wave function describes a physical system itself (realistic interpretation), or rather that it contains only the information that we may have on it (positivistic interpretation), presumably in some sense that is more subtle than a classical distribution function? This is not an easy question, and various authors answer the question with different nuances; we will come back to it question in § 2.2, in particular in the discussion of the “Schrödinger cat paradox”. Even if it not so easy to be sure about what the perfectly orthodox interpretation is, we could probably express it by quoting Peres [29]: “a state vector is not a property of a physical system, but rather represents an experimental procedure for preparing or testing one or more physical systems”; we could then add another quotation from the same article, as a general comment: “quantum theory is incompatible with the proposition that measurements are processes by which we discover some unknown and preexisting property”. In this context, a wave function is an absolute representation, but of a preparation procedure rather than of the isolated physical system itself; nevertheless, but, since this procedure may also imply some information on the system itself (for instance, in the case of repeated measurements of the same physical quantity), we have a sort of intermediate situation where none of the answers above is completely correct, but where they are combined in a way that emphasizes the role of the whole experimental setup.

1.2.2 An illustration

Just as an illustration of the fact that the debate is not closed, we take a quotation from a recent article [30] which, even if taken out of its context, provides an interesting illustration of the variety of nuances that can exist within the Copenhagen interpretation (from the context, it seems clear that the authors adhere to this interpretation); after criticizing erroneous claims of colleagues concerning the proper use of quantum concepts, they write: “(One) is led astray by regarding state reductions as physical processes, rather than accepting that they are nothing but mental processes”. The authors do not expand much more on this sentence, which they relate on

⁸Normally, in physics, information (or probabilities) is about something! (meaning about something which has an independent reality, see for instance § VII of [28]).

a “minimalistic interpretation of quantum mechanics”; actually they even give a general warning that it is dangerous to go beyond it (“Van Kampen’s caveat”). Nevertheless, let us try to be bold and to cross this dangerous line for a minute; what is the situation then? We then see that two different attitudes become possible, depending on the properties that we attribute to the Schrödinger evolution itself: is it also a “purely mental process”, or is it of completely different nature and associated more closely with an external reality? Implicitly, the authors of [30] seem to favor the second possibility - otherwise, they would probably have made a more general statement about all evolutions of the state vector - but let us examine both possibilities anyway. In the first case, the relation of the wave function to physical reality is completely lost and we meet all the difficulties mentioned in the preceding paragraph as well as some of the next section; we have to accept the idea that quantum mechanics has nothing to say about reality through the wave function (if the word reality even refers to any well-defined notion!). In the second case, we meet the conceptual difficulties related to the co-existence of two processes of completely different nature for the evolution of the state vector, as discussed in the next section. What is interesting is to note that Peres’s point of view (end of the preceding subsection), while also orthodox, corresponds to neither possibilities: it never refers to mental process, but just to preparation and tests on physical systems, which is clearly different; this illustrates the flexibility of the Copenhagen interpretation and the variety of ways that different physicists use to describe it.

Another illustration of the possible nuances is provided by a recent note published by the same author together with Fuchs [31] entitled “Quantum theory needs no ‘interpretation’”. These authors take explicitly a point of view where the wave function is not absolute, but observer dependent: “it is only a mathematical expression for evaluating probabilities and depends on the knowledge of whoever is doing the computing”. The wave function becomes similar to a classical probability distribution which, obviously, depends on the knowledge of the experimenter, so that several different distributions can be associated with the same physical system (if there are several observers). On the other hand, as mentioned above, associating several different wave functions with one single system is not part of what is usually called the orthodox interpretation (except, of course, for a trivial phase factor).

To summarize, the orthodox status of the wave function is indeed a subtle mixture between different, if not opposite, concepts concerning reality and the knowledge that we have of this reality. Bohr is generally considered more as a realist than a positivist or an operationalist [19]; he would probably have said that the wave function is indeed a useful tool, but that the concept of reality can not properly be defined at a microscopic level only; it has to include all macroscopic measurement apparatuses that are used to have access to microscopic information (we come back to this point in more detail

in § 3.2.3). In this context, it is understandable why he once even stated that “there is no quantum concept” [32]!

2 Difficulties, paradoxes

We have seen that, in most cases, the wave function evolves gently, in a perfectly predictable and continuous way, according to the Schrödinger equation; in some cases only (as soon as a measurement is performed), unpredictable changes take place, according to the postulate of wave packet reduction. Obviously, having two different postulates for the evolution of the same mathematical object is unusual in physics; the notion was a complete novelty when it was introduced, and still remains unique in physics, as well as the source of difficulties. Why are two separate postulates necessary? Where exactly does the range of application of the first stop in favor of the second? More precisely, among all the interactions - or perturbations- that a physical system can undergo, which ones should be considered as normal (Schrödinger evolution), which ones are a measurement (wave packet reduction)? Logically, we are faced with a problem that did not exist before, when nobody thought that measurements should be treated as special processes in physics. We learn from Bohr that we should not try to transpose our experience of everyday’s world to microscopic systems; this is fine, but where exactly is the limit between the two worlds? Is it sufficient to reply that there is so much room between macroscopic and microscopic sizes that the exact position of the border does not matter⁹?

Moreover, can we accept that, in modern physics, the “observer” should play such a central role, giving to the theory an unexpected anthropocentric foundation, as in astronomy in the middle ages? Should we really refuse as unscientific to consider isolated (unobserved) systems, because we are not observing them? These questions are difficult, almost philosophical, and we will not attempt to answer them here. Rather, we will give a few characteristic quotations, which illustrate¹⁰ various positions:

(i) Bohr (second ref. [19], page 204): “There is no quantum world.....it is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature”.

(ii) Heisenberg (same ref. page 205): “But the atoms or the elementary particles are not real; they form a world of potentialities or possibilities rather than one of things and facts”¹¹.

⁹Proponents of the orthodox interpretation often remark that one is led to the same experimental predictions, independently of the exact position of this border, so that any conflict with the experiments can be avoided.

¹⁰With, of course, the usual proviso: short quotations taken out of their context may, sometimes, give a superficial view on the position of their authors.

¹¹Later, Heisenberg took a more moderate attitude and no longer completely rejected the idea of a wave functions describing some physical reality.

(iii) Jordan (as quoted by Bell in [33]): “observations not only disturb what has to be measured, they *produce* it. In a measurement of position, the electron is forced to a decision. We compel it to assume a definite position; previously it was neither here nor there, it had not yet made its decision for a definite position.. ”.

(iv) Mermin [6], summarizing the “fundamental quantum doctrine” (orthodox interpretation): “the outcome of a measurement is brought into being by the act of measurement itself, a joint manifestation of the state of the probed system and the probing apparatus. Precisely how the particular result of an individual measurement is obtained - Heisenberg’s transition from the possible to the actual - is inherently unknowable”.

(v) Bell [34], speaking of “modern” quantum theory (Copenhagen interpretation): “it never speaks of events in the system, but only of outcomes of observations upon the system, implying the existence of external equipment¹²” (how, then, do we describe the whole universe, since there can be no external equipment in this case?).

(vi) Shimony [8]: “According to the interpretation proposed by Bohr, the change of state is a consequence of the fundamental assumption that the description of any physical phenomenon requires reference to the experimental arrangement”.

(vii) Rosenfeld [35]: “the human observer, whom we have been at pains to keep out of the picture, seems irresistibly to intrude into it,...”.

(viii) Stapp [28] “The interpretation of quantum theory is clouded by the following points: (1) Invalid classical concepts are ascribed fundamental status; (2) The process of measurement is not describable within the framework of the theory; (3) The subject-object distinction is blurred; (4) The observed system is required to be isolated in order to be defined, yet interacting to be observed”.

2.1 Von Neumann’s infinite regress

In this section, we introduce the notion of the Von Neumann regress, or Von Neumann chain, a process that is at the source of phenomenon of decoherence. Both actually correspond to the same basic physical process, but the word decoherence usually refers to its initial stage, when the number of degrees of freedom involved in the process is still relatively limited. The Von Neumann chain, on the other hand, is more general since it includes this initial stage as well as its continuation, which goes on until it reaches the other extreme where it really becomes paradoxical: the Schrödinger cat, the symbol of a macroscopic system, with an enormous number of degrees of freedom, in a impossible state (Schrödinger uses the word “ridiculous” to describe it). Decoherence in itself is an interesting physical phenomenon

¹²One could add “and of external observers”.

that is contained in the Schrödinger equation and introduces no particular conceptual problems; the word is relatively recent, and so is the observation of the process in beautiful experiments in atomic physics [36] - for more details on decoherence, see §5.3.2. Since for the moment we are at the stage of an historical introduction of the difficulties of quantum mechanics, we will not discuss microscopic decoherence further, but focus the interest on macroscopic systems, where serious conceptual difficulties do appear.

Assume that we take a simple system, such as a spin $1/2$ atom, which enters into a Stern-Gerlach spin analyzer. If the initial direction of the spin is transverse (with respect to the magnetic field which defines the eigenstates associated with the apparatus), the wave function of the atom will split into two different wave packets, one which is pulled upwards, the other pushed downwards; this is an elementary consequence of the linearity of the Schrödinger equation. Propagating further, each of the two wave packets may strike a detector, with which they interact by modifying its state as well as theirs; for instance, the incoming spin $1/2$ atoms are ionized and produce electrons; as a consequence, the initial coherent superposition now encompasses new particles. Moreover, when a whole cascade of electrons is produced in photomultipliers, all these additional electrons also become part of the superposition. In fact, there is no intrinsic limit in what soon becomes an almost infinite chain: rapidly, the linearity of the Schrödinger equation leads to a state vector which is the coherent superposition of states including a macroscopic number of particles, macroscopic currents and, maybe, pointers or recorders which have already printed zeros or ones on a piece of paper! If we stick to the Schrödinger equation, there is nothing to stop this “infinite Von Neumann regress”, which has its seed in the microscopic world but rapidly develops into a macroscopic consequence. Can we for instance accept the idea that, at the end, it is the brain of the experimenter (who becomes aware of the results) and therefore a human being, which enters into such a superposition?

Needless to say, no-one has ever observed two contradictory results at the same time, and the very notion is not even very clear: it would presumably correspond to an experimental result printed on paper looking more or less like two superimposed slides, or a double exposure of a photograph. But in practice we know that we always observe only one single result in a single experiment; linear superpositions somehow resolve themselves before they become sufficiently macroscopic to involve measurement apparatuses and ourselves. It therefore seems obvious¹³ that a proper theory should break the Von Neumann chain, and stop the regress when (or maybe before) it reaches the macroscopic world. But when exactly and how precisely?

¹³Maybe not so obvious after all? There is an interpretation of quantum mechanics that precisely rests on the idea of never breaking this chain: the Everett interpretation, which will be discussed in § 6.5.

2.2 Wigner’s friend

The question can also be asked differently: in a theory where the observer plays such an essential role, who is entitled to play it? Wigner discusses the role of a friend, who has been asked to perform an experiment, a Stern-Gerlach measurement for instance [37]; the friend may be working in a closed laboratory so that an outside observer will not be aware of the result before he/she opens the door. What happens just after the particle has emerged from the analyzer and when its position has been observed inside the laboratory, but is not yet known outside? For the outside observer, it is natural to consider the whole ensemble of the closed laboratory, containing the experiment as well as his friend, as the “system” to be described by a big wave function. As long as the door of the laboratory remains closed and the result of the measurement unknown, this wave function will continue to contain a superposition of the two possible results; it is only later, when the result becomes known outside, that the wave packet reduction should be applied. But, clearly, for Wigner’s friend who is inside the laboratory, this reasoning is just absurd! He/she will much prefer to consider that the wave function is reduced as soon as the result of the experiment is observed inside the laboratory. We are then back to a point that we already discussed, the absolute/relative character of the wave function: does this contradiction mean that we should consider two state vectors, one reduced, one not reduced, during the intermediate period of the experiment? For a discussion by Wigner of the problem of the measurement, see [38].

An unconventional interpretation, sometimes associated with Wigner’s name¹⁴, assumes that the reduction of the wave packet is a real effect which takes place when a human mind interacts with the surrounding physical world and acquires some consciousness of its state; in other words, the electrical currents in the human brain may be associated with a reduction of the state vector of measured objects, by some yet unknown physical process. Of course, in this view, the reduction takes place under the influence of the experimentalist inside the laboratory and the question of the preceding paragraph is settled. But, even if one accepts the somewhat provocative idea of possible action of the mind (or consciousness) on the environment, this point of view does not suppress all logical difficulties: what is a human mind, what level of consciousness is necessary to reduce the wave packet, etc.?

¹⁴The title of ref [37] is indeed suggestive of this sort of interpretation; moreover, Wigner writes in this reference that “it follows (from the Wigner friend argument) that the quantum description of objects is influenced by impressions entering my consciousness”. At the end of the article, he also discusses the influence of non-linearities which would put a limit on the validity of the Schrödinger equation, and be indications of life.

2.3 Schrödinger's cat

The famous story of the Schrödinger cat [39] [40] illustrates the problem in a different way; it is probably too well known to be described once more in detail here. Let us then just summarize it: the story illustrates how a living creature could be put into a very strange state, containing life and death, by correlation with a decaying radioactive atom, and through a Von Neumann chain; the chain includes a gamma ray detector, electronic amplification, and finally a mechanical system that automatically opens a bottle of poisonous gas if the atomic decay takes place. The resulting paradox may be seen as an illustration of the question: does an animal such a cat have the intellectual abilities that are necessary to perform a measurement and resolve all Von Neumann branches into one? Can it perceive its own state, projecting itself onto one of the alive or dead states? Or do humans only have access to a sufficient level of introspection to become conscious of their own observations, and to reduce the wave function? In that case, when the wave function includes a cat component, the animal could remain simultaneously dead and alive for an arbitrarily long period of time, a paradoxical situation indeed.

Another view on the paradox is obtained if one just considers the cat as a symbol of any macroscopic object; such objects can obviously never be in a “blurred” state containing possibilities that are obviously contradictory (open and closed bottle, dead and alive cat, etc.). Schrödinger considers this as a “quite ridiculous case”, which emerges from the linearity of his equation, but should clearly be excluded from any reasonable theory - or at best considered as the result of some incomplete physical description. In Schrödinger's words: “an indeterminacy originally restricted to the atomic domain becomes transformed into a macroscopic indeterminacy”. The message is simple: standard quantum mechanics is not only incapable of avoiding these ridiculous cases, it actually provides a recipe for creating them; one obviously needs some additional ingredients in the theory in order to resolve the Von Neumann regress, select one of its branches, and avoid stupid macroscopic superpositions. It is amusing to note in passing that Schrödinger's name is associated to two contradictory concepts that are actually mutually exclusive, a continuous equation of evolution and the symbolic cat, a limit that the equation should never reach! Needless to say, the limit of validity of the linear equation does not have to be related to the cat itself: the branch selection process may perfectly take place before the linear superposition reaches the animal. But the real question is that the reduction process has to take place somewhere, and where exactly?

Is this paradox related to decoherence? Not really. Coherence is completely irrelevant for Schrödinger, since the cat is actually just a symbol of a macroscopic object that is in an impossible blurred state, encompassing two possibilities that are incompatible in ordinary life; the state in question is

not necessarily a pure state (only pure states are sensitive to decoherence) but can also be a statistical mixture. Actually, in the story, the cat is never in a coherent superposition, since its blurred state is precisely created by correlation with some parts of the environment (the bottle of poison for instance); the cat is just another part of the environment of the radioactive atom. In other words, the cat is not the seed of a Von Neumann chain; it is actually trapped into two (or more) of its branches, in a tree that has already expanded into the macroscopic world after decoherence has already taken place at a microscopic level (radioactive atom and radiation detector), and will continue to expand after it has captured the cat. Decoherence is irrelevant for Schrödinger since his point is not to discuss the features of the Von Neumann chain, but to emphasize the necessity to break it: the question is not to have a coherent or a statistical superposition of macroscopically different states, it is to have no superposition at all¹⁵!

So the cat is the symbol of an impossibility, an animal that can never exist (a Schrödinger gargoyle?), and a tool for a “reductio ad absurdum” reasoning that puts into light the limitations of the description of a physical system by a Schrödinger wave function only. Nevertheless, in the recent literature in quantum electronics, it has become more and more frequent to weaken the concept, and to call “Schrödinger cat (SC)” any coherent superposition of states that can be distinguished macroscopically, independently of the numbers of degree of freedom of the system. SC states can then be observed (for instance an ion located in two different places in a trap), but often undergo rapid decoherence through correlation to the environment. Moreover, the Schrödinger equation can be used to calculate how the initial stages of the Von Neumann chain take place, and how rapidly the solution of the equation tends to ramify into branches containing the environment. Since this use of the words SC has now become rather common in a subfield of physics, one has to accept it; it is, after all, just a matter of convention to associate them with microscopic systems - any convention is acceptable as long as it does not create confusion. But it would be an insult to Schrödinger to believe that decoherence can be invoked as the solution of his initial cat paradox: Schrödinger was indeed aware of the properties of entanglement in quantum mechanics, a word that he introduced (and uses explicitly in the article on the cat), and he was not sufficiently naive to believe that standard quantum mechanics would predict possible interferences between dead and alive cats!

To summarize, the crux of most of our difficulties with quantum mechanics is the question: what is exactly the process that forces Nature to break the regress and to make its choice among the various possibilities

¹⁵This is for instance the purpose of theories with a modified non-linear Schrödinger dynamics: providing equations of motion where during measurements all probabilities dynamically go to zero, except one that goes to 1.

for the results of experiments? Indeed, the emergence of a single result in a single experiment, in other words the disappearance of macroscopic superpositions, is a major issue; the fact that such superpositions cannot be resolved at any stage within the linear Schrödinger equation may be seen as the major difficulty of quantum mechanics. As Pearle nicely expresses it [12], the problem is to “explain why events occur”!

2.4 Unconvincing arguments

We have already emphasized that the invention of the Copenhagen interpretation of quantum mechanics has been, and remains, one of the big achievements of physics. One can admire even more, in retrospect, how early its founders conceived it, at a time when experimental data were relatively scarce. Since that time, numerous ingenious experiments have been performed, precisely with the hope of seeing the limits of this interpretation but, until now, not a single fact has disproved the theory. It is really a wonder of pure logic that has allowed the early emergence of such an intellectual construction.

This being said, one has to admit that, in some cases, the brilliant authors of this construction may sometimes have gone too far, pushed by their great desire to convince. For instance, authoritative statements have been made concerning the absolute necessity of the orthodox interpretation which now, in retrospect, seem exaggerated - to say the least. According to these statements, the orthodox interpretation would give the only ultimate description of physical reality; no finer description would ever become possible. In this line of thought, the fundamental probabilistic character of microscopic phenomena should be considered as a proven fact, a rule that should be carved into marble and accepted forever by scientists. But, now, we know that this is not proven to be true: yes, one may prefer the orthodox interpretation if one wishes, but this is only a matter of taste; other interpretations are still perfectly possible; determinism in itself is not disproved at all. As discussed for instance in [6], and initially clarified by Bell [3] [7] and Bohm [4] [5], the “impossibility proofs” put forward by the proponents of the Copenhagen interpretation are logically unsatisfactory for a simple reason: they arbitrarily impose conditions that may be relevant to quantum mechanics (linearity), but not to the theories that they aim to dismiss - any theory with additional variables such as the Bohm theory, for instance. Because of the exceptional stature of the authors of the impossibility theorems, it took a long time to the physics community to realize that they were irrelevant; now, this is more widely recognized so that the plurality of interpretations is more easily accepted.

3 Einstein, Podolsky and Rosen

It is sometimes said that the article by Einstein, Podolsky and Rosen (EPR) [41] is, by far, that which has collected the largest number of quotations in the literature; the statement sounds very likely to be true. There is some irony in this situation since, so often, the EPR reasoning has been misinterpreted, even by prominent physicists! A striking example is given in the Einstein-Born correspondence [42] where Born, even in comments that he wrote after Einstein's death, still clearly shows that he never really understood the nature of the objections raised by EPR. Born went on thinking that the point of Einstein was an a priori rejection of indeterminism ("look, Einstein, indeterminism is not so bad"), while actually the major concern of EPR was locality and/or separability (we come back later to these terms, which are related to the notion of space-time). If giants like Born could be misled in this way, no surprise that, later on, many others made similar mistakes! This is why, in what follows, we will take an approach that may look elementary, but at least has the advantage of putting the emphasis on the logical structure of the arguments.

3.1 A theorem

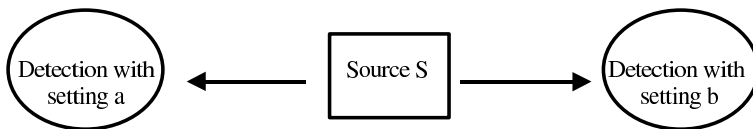
One often speaks of the "EPR paradox", but the word "paradox" is not really appropriate in this case. For Einstein, the basic motivation was not to invent paradoxes or to entertain colleagues inclined to philosophy; it was to build a strong logical reasoning which, starting from well defined assumptions (roughly speaking: locality and some form of realism), would lead ineluctably to a clear conclusion (quantum mechanics is incomplete, and even: physics is deterministic¹⁶). To emphasize this logical structure, we will speak here of the "EPR theorem", which formally could be stated as follows:

Theorem: If the predictions of quantum mechanics are correct (even for systems made of remote correlated particles) and if physical reality can be described in a local (or separable) way, then quantum mechanics is necessarily incomplete: some "elements of reality"¹⁷ exist in Nature that are ignored by this theory.

The theorem is valid, and has been scrutinized by many scientists who have found no flaw in its derivation; indeed, the logic which leads from the assumptions to the conclusion is perfectly sound. It would therefore be an error to repeat (a classical mistake!) "the theorem was shown by Bohr to be incorrect" or, even worse, "the theorem is incorrect since experimental

¹⁶Born's mistake, therefore, was to confuse assumptions and conclusions.

¹⁷These words are carefully defined by the authors of the theorem; see the beginning of § 3.2.3.



results are in contradiction with it¹⁸. Bohr himself, of course, did not make the error: in his reply to EPR [43], he explains why he thinks that the assumptions on which the theorem is based are not relevant to the quantum world, which makes it inapplicable to a discussion on quantum mechanics; more precisely, he uses the word “ambiguous” to characterize these assumptions, but he never claims that the reasoning is faulty (for more details, see § 3.2.3). A theorem which is not applicable in a particular case is not necessarily incorrect: theorems of Euclidean geometry are not wrong, or even useless, because one can also build a consistent non-Euclidean geometry! Concerning possible contradictions with experimental results we will see that, in a sense, they make a theorem even more interesting, mostly because it can then be used within a “reductio ad absurdum” reasoning.

Goods texts on the EPR argument are abundant; for instance, a classic is the wonderful little article by Bell [33]; another excellent introductory text is, for instance, ref. [44], which contains a complete description of the scheme (in the particular case where two settings only are used) and provides an excellent general discussion of many aspects of the problem; for a detailed source of references, see for instance [45]. Most readers are probably already familiar with the basic scheme considered, which is summarized in figure 1: a source S emits two correlated particles, which propagate towards two remote regions of space where they undergo measurements; the type of these measurements are defined by “settings”, or “parameters”¹⁹ (typically orientations of Stern-Gerlach analyzers, often noted a and b), which are at the choice of the experimentalists; in each region, a result is obtained, which can take only two values symbolized by ± 1 in the usual notation; finally, we will assume that, every time both settings are chosen to be the same value, the results of both measurements are always the same.

Here, rather than trying to paraphrase the good texts on EPR with more or less success, we will purposefully take a different presentation, based on a comparison, a sort of a parable. Our purpose is to emphasize a feature of the reasoning: the essence of the EPR reasoning is actually nothing but what is usually called “the scientific method” in the sense discussed by Francis Bacon and Claude Bernard. For this purpose, we will leave pure physics for

¹⁸The contradiction in question occurs through the Bell theorem (which is therefore sometimes criticized for the same reason), which was introduced as a continuation of the EPR theorem.

¹⁹Here we will use the words “settings” and “parameters” indifferently.

botany! Indeed, in both disciplines, one needs rigorous scientific procedures in order to prove the existence of relations and causes, which is precisely what we want to do.

3.2 Of peas, pods and genes

When a physicist attempts to infer the properties of microscopic objects from macroscopic observations, ingenuity (in order to design meaningful experiments) must be combined with a good deal of logic (in order to deduce these microscopic properties from the macroscopic results). Obviously, some abstract reasoning is indispensable, merely because it is impossible to observe with the naked eye, or to take in one's hand, an electron or even a macromolecule for instance. The scientist of past centuries who, like Mendel, was trying to determine the genetic properties of plants, had exactly the same problem: he did not have access to any direct observation of the DNA molecules, so that he had to base his reasoning on adequate experiments and on the observation of their macroscopic outcome. In our parable, the scientist will observe the color of flowers (the "result" of the measurement, +1 for red, -1 for blue) as a function of the condition in which the peas are grown (these conditions are the "experimental settings" a and b , which determine the nature of the measurement). The basic purpose is to infer the intrinsic properties of the peas (the EPR "element of reality") from these observations.

3.2.1 Simple experiments; no conclusion yet.

It is clear that many external parameters such as temperature, humidity, amount of light, etc. may influence the growth of vegetables and, therefore, the color of a flower; it seems very difficult in a practical experiment to be sure that all the relevant parameters have been identified and controlled with a sufficient accuracy. Consequently, if one observes that the flowers which grow in a series of experiments are sometimes blue, sometimes red, it is impossible to identify the reason behind these fluctuation; it may reflect some trivial irreproducibility of the conditions of the experiment, or something more fundamental. In more abstract terms, a completely random character of the result of the experiments may originate either from the fluctuations of uncontrolled external perturbations, or from some intrinsic property that the measured system (the pea) initially possesses, or even from the fact that the growth of a flower (or, more generally, life?) is fundamentally an indeterministic process - needless to say, all three reasons can be combined in any complicated way. Transposing the issue to quantum physics leads to the following formulation of the question: are the results of the experiments random because of the fluctuation of some uncontrolled influence taking place in the macroscopic apparatus, of some microscopic property of the measured

particles, or of some more fundamental process?

The scientist may repeat the “experiment” a thousand times and even more: if the results are always totally random, there is no way to decide which interpretation should be selected; it is just a matter of personal taste. Of course, philosophical arguments might be built to favor or reject one of them, but from a pure scientific point of view, at this stage, there is no compelling argument for a choice or another. Such was the situation of quantum physics before the EPR argument.

3.2.2 Correlations; causes unveiled.

The stroke of genius of EPR was to realize that correlations could allow a big step further in the discussion. They exploit the fact that, when the choice of the settings are the same, the observed results turn out to be always identical; in our botanical analogy, we will assume that our botanist observes correlations between colors of flowers. Peas come together in pods, so that it is possible to grow peas taken from the same pod and observe their flowers in remote places. It is then natural to expect that, when no special care is taken to give equal values to the experimental parameters (temperature, etc.), nothing special is observed in this new experiment. But assume that, every time the parameters are chosen to the same values, the colors are systematically the same; what can we then conclude? Since the peas grow in remote places, there is no way that they can be influenced by the any single uncontrolled fluctuating phenomenon, or that they can somehow influence each other in the determination of the colors. If we believe that causes always act locally, we are led to the following conclusion: the only possible explanation of the common color is the existence of some common property of both peas, which determines the color; the property in question may be very difficult to detect directly, since it is presumably encoded inside some tiny part of a biological molecule, but it is sufficient to determine the results of the experiments.

Since this is the essence of the argument, let us make every step of the EPR reasoning completely explicit, when transposed to botany. The key idea is that the nature and the number of “elements of reality” associated with each pea can not vary under the influence of some remote experiment, performed on the other pea. For clarity, let us first assume that the two experiments are performed at different times: one week, the experimenter grows a pea, then only next week another pea from the same pod; we assume that perfect correlations of the colors are always observed, without any special influence of the delay between the experiments. Just after completion of the first experiment (observation of the first color), but still before the second experiment, the result of that future experiment has a perfectly determined value; therefore, there must already exist one element of reality attached to the second pea that corresponds to this fact - clearly, it can

not be attached to any other object than the pea, for instance one of the measurement apparatuses, since the observation of perfect correlations only arises when making measurements with peas taken from the same pod. Symmetrically, the first pod also had an element of reality attached to it which ensured that its measurement would always provide a result that coincides with that of the future measurement. The simplest idea that comes to mind is to assume that the elements of reality associated with both peas are coded in some genetic information, and that the values of the codes are exactly the same for all peas coming from the same pod; but other possibilities exist and the precise nature and mechanism involved in the elements of reality does not really matter here. The important point is that, since these elements of reality can not appear by any action at a distance, they necessarily also existed before any measurement was performed - presumably even before the two peas were separated.

Finally, let us consider any pair of peas, when they are already spatially separated, but before the experimentalist decides what type of measurements they will undergo (values of the parameters, delay or not, etc.). We know that, if the decision turns out to favor time separated measurements with exactly the same parameter, perfect correlations will always be observed. Since elements of reality can not appear, or change their values, depending of experiments that are performed in a remote place, the two peas necessarily carry some elements of reality with them which completely determine the color of the flowers; any theory which ignores these elements of reality is incomplete. This completes the proof.

It seems difficult not to agree that the method which led to these conclusions is indeed the scientific method; no tribunal or detective would believe that, in any circumstance, perfect correlations could be observed in remote places without being the consequence of some common characteristics shared by both objects. Such perfect correlations can then only reveal the initial common value of some variable attached to them, which is in turn a consequence of some fluctuating common cause in the past (a random choice of pods in a bag for instance). To express things in technical terms, let us for instance assume that we use the most elaborate technology available to build elaborate automata, containing powerful modern computers²⁰ if necessary, for the purpose of reproducing the results of the remote experiments: whatever we do, we must ensure that, somehow, the memory of each computer contains the encoded information concerning all the results that it might have to provide in the future (for any type of measurement that might be made).

To summarize this section, we have shown that each result of a measure-

²⁰We are assuming here that the computers are not quantum computers (if quantum computers can ever be built, which is another question).

ment may be a function of two kinds of variables²¹:

(i) intrinsic properties of the peas, which they carry along with them.

(ii) the local setting of the experiment (temperature, humidity, etc.); clearly, a given pair that turned out to provide two blue flowers could have provided red flowers in other experimental conditions.

We may also add that:

(iii) the results are well-defined functions, in other words that no fundamentally indeterministic process takes place in the experiments.

(iv) when taken from its pod, a pea cannot “know in advance” to which sort of experiment it will be submitted, since the decision may not yet have been made by the experimenters; when separated, the two peas therefore have to take with them all the information necessary to determine the color of flowers for any kind of experimental conditions. What we have shown actually is that each pea carries with it as many elements of reality as necessary to provide “the correct answer”²² to all possible questions it might be submitted to.

3.2.3 Transposition to physics

The starting point of EPR is to assume that quantum mechanics provides correct predictions for all results of experiments; this is why we have built the parable of the peas in a way that exactly mimics the quantum predictions for measurements performed on two spin 1/2 particles for some initial quantum state: the red/blue color is obviously the analogue to the result that can be obtained for a spin in a Stern-Gerlach apparatus, the parameters (or settings) are the analogous to the orientation of these apparatuses (rotation around the axis of propagation of the particles). Quantum mechanics predicts that the distance and times at which the spin measurements are performed is completely irrelevant, so that the correlations will remain the same if they take place in very remote places.

Another ingredient of the EPR reasoning is the notion of “elements of reality”; EPR first remark that these elements cannot be found by a priori philosophical considerations, but must be found by an appeal to results of experiments and measurements. They then propose the following criterion: “if, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity”. In other words, certainty can not emerge from nothing: an experimental result that is known in advance is necessarily the consequence of some pre-existing physical property. In our

²¹In Bell’s notation, the A functions depend on the settings a and b as well as on λ .

²²Schrödinger used to remark that, if all students of a group always give the right answer to a question chosen randomly by the professor among two, they all necessarily knew the answer to both questions (and not only the one they actually answer).

botanical analogy, we implicitly made use of this idea in the reasoning of § 3.2.2.

A last, but essential, ingredient of the EPR reasoning is the notion of space-time and locality: the elements of reality in question are attached to the region of space where the experiment takes place, and they cannot vary suddenly (or even less appear) under the influence of events taking place in very distant region of space. The peas of the parable were in fact not so much the symbol of some microscopic object, electrons or spin 1/2 atoms for instance. Rather, they symbolize regions of space where we just know that “something is propagating”; it can be a particle, a field, or anything else, with absolutely no assumption on its structure or physical description. Actually, in the EPR quotation of the preceding paragraph, one may replace the word “system” by “region of space”, without altering the rest of the reasoning. One may summarize the situation by saying that the basic belief of EPR is that regions of space can contain elements of reality attached to them (attaching distinct elements of reality to separate regions of space is sometimes called “separability”) and that they evolve locally. From these assumptions, EPR prove that the results of the measurements are functions of:

(i) intrinsic properties of the spins that they carry with them (the EPR elements of reality)

(ii) of course, also of the orientations of the Stern-Gerlach analyzers

In addition, they show that:

(iii) the functions giving the results are well-defined functions, which implies that no indeterministic process is taking place; in other words, a particle with spin carries along with it all the information necessary to provide the result to any possible measurement.

(iv) since it is possible to envisage future measurements of observables that are called “incompatible” in quantum mechanics, as a matter of fact, incompatible observables can simultaneously have a perfectly well defined value.

Item (i) may be called the EPR-1 result: quantum mechanics is incomplete (EPR require from a complete theory that “every element of physical reality must have a counterpart in the physical theory”); in other words, the state vector may be a sufficient description for a statistical ensemble of pairs, but for one single pair of spins, it should be completed by some additional information; in still other words, inside the ensemble of all pairs, one can distinguish between sub-ensembles with different physical properties. Item (iii) may be called EPR-2, and establishes the validity of determinism from a locality assumption. Item (iv), EPR-3 result, shows that the notion of incompatible observables is not fundamental, but just a consequence of the incomplete character of the theory; it actually provides a reason to reject complementarity. Curiously, EPR-3 is often presented as the major EPR result, sometimes even with no mention of the two others; actually, the re-

jection of complementarity is almost marginal or, at least, less important for EPR than the proof of incompleteness. In fact, in all that follows in this article, we will only need EPR-1,2.

Niels Bohr, in his reply to the EPR article [43], stated that their criterion for physical reality contains an essential ambiguity when it is applied to quantum phenomena. A more extensive quotation of Bohr's reply is the following:

“The wording of the above mentioned criterion (the EPR criterion for elements of reality)... contains an ambiguity as regards the expression 'without in any way disturbing a system'. Of course there is in a case like that considered (by EPR) no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of an influence of the very conditions which define the possible types of predictions regarding the future behavior of the system.... the quantum description may be characterized as a rational utilization of all possibilities of unambiguous interpretation of measurements, compatible with the finite and uncontrollable interactions between the objects and the measuring instruments in the field of quantum theory”.

Indeed, in Bohr's view, physical reality cannot be properly defined without reference to a complete and well-defined experiment. This includes, not only the systems to be measured (the microscopic particles), but also all the measurement apparatuses: “these (experimental) conditions must be considered as an inherent element of any phenomenon to which the term physical reality can be unambiguously applied”. Therefore EPR's attempt to assign elements of reality to one of the spins only, or to a region of space containing it, is incompatible with orthodox quantum mechanics²³ - even if the region in question is very large and isolated from the rest of the world. Expressed differently, a physical system that is extended over a large region of space is to be considered as a single entity, within which no attempt should be made to distinguish physical subsystems or any substructure; trying to attach physical reality to regions of space is then automatically bound to failure. In terms of our Leitmotiv of § 1.1.3, the difference between ordinary space and configuration space, we could say the following: the system has a single wave function for both particles that propagates in a configuration space with more than 3 dimensions, and this should be taken very seriously; no attempt should be made to come back to three dimensions and implement locality arguments in a smaller space.

Bohr's point of view is, of course, not contradictory with relativity, but

²³One could add that the EPR disproof of the notion of incompatible observables implies that, at least, two different settings are considered for one of the measurement apparatuses; this should correspond, in Bohr's view, to two different physical realities (every different couple a,b actually corresponds to a different physical reality), and not to a single one as assumed in the EPR reasoning.

since it minimizes the impact of basic notions such as space-time, or events (a measurement process in quantum mechanics is not local; therefore it is not an event *stricto sensu*), it does not fit very well with it. One could add that Bohr's article is difficult to understand; many physicists admit that a precise characterization of his attitude, in terms for instance of exactly what traditional principles of physics should be given up, is delicate (see for example the discussion of ref. [8]). In Pearle's words: "Bohr's rebuttal was essentially that Einstein's opinion disagreed with his own" [46]. It is true that, when scrutinizing Bohr's texts, one never gets completely sure to what extent he fully realized all the consequences of his position. Actually, in most of his reply to EPR [43] in *Physical Review*, he just repeats the orthodox point of view in the case of a single particle submitted to incompatible measurements, and even goes through considerations that are not obviously related to the EPR argument, as if he did not appreciate how interesting the discussion becomes for two remote correlated particles; the relation to locality is not explicitly discussed, as if this was an unimportant issue (while it was the starting point of further important work, the Bell theorem for instance²⁴). The precise reply to EPR is actually contained in only a short paragraph of this article, from which the quotations given above have been taken. Even Bell confessed that he had strong difficulties understanding Bohr ("I have very little idea what this means.." - see the appendix of ref. [33])!

4 Quantitative theorems: Bell, GHZ, Hardy, BKS

The Bell theorem [47] may be seen in many different ways. In fact, Bell initially invented it as a logical continuation of the EPR theorem: the idea is to take completely seriously the existence of the EPR elements of reality, and introduce them into the mathematics with the notation λ ; one then proceeds to study all possible kinds of correlations that can be obtained from the fluctuations of the λ 's, making the condition of locality explicit in the mathematics (locality was already useful in the EPR theorem, but not used in equations). As a continuation of EPR, the reasoning necessarily develops from a deterministic framework and deals with classical probabilities; it studies in a completely general way all kinds of correlation that can be predicted from the fluctuations in the past of some classical common cause - if one prefers, from some uncertainty concerning the initial state of the system. This leads to the famous inequalities. But subsequent studies have shown that the scope of the Bell theorem is not limited to determinism; for instance, the λ 's may influence the results of future experiments by fixing the values of probabilities of the results, instead of these results themselves (see appendix I). We postpone the discussion of the various possible general-

²⁴If Bohr had known the Bell theorem, he could merely have replied to EPR that their logical system was inconsistent (see § 4.1.3)!

izations to § 4.1.4 and, for the moment, we just emphasize that the essential condition for the validity of the Bell theorem is locality: all kinds of fluctuations can be assumed, but their effect must affect physics only locally. If we assume that throwing dice in Paris may influence physical events taking place in Tokyo, or even in other galaxies, the proof of the theorem is no longer possible. For non-specialized discussions of the Bell theorem, see for instance [33] [44] [48] [49].

4.1 Bell inequalities

The Bell inequalities are relations satisfied by the average values of product of random variables that are correlated classically (their correlations arise from the fluctuations of some common cause in the past, as above for the peas). As we will see, the inequalities are especially interesting in cases where they are contradictory with quantum mechanics; one of these situations occurs in the EPRB (B for Bohm [50]) version of the EPR argument, where two spin 1/2 particles undergo measurements. This is why we begin this section by briefly recalling the predictions of quantum mechanics for such a physical system - but the only ingredient we need from quantum mechanics at this stage is the predictions concerning the probabilities of results. Then we leave again standard quantum mechanics and come back to the EPR-Bell argument, discuss its contradictions with quantum mechanics, and finally emphasize the generality of the theorem.

4.1.1 Two spins in a quantum singlet state

We assume that two spin 1/2 particles propagate in opposite directions after leaving a source which has emitted them in a singlet spin state. Their spin state is then described by:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|+, -\rangle - |-, +\rangle] \quad (1)$$

When they reach distant locations, they are then submitted to spin measurements, with Stern-Gerlach apparatuses oriented along angles a and b around the direction of propagation. If θ is the angle between a and b , quantum mechanics predicts that the probability for a double detection of results $+1, +1$ (or of $-1, -1$) is:

$$\mathcal{P}_{+,+} = \mathcal{P}_{-,-} = \frac{1}{2} \sin^2 \frac{\theta}{2} \quad (2)$$

while the probability of two opposite results is:

$$\mathcal{P}_{+,-} = \mathcal{P}_{-,+} = \frac{1}{2} \cos^2 \frac{\theta}{2} \quad (3)$$

This is all that we want to know, for the moment, of quantum mechanics: probability of the results of measurements. We note in passing that, if $\theta = 0$, (when the orientations of the measurements apparatuses are parallel) the formulas predict that one of the probabilities vanishes, while the other is equal to one; therefore the condition of perfect correlations required by the EPR reasoning is fulfilled (in fact, the results of the experiments are always opposed, instead of equal, but it is easy to convince oneself that this does not have any impact on the reasoning).

4.1.2 Proof

We now come back to the line of the EPR theorem. In the framework of strict deterministic theories, the proof of the Bell theorem is the matter of a few lines; the longest part is actually the definition of the notation. Following Bell, we assume that λ represents all “elements of reality” associated to the spins; it should be understood that λ is only a concise notation which may summarize a vector with many components, so that we are not introducing any limitation here. In fact, one can even include in λ components which play no special role in the problem; the only thing which matters it that λ does contain all the information concerning the results of possible measurements performed on the spins. We use another classical notation, A and B , for these results, and small letters a and b for the settings (parameters) of the corresponding apparatuses. Clearly A and B may depend, not only on λ , but also on the settings a and b ; nevertheless locality requests that b has no influence on the result A (since the distance between the locations of the measurements can be arbitrarily large); conversely, a has no influence on result B . We therefore call $A(a, \lambda)$ and $B(b, \lambda)$ the corresponding functions (their values are either $+1$ or -1).

In what follows, it is sufficient to consider two directions only for each separate measurement; we then use the simpler notation:

$$A(a, \lambda) = A \quad ; \quad A(a', \lambda) = A' \quad (4)$$

and:

$$B(b, \lambda) = B \quad ; \quad B(b', \lambda) = B' \quad (5)$$

For each pair of particles, λ is fixed, and the four numbers have well-defined values (which can only be ± 1). With Eberhard [51] we notice that the product:

$$M = AB + AB' - A'B + A'B' = (A - A')B + (A + A')B' \quad (6)$$

is always equal to either $+2$, or to -2 ; this is because one of the brackets in the right hand side of this equation always vanishes, while the other is ± 2 . Now, if we take the average value of M over a large number of emitted pairs

(average over λ), since each instance of M is limited to these two values, we necessarily have:

$$-2 \leq \langle M \rangle \leq +2 \tag{7}$$

This is the so called BCHSH form [52] of the Bell theorem: the average values of all possible kinds of measurements that provide random results, whatever the mechanism behind them may be (as long as the randomness is local and arises from the effect of some common fluctuating cause in the past), necessarily obey this strict inequality.

4.1.3 Contradiction with quantum mechanics and with experiments

The generality of the proof is such that one could reasonably expect that any sensible physical theory will automatically give predictions that also obey this inequality; the big surprise was to realize that quantum mechanics does not: it turns out that, for some appropriate choices of the four directions a, a', b, b' (the precise values do not matter for the discussion here), the inequality is violated by a factor $\sqrt{2}$, which is more than 40%. Therefore, the EPR-Bell reasoning leads to a quantitative contradiction with quantum mechanics; indeed, the latter is not a local realistic theory in the EPR sense. How is this contradiction possible, and how can a reasoning that is so simple be incorrect within quantum mechanics? The answer is the following: what is wrong, if we believe quantum mechanics, is to attribute well-defined values A, A', B, B' to each emitted pair; because only two of them at maximum can be measured in any experiment, we can not speak of these four quantities, or reason on them, even as unknown quantities. As nicely emphasized by Peres in an excellent short article [53], “unperformed experiments have no result”, that is all!

Wheeler expresses a similar idea when he writes: “No elementary quantum phenomenon is a phenomenon until it is a recorded phenomenon” [54]. As for Wigner, he emphasizes in [55] that the proof of the Bell inequalities relies on a very simple notion: the number of categories into which one can classify all pairs of particles²⁵. Each category is associated with well-defined results of measurements, for the various choices of the settings a and b that are considered; in any sequence of repeated experiments, each category will contribute with some given weight, its probability of occurrence, which has to be positive or zero. Wigner then notes that, if one introduces the notion of locality, each category becomes the intersection of a sub-ensemble that depends on a only, by another sub-ensemble that depends on b only. This

²⁵In this reference, Wigner actually reasons explicitly in terms of hidden variables; he considers domains for these variables, which correspond to given results for several possible choices of the settings. But these domains also correspond to categories of pairs of particles, which is why, here, we use the notion of categories.

operation immediately reduces the number of categories: in a specific example (involving three possible values of each setting), he shows that their number reduces from 4^9 to $(2^3)^2 = 2^6$; with two values only for each setting, the reduction would be from 4^4 to $(2^2)^2 = 2^4$. The mathematical origin of the Bell inequalities lies precisely in the possibility of distributing all pairs into this smaller number of categories, with positive probabilities.

A general way to express the Bell theorem in logical terms is to state that the following system of three assumptions (which could be called the EPR assumptions) is self-contradictory:

1. validity of their notion of “elements of reality”
2. locality
3. the predictions of quantum mechanics are always correct.

The Bell theorem then becomes a useful tool to build a “*reductio ad absurdum*” reasoning: it shows that, among all three assumptions, one (at least) has to be given up. The motivation of the experimental tests of the Bell inequalities was precisely to check if it was not the third which should be abandoned. Maybe, after all, the Bell theorem is nothing but an accurate pointer towards exotic situations where the predictions of quantum mechanics are so paradoxical that they are actually wrong? Such was the hope of some theorists, as well as the exciting challenge to experimentalists.

Experiments were performed in the seventies, initially with photons [56] [57] where they already gave very clear results, as well as with protons [58]; in the eighties, they were made more and more precise and convincing [59] - see also [60]; ever since, they have been constantly improved (see for instance [61], but the list of references is too long to be given here); all these results have clearly shown that, in this conflict between local realism and quantum mechanics, the latter wins completely. A fair summary of the situation is that, even in these most intricate situations invented and tested by the experimentalists, no one has been able to disprove quantum mechanics. In this sense, we can say that Nature obeys laws which are non-local, or non-realist, or both. It goes without saying that no experiment in physics is perfect, and it is always possible to invent ad hoc scenarios where some physical processes, for the moment totally unknown, “conspire” in order to give us the illusion of correct predictions of quantum mechanics - we come back to this point in § 5.1 - but the quality and the number of the experimental results does not make this attitude very attractive intellectually.

4.1.4 Generality of the theorem

We have already mentioned that several generalizations of the Bell theorem are possible; they are at the same time mathematically simple and conceptually interesting. For instance, in some of these generalizations, it is assumed that the result of an experiment becomes a function of several fluctuating causes: the fluctuations taking place in the source as usual, but also fluctu-

ations taking place in the measuring apparatuses [62], or/and perturbations acting on the particles during their motion towards the apparatuses; actually, even fundamentally indeterministic (but local) processes may influence the results. The two former cases are almost trivial since they just require the addition of more dimensions to the vector variable λ ; the latter requires replacing the deterministic functions A and B by probabilities, but this is also relatively straightforward [49] (see also footnote 10 in [62] and appendix I of this article). Moreover, one should realize that the role of the A and B functions is just to relate the conditions of production of a pair of particles (or of their propagation) to their behavior when they reach the measurements apparatuses (and to the effects that they produce on them); they are, so to say, solutions of the equation of motion whatever these are. The important point is that they may perfectly include, in a condensed notation, a large variety of physical phenomena: propagation of point particles, propagation of one or several fields from the source to the detectors (see for instance the discussion in §4 of [33]), particles and fields in interaction, or whatever process one may have in mind (even random propagations can be included) - as long as they do not depend on the other setting (A is supposed to be a function of a , not of b). The exact mathematical form of the equations of propagation is irrelevant; the essential thing is that the functions exist.

Indeed, what really matters for the proof of the Bell theorem is the dependence with respect to the settings a and b : the function A must depend on a only, while B must depend on b only. Locality expressed mathematically in terms of a and b is the crucial ingredient. For instance we could, if we wished, assume that the result A of one measurement is also function of fluctuating random variables attached to the other apparatus, which introduces a non-local process; but this does not create any mathematical problem for the proof (as long as these variables are not affected by setting b). On the other hand, if A becomes a function of a and b (and/or the same for B), it is easy to see that the situation is radically changed: in the reasoning of § 4.1.2 we must now associate 8 numbers to each pair (since there are two results to specify for each of the 4 different combinations of settings), instead of 4, so that the proof miserably collapses. Appendix I gives another concrete illustration showing that it is locality, not determinism, which is at stake; see also the appendix of [49]).

Needless to say, the independence of A of b does not mean that the result observed on one side, A , is independent of the outcome at the other side, B : one should not confuse setting and outcome dependences! It is actually clear that, in any theory, the correlations would disappear if outcome dependence was totally excluded. We should also mention that the setting dependence is subject to some constraints, if the theory is to remain compatible with relativity. If, for instance, the probability of observation of the results on one side, which is a sum of probabilities over the various possible outcomes on the other side, was still a function of the other setting, one would run

into incompatibility; this is because one could use the device to send signals without any fundamental delay, thus violating the constraints of relativity. See refs. [63] and [64] for a discussion in terms of “strong locality” and “predictive completeness” (or “parameter independence” and of “outcome independence” in ref.[65]). Appendix IV discusses how the general formalism of quantum mechanics manages to ensure compatibility with relativity.

An interesting generalization of the Bell theorem, where time replaces the settings, has been proposed by Franson [66] and implemented in experiments for an observation of a violation of the Bell inequalities (see for instance [67]); another generalization shows that a violation of the Bell inequalities is not limited to a few quantum states (singlet for instance), but includes all states that are not products [68] [69]. For a general discussion of the conceptual impact of a violation of the inequalities, we refer to the book collecting Bell’s articles [7].

We wish to conclude this section by emphasizing that the Bell theorem is much more general than many people think. All potential authors on the subject should think twice and remember this carefully before taking their pen and sending a manuscript to a physics journal: every year a large number of them is submitted, with the purpose of introducing “new” ways to escape the constraints of the Bell theorem, and to “explain” why the experiments have provided results that are in contradiction with the inequalities. According to them, the non-local correlations would originate from some new sort of statistics, or from perturbations created by cosmic rays, gas collisions with fluctuating impact parameters, etc. The imagination is the only limit of the variety of the processes that can be invoked, but we know from the beginning that all these attempts are doomed to failure. The situation is analogous to the attempts of past centuries to invent “perpetuum mobile” devices: even if some of these inventions were extremely clever, and if it is sometimes difficult to find the exact reason why they can not work, it remains true that the law of energy conservation allows us to know at once that they cannot. In the same way, some of these statistical “Bell beating schemes” may be extremely clever, but we know that the theorem is a very general theorem in statistics: in all situations that can be accommodated by the mathematics of the λ ’s and the A and B functions (and there are many!), it is impossible to escape the inequalities. No, non-local correlations can not be explained cheaply; yes, a violation of the inequalities is therefore a very, very, rare situation. In fact, until now, it has never been observed, except of course in experiments designed precisely for this purpose. In other words, if we wanted to build automata including arbitrarily complex mechanical systems and computers, we could never mimic the results predicted by quantum mechanics (at least for remote measurements); this will remain impossible forever, or at least until completely different computers working

on purely quantum principles are built²⁶.

4.2 Hardy's impossibilities

Another scheme of the same conceptual type was introduced recently by Hardy [70]; it also considers two particles but it is nevertheless completely different since it involves, instead of mathematical constraints on correlation rates, the very possibility of occurrence for some type of events - see also [71] for a general discussion of this interesting contradiction. As in § 4.1.2, we assume that the first particle may undergo two kinds of measurements, characterized by two values a and a' of the first setting; if we reason as in the second half of § 4.1.2, within the frame of local realism, we can call A and A' the corresponding results. Similar measurements can be performed on the second particle, and we call B and B' the results.

Let us now consider three types of situations:

(i) settings without prime: we assume that the result $A = 1, B = 1$ is sometimes obtained.

(ii) one prime only: we assume that the “double one” is impossible, in other words that one never gets $A = 1, B' = 1$, and never $A' = 1, B = 1$ either.

(iii) double prime settings: we assume that “double minus one” is impossible, in other words that $A' = -1, B' = -1$ is never observed.

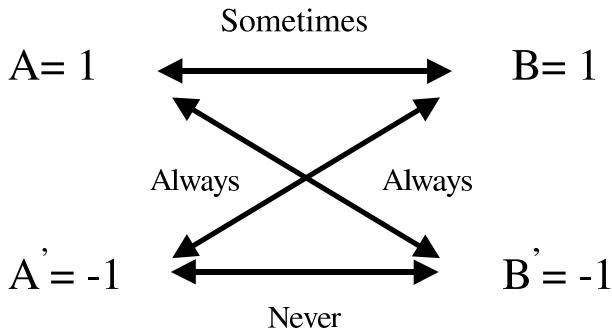
A closer inspection shows that these three assumptions are in fact incompatible. To see why, let us for instance consider the logical scheme of figure 2, where the upper part corresponds to the possibility opened by statement (i); statement (ii) then implies that, if $A = 1$, one necessarily has $B' = -1$, which explains the first diagonal in the figure; the second diagonal follows by symmetry. Then we see that all events corresponding to the results $A = B = 1$ also necessarily correspond to $A' = B' = -1$, so that a contradiction with statement (iii) appears: the three propositions are in fact incompatible. A way to express it is to say that the “sometimes” of (i) is contradictory with the “never” of proposition (iii).

But it turns out that quantum mechanics does allow a simultaneous realization of all three propositions! To see how, let us for instance consider a two-spin state vector of the form:

$$|\Psi\rangle = \alpha |+, -\rangle + \beta |-, +\rangle + \gamma |+, +\rangle \quad (8)$$

where the $|\pm, \pm\rangle$ refer to eigenstates of A' and B' (NB: axis Oz is chosen as the direction of measurement associated with primed operators). From the beginning, the absence of any $|\Psi\rangle$ component on $|-, -\rangle$ ensures that

²⁶In terms of the Mendel parable: an observation of a violation of the Bell inequalities would imply that something inside both peas (maybe a pair of DNA molecules?) remains in a coherent quantum superposition, without decoherence, even if the distance between the peas is large.



proposition (iii) is true. As for the measurements without prime, we assume that they are both performed along a direction in the plane xOz that makes an angle 2θ with Oz ; the eigenstate with eigenvalue $+1$ associated in the single-spin state is then merely:

$$\cos \theta | + \rangle + \sin \theta | - \rangle \quad (9)$$

The first state excluded by proposition (ii) (diagonal in figure 2) is then the two-spin state:

$$\cos \theta | +, + \rangle + \sin \theta | +, - \rangle \quad (10)$$

while the second is:

$$\cos \theta | +, + \rangle + \sin \theta | -, + \rangle \quad (11)$$

so that the two exclusion conditions are equivalent to the conditions:

$$\alpha \sin \theta + \gamma \cos \theta = \beta \sin \theta + \gamma \cos \theta = 0 \quad (12)$$

or, within a proportionality coefficient:

$$\alpha = \beta = -\gamma \cot \theta \quad (13)$$

This arbitrary coefficient may be used to write $|\Psi\rangle$ in the form:

$$|\Psi\rangle = -\cos \theta (| +, - \rangle + | -, + \rangle) + \sin \theta | +, + \rangle \quad (14)$$

The last thing to do is to get the scalar product of this ket by that where the two spins are in the state (9); we get the result:

$$-\sin \theta \cos^2 \theta \quad (15)$$

The final step is to divide this result by the square of the norm of ket (14) in order to obtain the probability of the process considered in (iii); this is a straightforward calculation (see appendix II), but here we just need to point

out that the probability is not zero; the precise value of its θ maximum found in appendix II is about 9%. This proves that the pair of results considered in proposition (i) can sometimes be obtained together with (ii) and (iii): indeed, in 9% of the cases, the predictions of quantum mechanics are in complete contradiction with those of a local realist reasoning.

An interesting aspect of the above propositions is that they can be generalized to an arbitrary number of measurements [72]; it turns out that this allows a significant increase of the percentage of “impossible events” (impossible within local realism) predicted by quantum mechanics - from 9% to almost 50%! The generalization involves a chain, which keeps the two first lines (i) and (ii) unchanged, and iterates the second in a recurrent way, by assuming that:

(iii) for measurements of the type (a', b'') or (a'', b') , one never gets opposite results²⁷.

(iv) for measurements of the type (a'', b''') or (a''', b'') , one never gets opposite results.

etc..

(n) finally, for measurement of the type (a^n, b^n) , one never gets -1 and -1 .

The incompatibility proof is very similar to that given above; it is summarized in figure 3. In both cases, the way to resolve the contradiction is the same as for the Bell theorem: in quantum mechanics, it is not correct to reason on all 4 quantities A, A', B and B' for a given pair of spins, even as quantities that are unknown and that could be determined in a future experiment. This is simply because, with a given pair, it is obviously impossible to design an experiment that will measure all of them: they are incompatible. If we insisted on introducing similar quantities to reproduce the results of quantum mechanics, we would have to consider 8 quantities instead of 4 (see second paragraph of § 4.1.4). For a discussion of non-local effects with other states, see [73].

4.3 GHZ equality

For many years, everyone thought that Bell had basically exhausted the subject by considering all really interesting situations, in other words that two-spin systems provided the most spectacular quantum violations of local realism. It therefore came as a surprise to many when in 1989 Greenberger, Horne and Zeilinger (GHZ) showed that this is not true: as soon as one considers systems containing more than two correlated particles, even more dramatic violations of local realism become possible in quantum mechanics, and even without involving inequalities. Here, we limit ourselves to the

²⁷In fact, the reasoning just requires that the pair $-1, +1$ is never obtained, and does not require any statement about $+1, -1$.

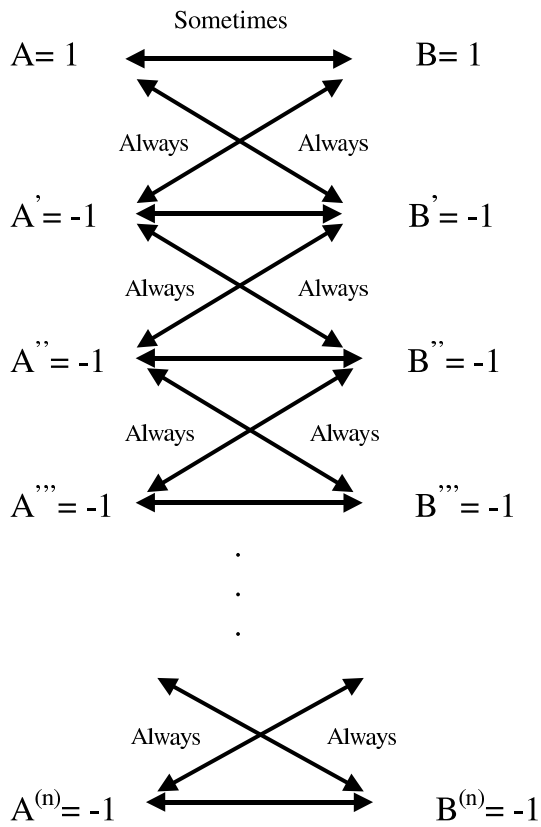


Figure 1:

discussion of three particle systems, as in the original articles [74] [75], but generalization to N particles are possible; see for instance § 5.3.1 or [76]. While [75] discussed the properties of three correlated photons, each emitted through two pinholes and impinging beam splitters, we will follow ref. [77] and consider a system of three $1/2$ spins (external variables play no role here); we assume that the system is described by the quantum state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|+, +, +\rangle - |-, -, -\rangle] \quad (16)$$

where the $|\pm\rangle$ states are the eigenstates of the spins along the Oz axis of an orthonormal frame $Oxyz$. We now calculate the quantum probabilities of measurements of the spins $\sigma_{1,2,3}$ of the three particles, either along direction Ox , or along direction Oy which is perpendicular. More precisely, we assume that what is measured is not individual spin components, but only the product of three of these components, for instance $\sigma_{1y} \times \sigma_{2y} \times \sigma_{3x}$. A straightforward calculation (see appendix III) shows that:

$$\begin{aligned} \mathcal{P}(\sigma_{1y} \times \sigma_{2y} \times \sigma_{3x} = 1) &= +1 \\ \mathcal{P}(\sigma_{1x} \times \sigma_{2y} \times \sigma_{3y} = 1) &= +1 \\ \mathcal{P}(\sigma_{1y} \times \sigma_{2x} \times \sigma_{3y} = 1) &= +1 \end{aligned} \quad (17)$$

In fact, the state vector written in (16) turns out to be a common eigenstate to all three operator products, so that each of them takes a value $+1$ that is known before the measurement²⁸. Now, if we consider the product of three spin components along Ox , it is easy to check (appendix III) that the same state vector is also an eigenstate of the product operator $\sigma_{1x} \times \sigma_{2x} \times \sigma_{3x}$, but now with eigenvalue -1 , so that:

$$\mathcal{P}(\sigma_{1x} \times \sigma_{2x} \times \sigma_{3x} = -1) = 1 \quad (18)$$

This time the result is -1 , with probability 1, that is with certainty.

Let us now investigate the predictions of a local realist EPR type point of view in this kind of situation. Since the quantum calculation is so straightforward, it may seem useless: indeed, no one before GHZ suspected that anything interesting could occur in such a simple case, where the initial state is an eigenstate of all observables considered, so that the results are perfectly certain. But, actually, we will see that a complete contradiction emerges from this analysis! The local realist reasoning is a simple generalization of that given in 4.1.2; we call $A_{x,y}$ the results that the first spin will give for a measurement, either along Ox , or Oy ; similar letters B and C are used for the measurement on the two other spins. From the three equalities

²⁸But, if the product is fixed, each of the individual components still fluctuates with a 100% amplitude, between results $+1$ and -1 .

written in (17) we then get:

$$\begin{aligned} A_y B_y C_x &= 1 \\ A_x B_y C_y &= 1 \\ A_y B_x C_y &= 1 \end{aligned} \tag{19}$$

Now, if we assume that the observations of the three spins are performed in three remote regions of space, locality implies that the values measured for each spin should be independent of the type of observation performed on the two other spins. This means that the same values of A , B and C can be used again for the experiment where the three Ox components are measured: the result is merely the product $A_x B_x C_x$. But, since the squares A_y^2 etc. are always equal to $+1$, we can obtain this result by multiplying all three lines of equation (19), which provides:

$$A_x B_x C_x = +1 \tag{20}$$

But equality (18) predicts the opposite sign!

Here we obtain a contradiction that looks even more dramatic than for the Bell inequalities: the two predictions do not differ by some significant fraction (about 40%), they are just completely opposite. In addition, all fluctuations are eliminated since all of the results (the products of the three components) are perfectly known before measurement: the 100% contradiction is obtained with 100% certainty! Unfortunately, this does not mean that, experimentally, tests of the GHZ equality are easy. Three particles are involved, which must be put in state (16), surely a non-trivial task; moreover one has to design apparatuses that measure the product of three spin components. To our knowledge, no experiment analogous to the Bell inequality experiments has been performed on the GHZ equality yet, at least with macroscopic distances; only microscopic analogues have been observed, in NMR experiments [78] - for recent proposals, see for instance [79] [80]. Nevertheless constant progress in the techniques of quantum electronics is taking place, and GHZ entanglement has already been observed [81] [82], so that one gets the impression that a full experiment is not too far away in the future.

In a GHZ situation, how precisely is the conflict between the reasoning above and quantum mechanics resolved? There are different stages at which this reasoning can be put into question. First, we have assumed locality, which here takes the form of non-contextuality (see § 4.4): each of the results is supposed to be independent of the nature of the measurements that are performed on the others, because they take place in remote regions of space. Clearly, there is no special reason why this should necessarily be true within quantum mechanics. Second, we have also made assumptions concerning the nature of the “elements of reality” attached to the particles. In this respect, it is interesting to note that the situation is very different from

the EPR-Bell or Hardy cases: Bohr could not have replied that different elements of reality should be attached to different experimental setups! In the GHZ argument, it turns out that all four quantum operators corresponding to the measurements commute, so that there is in principle no impossibility of measuring all of them with a single setup. But the local realist reasoning also assumes that a measurement of the product of three operators is equivalent to a separate measurement of each of them, which attributes to them separate elements of reality. In the formalism of quantum mechanics, the question is more subtle. It turns out that the measurement of a single product of commuting operators is indeed equivalent to the measurement of each of them; but this is no longer the case for several product operators, as precisely illustrated by those introduced above: clearly, all 6 spin component operators appearing in the formulas do not commute with each other. It is therefore impossible to design a single experimental setup to have access to all 6 quantities $A_{x,y}$, $B_{x,y}$ and $C_{x,y}$ that we have used in the local realist proof²⁹.

When the measurements are imperfect, the GHZ equality can give rise to inequalities (as in the BCHSH theorem), as discussed in [75] and [83]; this latter reference also presents a generalization to an arbitrary number N of particles; in the same line, ref. [76] provides a discussion of the N -particle correlation function with varying angles for the analyzers, which we partially reproduce in § 5.3.1.

4.4 Bell-Kochen-Specker; contextuality.

Another theorem was introduced also by Bell [3] as well as (independently and very shortly after) by Kochen and Specker [84], hence the name “BKS theorem” that is often used for it. This theorem is not particularly related to locality, as opposed to those that we have already discussed in the preceding subsections. It is actually related to another notion, called “contextuality”: an additional variable attached to a physical system is called “contextual” if its value depends, not only of the physical quantity that it describes, but also of the other physical quantities that can be measured at the same time on the same system (in quantum mechanics they correspond to commuting observables). If, on the other hand, its value is completely independent of all the other observables that the experimenter may decide to measure at the same time, the additional variable is called “non-contextual”; one can then say that it describes a property of the physical system only, and

²⁹The ideal GHZ experiment would therefore involve only measurements of commuting observables, i.e. products measured directly without measuring each factor separately. In practice, it is probably easier to measure each factor in the product; if all four products are needed, this necessarily implies successive measurements of incompatible observables with different experimental setups; the price to pay, then, is that loopholes such as the “biased sample loophole” (§ 5.1) may be opened again in the interpretation of the results.

not a combined property of the system and the measurement apparatus; it may have pre-existed in the system before any measurement. The notion of distance is no longer relevant in this context; for instance, the theorem applies to a single system with no extension in space.

Let us first consider a spin 1 particle in quantum mechanics, with three quantum states $|-1\rangle$, $|0\rangle$ and $|+1\rangle$ as a basis of a state space of the dimension 3. The three components S_x , S_y , and S_z , do not commute (they obey the usual commutation relation for the angular momentum), but it is easy to show that all the squares of all these three operators do commute; this is a specific property of angular momentum 1, and can be seen by an elementary matrix calculation with the usual operators S_{\pm} . Moreover, the sum of these squares is a constant (a c-number) since:

$$S_x^2 + S_y^2 + S_z^2 = 2\hbar^2 \quad (21)$$

It is not against any fundamental principle of quantum mechanics, therefore, to imagine a triple measurement of the observables S_x^2 , S_y^2 , and S_z^2 ; we know that the sum of the three results will always be 2 (from now on we drop the factor \hbar^2 , which plays no role in the discussion). Needless to say, the choice of the three orthogonal directions is completely arbitrary, and the compatibility is ensured for any choice of this triad, but not more than one: the measurements for different choices remain totally incompatible.

In passing, we note that the measurement of the square S_x^2 of one component cannot merely be seen as a measurement of S_x followed by a squaring calculation made afterwards by the experimentalist! Ignoring information is not equivalent to not measuring it (we come to this point in more detail, in terms of interferences and decoherence, at the end of § 6.1). There is indeed less information in S_x^2 than in S_x itself, since the former has only two eigenvalues (1 and 0), while the latter has three (-1 is also a possible result). What is needed to measure S_x^2 is, for instance, a modified Stern-Gerlach system where the components of the wave function corresponding to results ± 1 are not separated, or where they are separated but subsequently grouped together in a way they makes them impossible to distinguish. Generally speaking, in quantum mechanics, measuring the square of an operator is certainly not the same physical process as measuring the operator itself!

Now, suppose that we try to attach to each individual spin an EPR element of reality/additional variable that corresponds to the result of measurement of S_x^2 ; by symmetry, we will do the same for the two other components, so that each spin now gets three additional variables λ to which we may attribute values that determine the possible results: 1 or 0. The results are described by functions of these variables, which we note $A_{x,y,z}$:

$$A_x = 0 \text{ or } 1; \quad A_y = 0 \text{ or } 1; \quad A_z = 0 \text{ or } 1 \quad (22)$$

At first sight, this seems to provide a total number of 8 possibilities; but, if we want to preserve relation (21), we have to select among these 8 possibilities

only those three for which two A 's are one, one is zero. As traditional, for this particular spin we then attribute colors to the three orthogonal directions Ox , Oy and Oz : the two directions that get an $A = 1$ are painted in red, the last in blue [85].

The same operation can obviously be made for all possible choices of the triplet of directions $Oxyz$. A question which then naturally arises is: for an arbitrary direction, can one attribute a given color (a given value for A_x) that remains independent of the context in which it was defined? Indeed, we did not define the value as a property of an Ox direction only, but in the context of two other directions Oy and Oz ; the possibility of a context independent coloring is therefore not obvious. Can we for instance fix Oz and rotate Ox and Oy around it, and still keep the same color for Oz ? We are now facing an amusing little problem of geometry that we might call "ternary coloring of all space directions". Bell as well as Kochen and Specker showed that this is actually impossible; for a proof see either the original articles, or the excellent review [6] given by Mermin.

In the same article, this author shows how the complications of the geometrical problem may be entirely avoided by going to a space of states of dimension 4 instead as 3. He considers two spin 1/2 particles and the following table of 9 quantum variables (we use the same notation as in §4.3):

$$\begin{array}{ccc} \sigma_x^1 & \sigma_x^2 & \sigma_x^1\sigma_x^2 \\ \sigma_y^2 & \sigma_y^1 & \sigma_y^1\sigma_y^2 \\ \sigma_x^1\sigma_y^2 & \sigma_y^1\sigma_x^2 & \sigma_z^1\sigma_z^2 \end{array} \quad (23)$$

All operators have eigenvalues ± 1 . It is easy to see why all three operators belonging to the same line, or to the same column, always commute (the products of two σ 's that anti-commute are commuting operators, since the commutation introduces two -1 signs, with cancelling effects). Moreover, the products of all three operators is always $+1$, except the last column for which it is -1 ³⁰. Here, instead of an infinite number of triplet of directions in space, we have 6 groups of three operators, but the same question as above arises: can we attribute a color to each of the 9 elements of matrix (23), red for result $+1$ and yellow for result -1 , in a way that is consistent with the results of quantum mechanics? For this consistency to be satisfied, all lines and columns should either contain three red cases, or one red and two yellow, except the last column that will contain one or three yellow cases (in order to correspond to -1 instead of $+1$).

This little matrix coloring problem is much simpler than the geometrical coloring problem mentioned above: it is obviously impossible to find 9 num-

³⁰This can easily be checked from the well-known properties of the Pauli matrices; the minus sign for the third column comes from the product of the two i 's, arising from the relation $\sigma_x\sigma_y = i\sigma_z$; on the other hand, in the third line one gets $i \times (-i) = 1$ because of the change of order of the operators.

bers with a product that is at the same time equal to 1, condition on rows, and -1 , condition on columns (we note in passing that Mermin’s reasoning is very close to that of § 4.3, which illustrates how similar the GHZ theorem and this form of the BKS theorem are). Here, as in the three direction problem, non-contextuality leads us to an impossible coloring problem. For another illustration of the impossibility, see also § VI of ref. [6] which deals with three $1/2$ spins instead of two.

What can we conclude from this contradiction? Certainly that the predictions of quantum mechanics are incompatible with a non-contextual view on the EPR elements of reality/additional variables, where the result of the measurement should depend solely of the system measured - see for instance the discussion given in ref. [86]. But is this a good argument against these elements of reality, or at least an indication that, if they exist, their properties are completely unexpected? Not really. As Bell noted [3], “the result of an observation may reasonably depend not only on the state of the system (including hidden/additional variables) but also on the complete disposition of the apparatus”. There is for instance no special conceptual difficulty in building a theory where additional variables are attributed to the apparatuses, and where both kinds of additional variables collaborate in order to determine the observed result. Violations of the Bell theorem by quantum mechanics are therefore generally considered as much more significant quantum manifestations than violations of the BKS theorem. For a general discussion of the status of the various “impossibility theorems” with emphasis on the BKS theorems, see ref. [6]

5 Non-locality and entanglement: where are we now?

In view of the locality theorems as well as their violation by the modern experimental results, which were not available when the orthodox interpretation of quantum mechanics was invented, some physicists conclude triumphantly: “Bohr was right!”, while others will claim with the same enthusiasm “Bohr was wrong!”. Both these opinions make sense, depending on what aspect of the debate one wishes to favor. We have already touched the question at the end of § 3.2.3; here, we will just add that, whether one personally feels closer to the orthodox camp or to local realism, it remains clear that the line initiated by Einstein and Bell had the decisive role in the last 50 years. In fact, they are the ones who pointed out the inadequate character of some impossibility theorems, as well as the crucial importance of the notion of locality in all these discussions. This resulted in much more progress and understanding than the simple re-statement of the orthodox position. For instance, even now, the introduction of the reduction of the state vector is sometimes “explained” by invoking the “unavoidable perturbations that the

measurement apparatus brings to the measured system” - see for instance the traditional discussion of the Heisenberg microscope which still appears in textbooks! But, precisely, the EPR-Bell argument shows us that this is only a cheap explanation: in fact, the quantum description of a particle can be modified without any mechanical perturbation acting on it, provided the particle in question was previously correlated with another particle. So, a trivial effect such as a classical recoil effect in a photon-electron collision cannot be the real explanation of the nature of the wave packet reduction! It is much more fundamentally quantum and may involve non-local effects.

Another lesson is that, even if quantum mechanics and relativity are not incompatible, they do not fit very well together: the notion of events in relativity, which are supposed to be point-like in space-time, or the idea of causality, are still basic notions, but not as universal as one could have thought before the Bell theorem. Indeed, quantum mechanics teaches us to take these notions “with a little grain of salt”. Still another aspect is related to the incredible progress that experiments have made in the 20th century, whether or not stimulated by fundamental quantum mechanics. One gets the impression that this progress is such that it will allow us to have access to objects at all kinds of scale, ranging from the macroscopic to the microscopic. Therefore, while at Bohr’s time one could argue that the precise definition of the border line between the macroscopic world of measurement apparatuses was not crucial, or even academic, the question may become of real importance; it may, perhaps, even give rise to experiments one day. All these changes, together, give the impression that the final stage of the theory is not necessarily reached and that conceptual revolutions are still possible.

In this section, we give a brief discussion of some issues that are related to quantum non-locality and entanglement, with some emphasis on those that are presently, or may soon be, the motivation of experiments (§ 5.2 is an exception, since it is purely theoretical). Going into details would certainly bring us beyond the scope of this article, so that we will limit ourselves to a few subjects, which we see as particularly relevant, even if our choice may be somewhat arbitrary. Our main purpose is just to show that, even if theoretically it is really difficult to add anything to what the founding fathers of quantum mechanics have already said long ago, it still remains possible to do interesting physics in the field of fundamental quantum mechanics! Even if we treat the subject somewhat superficially, the hope is that the reader will be motivated to get more precise information from the references.

5.1 Loopholes, conspiracies

One sometimes hears that the experiments that have been performed so far are not perfectly convincing, and that no one should claim that local realism à la Bell has been disproved. Strictly speaking, this is true: there are

indeed logical possibilities, traditionally called “loopholes”, which are still open for those who wish to restore local realism. One can for instance deny the existence of any real conflict between the experimental results and the Bell inequalities. First of all, of course, one can always invoke trivial errors, such as very unlikely statistical fluctuations, to explain why all experiments seem to “mimic” quantum mechanics so well; for instance some authors have introduced ad hoc fluctuations of the background noise of photomultipliers, which would magically correct the results in a way that would give the impression of exact agreement with quantum mechanics. But the number and variety of Bell type experiments supporting quantum mechanics with excellent accuracy is now large; in view of the results, very few physicists seem to take this explanation very seriously.

Then one could also think of more complicated scenarios: for instance, some local unknown physical variables may couple together in a way that will give the (false) impression of non-local results, while the mechanism behind them remains local. One possibility is that the polarization analyzers might, somehow, select a subclass of pairs which depend on their settings; then, for each choice (a, b) , only a small fraction of all emitted pairs would be detected; one could then assume that, when the orientation of the analyzers are changed by a few degrees, all the pairs that were detected before are eliminated, and replaced with a completely different category of physical systems with arbitrary properties. In this situation, everything becomes possible: one can ascribe to each category of pairs whatever ad hoc physical properties are needed to reproduce any result, including those of quantum mechanics, while remaining in a perfectly local context.

Indeed, in the derivation of the Bell inequalities, one assumes the existence of ensemble averages over a non-biased, well defined, ensemble of pairs, which are completely independent of the settings a and b . Various proofs of the Bell inequalities are possible, but in many of them one explicitly writes the averages with an integral containing a probability distribution $\rho(\lambda)$; this function mathematically defines the ensemble on which these averages are taken. The non-biasing assumption is equivalent to assuming that ρ is independent of a and b ; on the other hand, it is easy to convince oneself that the proof of the Bell inequalities is no longer possible if ρ becomes a function of a and b . In terms of the reasoning of § 4.1.2, where no function ρ was introduced, what we have assumed is that the four numbers A, A', B and B' are all attached to the same pair; if M was built from more numbers, such as numbers associated to different pairs, the algebra would clearly no longer hold, and the rest of the proof of the inequality would immediately collapse..

Of course, no problem occurs if every emitted pair is detected and provides two results ± 1 , one on each side, whatever the choice of a and b (and even if this choice is made after the emission of the pair). It then makes sense to obtain the ensemble average $\langle M \rangle$ from successive measurements

of four average values $\langle AB \rangle$, $\langle AB' \rangle$, etc.. But, if many pairs are undetected, one can not be completely sure that the detection efficiency remains independent of the settings a and b ; if it is not, the four averages may in principle correspond to different sub-ensembles, and there is no special reason why their combination by sum and difference should not exceed the limit of 2 given by the Bell theorem³¹. The important point is not necessarily to capture all pairs, since one could in theory redefine the ensemble in relation with detection; but what is essential, for any perfectly convincing experiment on the violation of the Bell inequalities, is to make sure that the sample of counted events is completely independent of their settings a and b (unbiased sample). This, in practice, implies some sort of selection (or detection) that is completely independent of the settings, which is certainly not the case in any experiment that detect only the particles that have crossed the analyzers.

An ideal situation would be provided by a device with a triggering button that could be used by an experimentalist, who could at will launch a pair of particles (with certainty); if the pair in question was always analyzed and detected, with 100% efficiency, the loophole would definitely be closed! When discussing thought experiments, Bell introduced in some of his talks the notion of “preliminary detectors” [87], devices which he sketched as cylinders through which any pair of particles would have to propagate before reaching both ends of the experiment (where the a and b dependent measurement apparatuses sit); the idea was that the preliminary detectors should signal the presence of pairs and that, later, the corresponding pairs would always be detected at both ends. The role of these cylinders was therefore to make the definition of the sample perfectly precise, even if initially the pairs were emitted by the source in all directions. Such class of systems, which allow a definition of an ensemble that is indeed totally independent of a and b , are sometimes called an “event ready detectors”. See also reference [88] where Bell imagines a combination of veto and go detectors associated with the first detected particles in a ternary emission, precisely for the purpose of better sample definition.

Needless to say, in practice, the situation is very different. First, one should realize that, in all experiments performed until now, most pairs are simply missed by the detectors. There are several reasons for this situation: in photon experiments, the particles are emitted in all directions, while the analyzers collect only a small solid angle and, therefore, only a tiny fraction of the pairs (this was especially true in the initial experiments using photon cascades; in more modern experiments [61], the use of parametric photon conversion processes introduces a strong correlation between the direction

³¹Another intuitive way to understand why experiments where most pairs go undetected are useless for a violation of the inequality is the following: if one associates 0 to the absence of result, the occurrence of many zeros in the results will bring the correlations rates closer to zero and the combination will never exceed 2.

of propagation of the photons and a much better collection efficiency, but it still remains low). Moreover, the transmission of the analyzers is less than one (it is actually less than $1/2$ if ordinary photon polarization filters are used, but experiments have also been performed with birefringent two-channel analyzers [59], which are not limited to 50% efficiency). Finally, the quantum efficiency of particle detectors (photomultipliers for photons) is not 100% either, so that pairs of particles are lost at the last stage too. There is no independent way to determine the sample of detected pairs, except of course the detection process itself, which is obviously a and b dependent; as a consequence, all experimental results become useful only if they are interpreted within a “no-biasing” assumption, considering that the settings of the analyzers does not bias the statistics of events. On the other hand, we should also mention that there is no known reason why such a sample biasing should take place in the experiments, and that the possibility remains speculative. For proposals of “loophole free” experiments³², see refs. [89] and [90]; actually, there now seems to be a reasonable hope that this loophole will be closed by the experiments within the next few years.

Other loopholes are also possible: even if experiments were done with 100% efficiency, one could also invoke some possibilities for local processes to artificially reproduce quantum mechanics. One of them is usually called the “conspiracy of the polarizers” (actually, “conspiracy of the analyzers” would be more appropriate; the word polarizer refers to the experiments performed with photons, where the spin orientation of the particles is measured with polarizing filters; but there is nothing specific of photons in the scenario, which can easily be transposed to massive spin $1/2$ particles) - or also “communication loophole”. The idea is the following: assume that, by some unknown process, each analyzer could become sensitive to the orientation of the other analyzer; it would then acquire a response function which depends on the other setting and the function A could acquire a dependence on both a and b . The only way to beat this process would be to choose the settings a and b at the very last moment, and to build an experiment with a large

³²A perfect correlation between the detections on each side (in an ideal experiment with parametric generation of photons for instance) would provide another possible scheme for a loophole free experiment - this, of course, implies that two channel detectors with a 100% efficiency are used on both ends of the experiment. In itself, the fact that any click at one side is always correlated with a click at the other, independently of the settings a and b , is not sufficient to exclude a setting dependence of the ensemble of detected pairs. But, if one assumes locality at this stage also, a simple reasoning shows that a perfect detection correlation is sufficient to ensure the independence: how could a particle on one side “know” that it belongs to the right sub-ensemble for the other particle to be detected, without knowing the other setting? In other words, locality arguments may be used, not only for the results of the apparatuses (the functions A and B), but also in order to specify the ensemble of observed pairs (the distribution function ρ). Under these conditions, the observation (in some future experiment) of a violation of the Bell inequalities with a perfect detection correlation would be sufficient to exclude local theories, and therefore to close the loophole.

distance between the two analyzers so that no information can propagate (at the speed of light) between the two of them. A first step in this direction was done by Aspect and coll. in 1982 [91], but more recent experiments have beautifully succeeded in excluding this possibility in an especially convincing way [92]. So there no longer exist a real conspiracy loophole; quantum mechanics seems to still work well under these more severe time dependent conditions.

Along a similar line is what is sometimes called the “fatalistic loophole” (or also “superdeterminism”). The idea is to put into question an implicit assumption of the reasoning that leads to the Bell theorem: the completely arbitrary choice of the settings a and b by the experimenters. Usually, a and b are indeed considered as free variables: their values that are not the consequence of any preliminary event that took place in the past, but those of a free human choice. On the other hand, it is true that there is always some an overlap between the past cones of two events, in this case the choice of the settings. It is therefore always possible in theory to assume that they have a common cause; a and b are then no longer free parameters, but variables that can fluctuate (in particular, if this cause itself fluctuates) with all kinds of correlations. In this case, it is easy to see that the proof of the Bell theorem is no longer possible³³, so that any contradiction with locality is avoided. What is then denied is the notion of free will of the experimenters, whose decisions are actually predetermined, without them being aware of this fact; expressed more technically, one excludes from the theory the notion of arbitrary external parameters, which usually define the experimental conditions. This price being paid, one could in theory build an interpretation of quantum mechanics that would remain at the same time realistic, local and (super)deterministic, and would include a sort of physical theory of human decision. This is, of course, a very unusual point of view, and the notion of arbitrary external parameters is generally accepted; in the words of Bell [93]: “A respectable class of theories, including quantum theory as it is practised, have free external variables in addition to those internal to and conditioned by the theory....They are invoked to represent the experimental conditions. They also provide a point of leverage for free willed experimenters, ...”. Needless to say, the fatalist attitude in science is even more subject to the difficulties of orthodox quantum mechanics concerning the impossibility to develop a theory without observers, etc..

We could not conclude honestly this section on loopholes without mentioning that, while most specialists acknowledge their existence, they do not take them too seriously because of their “ad hoc” character. Indeed, one should keep in mind that the explanations in question remain artificial,

³³For instance, in the proof that makes uses of a probability density $\rho(\lambda)$, if one assumes that a and b become two functions $a(\lambda)$ and $b(\lambda)$, it makes no sense to compare the average values for different fixed values of a and b .

inasmuch they do not rest on any precise theory: no-one has the slightest idea of the physical processes involved in the conspiracy, or of how pair selection would occur in a way that is sufficiently complex to perfectly reproduce quantum mechanics. By what kind of mysterious process would experiments mimic quantum mechanics so perfectly at low collection efficiencies, and cease to do so at some threshold of efficiency? Bell himself was probably the one who should have most liked to see that his inequalities could indeed be used as a logical tool to find the limits of quantum mechanics; nevertheless, he found these explanation too unaesthetic to be really plausible. But in any case logic remains logic: yes, there still remains a slight possibility that, when the experiments reach a level of efficiency in pair collection where the loophole becomes closed, the results concerning the correlation rates will progressively deviate from those of quantum mechanics to reach values compatible with local realism. Who knows?

5.2 Locality, contrafactuality

One can find in the literature various attitudes concerning the exact relation between quantum mechanics and locality. Some authors consider that the non-local character of quantum mechanics is a well-known fact, while for others quantum non-locality is an artefact created by the introduction into quantum mechanics of notions which are foreign to it (typically the EPR elements of reality). Lively discussions to decide whether or not quantum mechanics in itself is inherently non-local have taken place and are still active [94] [95] [96]; see also references [21] and [97] [98]. Delicate problems of logic are involved and we will not discuss the question in more detail here.

What is easier to grasp for the majority of physicists is the notion of “contrafactuality” [95]. A counterfactual reasoning consists in introducing the results of possible experiments that can be envisaged for the future as well-defined quantities, and valid mathematical functions to use in equations, even if they are still unknown - in algebra one writes unknown quantities in equations all the time. This is very natural: as remarked by d’Espagnat [99] [100] and by Griffiths [101], “counterfactuals seem a necessary part of any realistic version of quantum theory in which properties of microscopic systems are not created by the measurements”. One can also see the EPR reasoning as a justification of the existence of counterfactuals. But it also remains true that, in practice, it is never possible to realize more than one of these experiments: for a given pair, one has to choose a single orientation of the analyzers, so that all other orientations will remain forever in the domain of speculation. For instance, in the reasoning of § 4.1.2, at least some of the numbers A , A' , B and B' are counterfactuals, and we saw that using them led us to a contradiction with quantum mechanics through the Bell inequalities. One could conclude that contrafactuality should be put into question in quantum mechanics; alternatively, one could maintain counter-

factual reasoning, but then the price to pay is the explicit appearance of non-locality. We have already quoted a sentence by Peres [53] which wonderfully summarizes the situation as seen within orthodoxy: "unperformed experiments have no results"; as Bell once regretfully remarked [93]: "it is a great inconvenience that the real world is given to us once only"!

But, after all, one can also accept contrafactuality as well as explicit non-locality together, and obtain a perfectly consistent point of view; it would be a real misunderstanding to consider the Bell theorem as an impossibility theorem, either for contrafactuality, or for hidden variables. In other words, and despite the fact that the idea is still often expressed, it is not true that the Bell theorem is a new sort of Von Neumann theorem. The reason is simple: why require that theories with contrafactuality/additional variables should be explicitly local at all stages, while it is not required from standard quantum mechanics? Indeed, neither the wave packet reduction postulate, nor the calculation of correlation of experimental results in the correlation point of view (§ 6.1), nor again the expression of the state vector itself, correspond to mathematically local calculations. In other words, even if one can discuss whether or not quantum mechanics is local or not at a fundamental level, it is perfectly clear that its formalism is not; it would therefore be just absurd to request a local formalism from a non-orthodox theory - especially when the theory in question is built in order to reproduce all results of quantum mechanics! As an illustration of this point, as seen from theories with additional variables, we quote Goldstein [16]: "in recent years it has been common to find physicists failing to appreciate that what Bell demonstrated with his theorem was not the impossibility of Bohmian mechanics, but rather a more radical implication - namely non-locality - that is intrinsic to quantum theory itself".

5.3 "All-or-nothing coherent states"; decoherence

In this section, we first introduce many particle quantum states which have particularly remarkable correlation properties; then we discuss more precisely a phenomenon that we have already introduced above, decoherence, which tends to reduce their lifetime very efficiently, especially if the number of correlated particles is large.

5.3.1 Definition and properties of the states

The states that we will call "all-or-nothing coherent states" (or all-or-nothing states for short) could also be called "many-particle GHZ states" since they are generalizations of (16) to an arbitrary number N of particles:

$$|\Psi\rangle = \alpha |1 : +; 2 : +; \dots; N : +\rangle + \beta |1 : -; 2 : -; \dots; N : -\rangle \quad (24)$$

where α and β are arbitrary complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. In fact, the most interesting situations generally occur when α and β have

comparable modulus, meaning that there are comparable probabilities to find the system in states where all, or none, of the spins is flipped (hence the name we use for these states); when α and β are both equal to $1/\sqrt{2}$, these states are sometimes called “states of maximum entanglement” in the literature, but since the measure of entanglement for more than two particles is not trivial (several different definitions have actually been proposed in the literature), here we will use the words “all-or-nothing states” even in this case.

In order to avoid a frequent cause of confusion, and to better emphasize the peculiarities of these entangled states, let us first describe a preparation procedure that would NOT lead to such a state. Suppose that N spin $1/2$ particles oriented along direction Ox enter a Stern-Gerlach magnet oriented along direction Oz , or more generally that N particles cross a filter (polarization dependent beam splitter, Stern-Gerlach analyzer, etc.) while they are initially in a state which is a coherent superposition of the eigenstates of this filter, with coefficients α_1 and β_1 . The effect of the filter on the group of particles is to put them into a state which is a product of coherent superpositions of the two outputs of the filter, namely:

$$|\Psi\rangle = \left[\alpha |1: +\rangle + \beta |1: -\rangle \right] \otimes \left[\alpha |2: +\rangle + \beta |2: -\rangle \right] \otimes \dots \otimes \left[\alpha |N: +\rangle + \beta |N: -\rangle \right] \quad (25)$$

The point we wish to make is that this state is totally different from (24), since it contains many components of the state vector where some of the spins are up, some down. In (25), each particle is in a coherent superposition of the two spin states, a situation somewhat analogous to a Bose-Einstein condensate where all particles are in the same coherent state - for instance two states located on either sides of a potential barrier as in the Josephson effect. By contrast, in (24) all spins, or none, are flipped from one component to the other³⁴ so that the coherence is essentially a N -body coherence only; it involves entanglement and is much more subtle than in (25). For instance, one can show [76] that the coherence in question appears only “at the last moment”, when all particles are taken into account: as long as one considers any subsystem of particles, even $N - 1$, it exhibits no special property and the spins are correlated in an elementary way (as they would be in a classical magnet); it is only when the last particle is included that quantum coherence appears and introduces effects which are completely non-classical.

Assume for instance that:

$$\alpha = 1/\sqrt{2} \quad ; \quad \beta = e^{i\varphi}/\sqrt{2} \quad (26)$$

³⁴In an all-or-nothing coherent state, all spins are not necessarily up in the first component of the state vector, while they are down in the second; what matters is that every spin changes component from one component to the other and reaches an orthogonal state (the quantization axis of every spin is not even necessarily the same).

and that a measurement is performed of a component of each spin that belongs to the Oxy plane and is defined by its angle θ_1 with Ox for the first particle, θ_2 for the second,... θ_N for the last. It is an easy exercise of standard quantum mechanics to show that the product of all results has the following average value:

$$E(\theta_1, \theta_2, \dots, \theta_N) = \cos(\theta_1 + \theta_2 + \dots + \theta_N - \varphi) \quad (27)$$

(we take the usual convention where the results of each measurement is ± 1). For instance, each time the sum $\theta_1 + \theta_2 + \dots + \theta_N - \varphi$ is equal to an integer even multiple of π , the average is 1, indicating that the result is certain and free from any fluctuation (in the same way, an odd multiple of π would give a certain value, -1). Indeed, the result of the quantum calculation may look extremely simple and trivial; but it turns out that, once more, it is totally impossible to reproduce the oscillations contained in (27) within local realism. In the case $N = 2$, this is of course merely the consequence of the usual Bell theorem; as soon as N becomes 3 or takes a larger value, the contradiction becomes even more dramatic. Actually, if one assumes that a local probabilistic theory reproduces (27) only for some sets of particular value of the angles θ 's (those for which the result is certain), one can show [76] that the theory in question necessarily predicts that E is independent of all θ 's. The average keeps a perfectly constant value $+1$! Indeed, the very existence of the oscillation predicted by (27) can be seen as a purely quantum non-local effect (as soon as $N \geq 2$).

This is, by far, not the only remarkable property of all-or-nothing coherent states. For instance, it can be shown that they lead to exponential violations of the limits put by local realistic theories [83]; it has also been pointed out [102] that these states, when relation (26) is fulfilled (they are then called “maximally correlated states” in [102]), have interesting properties in terms of spectroscopic measurements: the frequency uncertainty of measurements decreases as $1/N$ for a given measurement time, and not as $1/\sqrt{N}$ as a naive reasoning would suggest. This is of course a more favorable situation, and the quantum correlation of these states may turn out to be, one day, the source of improved accuracy on frequency measurements. How to create such states with massive particles such as atoms, and not with photons as usual, was demonstrated experimentally by Hagley et al. in 1997 [103] in the case $N = 2$. We have already mentioned in § 4.3 that entanglement with $N = 3$ was reported in refs. [81] and [82]. Proposals for methods to generalize to larger values of N with ions in a trap were put forward by Molmer et al. [104]; the idea exploits the motion dependence of resonance frequencies for a system of several ions in the same trap, as well as on some partially destructive interference effects. The scheme was successfully put into practice in a very recent experiment by Sackett et al. [105] where “all-or-nothing states” were created for $N = 2$ as well as $N = 4$ ions in a trap.

5.3.2 Decoherence

We have defined in §2.1 decoherence as the initial part of the phenomenon associated with the Von Neumann infinite regress: coherent superpositions tend to constantly propagate towards the environment, they involve more and more complex correlations with it, so that they become rapidly completely impossible to detect in practice. To see more precisely how this happens, let us for instance consider again state (24); we now assume that the single particle states $| + \rangle$ and $| - \rangle$ are convenient notations for two states where a particle has different locations in space (instead of referring only to opposite spin directions): this will happen for instance if the particles cross a Stern-Gerlach analyzer which correlates the spin directions to the positions of the particles. Under these conditions, it is easy to see that the coherence contained in the state vector becomes extremely fragile to any interaction with environment. To see why, let us assume that an elementary particle (photon for instance), initially in state $| \mathbf{k}_0 \rangle$, interacts with the particles. It will then scatter into a quantum state that is completely different, depending on where the scattering event took place: if the scattering atoms are in the first state $| + \rangle$, the photon is scattered by the atoms into state $| \mathbf{k}_+ \rangle$; on the other hand, if it interacts with atoms in state $| - \rangle$, it is scattered into state $| \mathbf{k}_- \rangle$ ³⁵. As soon as the new particle becomes correlated with the atoms, the only state vector that can be used to describe the system must incorporate this new particle as well, and becomes:

$$| \Psi' \rangle = \alpha | 1 : + ; 2 : + ; \dots ; N : + \rangle \otimes | \mathbf{k}_+ \rangle + \beta | 1 : - ; 2 : - ; \dots ; N : - \rangle \otimes | \mathbf{k}_- \rangle \quad (28)$$

Assume now that we are interested only in the system of N atoms; the reason might be, for instance, that the scattered photon is impossible (or very difficult) to detect (e.g. it may be a far-infrared photon). It is then useful to calculate the partial trace over this photon in order to obtain the density operator which describes the atoms only. A straightforward calculation shows that this partial trace can be written, in the basis of the two states $| +, +, +, \dots \rangle$ and $| -, -, -, \dots \rangle$:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \langle \mathbf{k}_- | \mathbf{k}_+ \rangle \\ \alpha^*\beta \langle \mathbf{k}_+ | \mathbf{k}_- \rangle & |\beta|^2 \end{pmatrix} \quad (29)$$

(for the sake of simplicity we assume that the states $| \mathbf{k}_\pm \rangle$ are normalized). We see in this formula that, if the scalar product $\langle \mathbf{k}_- | \mathbf{k}_+ \rangle$ was equal to one, the density matrix of the atoms would not be affected at all by the scattering of the single photon. But this would assume that the photon is

³⁵We could also have assumed that the photon is focussed so that it can interact only with one sort of atoms, but is not scattered by the other, without changing the conclusion of this discussion.

scattered exactly into the same state, independently of the spatial location of the scatterers! This cannot be true if the distance between the locations is much larger than the photon wavelength. Actually, it is much more realistic to assume that this scalar product is close to zero, which means that the off-diagonal element of (29), in turn, becomes almost zero. We then conclude that the scattering of even a single particle destroys the coherence between atomic states, as soon as they are located at different places.

The situation becomes even worse when more and more photons (assumed to be all in the same initial state $|\mathbf{k}_0\rangle$) are scattered, since one then has to replace (28) by the state:

$$|\Psi'\rangle = \alpha |1: +; 2: +; \dots; N: +\rangle \otimes |\mathbf{k}_+\rangle |\mathbf{k}'_+\rangle |\mathbf{k}''_+\rangle \dots + \beta |1: -; 2: -; \dots; N: -\rangle \otimes |\mathbf{k}_-\rangle |\mathbf{k}'_-\rangle |\mathbf{k}''_-\rangle \dots \quad (30)$$

with obvious notation (the states with n primes correspond to the $n - 1$ th. scattered photon); the same calculation as above then provides the following value for ρ :

$$\left(\begin{array}{cc} |\alpha|^2 & \alpha\beta^* \langle \mathbf{k}_- | \mathbf{k}_+ \rangle \langle \mathbf{k}'_- | \mathbf{k}'_+ \rangle \dots \\ \alpha^*\beta \langle \mathbf{k}_+ | \mathbf{k}_- \rangle \langle \mathbf{k}'_+ | \mathbf{k}'_- \rangle \dots & |\beta|^2 \end{array} \right) \quad (31)$$

Since we now have, in the off-diagonal elements, the product of all single scalar product $\langle \mathbf{k}_- | \mathbf{k}_+ \rangle$, it is clear that these elements are even more negligible than when a single photon is scattered. Actually, as soon as the two states $|\mathbf{k}_+\rangle$ and $|\mathbf{k}_-\rangle$ are not strictly identical, they tend exponentially to zero with the number of scattering events.

This is a completely general property: objects (especially macroscopic objects) have a strong tendency to leave a trace in the environment by correlating themselves with any elementary particle which passes by; in the process, they lose their own coherence, which regresses into a coherence involving the environment and more and more complex correlations with it (the scattered photon, in turn, may correlate with other particles); soon it becomes practically impossible to detect. The phenomenon is unavoidable, unless the scattering properties of both states symbolized by $|+\rangle$ and $|-\rangle$ are exactly the same, which excludes any significant spatial separation between the states. In particular, it is impossible to imagine that a cat, whether dead or alive, will scatter photons exactly in the same way, otherwise we could not even see the difference! This shows how fragile coherent superpositions of macroscopic objects are, as soon as they involve states that can be seen as distinct.

We are now in a position where we can come back in more detail to some questions that we already discussed in passing in this text, and which are related to decoherence and/or the Schrödinger cat. The first question relates to the conceptual status of the phenomenon of decoherence. Some

authors invoke this phenomenon as a kind of “explanation” of the postulate of wave packet reduction: when the superposition of the initial system becomes incoherent, are we not in presence of a statistical mixture that resembles the description of a classical object with well defined (but ignored) properties? On this point, we do not have much to add to what was already said in § 2: this explanation is unsatisfactory because the purpose of the postulate of wave packet reduction is not to explain decoherence, which can already be understood from the Schrödinger equation, but the uniqueness of the result of the measurement - in fact, the effect of the wave packet reduction is sometimes to put back the measured sub-system into a pure state, which is the perfect opposite of a statistical mixture, so that the real question is to understand how the (re)emergence of a pure state should be possible [132]. Indeed, in common life, as well as in laboratories, one never observes superposition of results; we observe that Nature seems to operate in such a way that a single result always emerges from a single experiment; this will never be explained by the Schrödinger equation, since all that it can do is to endlessly extend its ramifications into the environment, without ever selecting one of them only.

Another way to say the same thing is to emphasize the logical structure of the question. The starting point is the necessity for some kind of limit of the validity of the linear Schrödinger equation, the initial reason being that a linear equation can never predict the emergence of a single result in an experiment. The difficulty is where and how to create this border. Logically, it is then clear that this problem will never be solved by invoking any process that is entirely contained in the linear Schrödinger equation, such as decoherence or any other similar linear process; common sense says that, if one stays in the middle of a country one never reaches its borders. Actually, no one seriously doubts that a typical measurement process will involve decoherence at some initial stage, but the real question is what happens after.

Pressed to this point, some physicists reply that one can always assume that, at some later stage, the superposition resolves into one of its branches only; this is of course true, but this would amount to first throwing a problem out by the door, and then letting it come back through the window! (see discussions above, for instance on the status of the state vector and the necessity to resolve the Wigner friend paradox). A more logical attitude, which is indeed sometimes proposed as a solution to the problem, is to consider that the natural complement of decoherence is the Everett interpretation of quantum mechanics (see § 6.5); indeed, this provides a consistent interpretation of quantum mechanics, where the emergence of a single result does not have to be explained, since it is assumed never to take place (the Schrödinger equation then has no limit of validity). But, of course, in this point of view, one has to deal with all the intrinsic difficulties of the Everett interpretation, which we will discuss later.

Concerning terminology, we have already mentioned in § 2.3 that, during the last few years, it has become rather frequent to read the words “Schrödinger cat” (SC) used in the context of states such as (24) for small values of N (actually even for a single ion, when $N = 1$). This is a redefinition of the words, since the essential property of the original cat was to have a macroscopic number of degree of freedom, which is not the case for a few atoms, ions or photons. But let us now assume that someone succeeded in preparing an all-or-nothing state with a very large value of N , would that be a much better realization of the Schrödinger cat as meant by its inventor? To some extent, yes, since the cat can be seen as a symbol of a system of many particles that change their state, when one goes from one component of the state vector to the other. Indeed, it is likely that many of the atoms of a cat take part in different chemical bonds if the cat is alive or dead, which puts them in a different quantum state. But it seems rather hard to invent a reason why every atom, every degree of freedom, should necessarily be in an orthogonal state in each case, while this is the essential property of “all-or-nothing states”. In a sense they do too much for realizing a standard Schrödinger cat, and the concepts remain somewhat different, even for large values of N .

From an experimental point of view, decoherence is an interesting physical phenomenon that is certainly worth studying in itself, as recent experiments have illustrated [36]; a result of these studies and of the related calculations, among others, is to specify the basis in the space of states that is relevant to the decoherence process, as a function of the coupling Hamiltonian, as well as the characteristic time constants that are associated. One can reasonably expect that more experiments on decoherence will follow this initial breakthrough and provide a more detailed understanding of many aspects of the phenomenon. Nevertheless decoherence is not to be confused with the measurement process itself; it is just the process which takes place just before: during decoherence, the off-diagonal elements of the density matrix³⁶ vanish (decoherence), while in a second step all diagonal elements but one should vanish (emergence of a single result).

5.4 Quantum cryptography, teleportation

In the two subsections below, we discuss two concepts that have some similarity, quantum cryptography and teleportation.

³⁶ The formalism of the density operator, or matrix, is elegant and compact, but precisely because it is compact it sometimes partially hides the physical origin of the mathematical terms. The density matrix allows one to treat in the same way classical probabilities, arising from non fundamental uncertainties and imperfect knowledge of a physical system, and purely quantum probabilities which are more fundamental and have nothing to do with any particular observer. But mathematical analogies should not obscure conceptual difficulties!

5.4.1 Sharing cryptographic keys by quantum measurements

A subject that is relatively new and related to the EPR correlations, is quantum cryptography [106] [107]. The basic idea is to design a perfectly safe system for the transmission at a distance of a cryptographic key - the word refers to a random series of numbers 0 or 1, which are used to code, and subsequently decode, a message. In a first step, the two remote correspondents A (traditionally called Alice) and B (traditionally called Bob) share this key; they then use it to code, or to decode, all the messages that they exchange; if the key is perfectly random, it becomes completely impossible for anyone who does not know it to decode any message, even if it is sent publicly. But if, during the initial stage of key exchange, someone can eavesdrop the message (the actor in question is traditionally called Eve), in the future he/she will be able to decode all messages sent with this key. Exchanging keys is therefore a really crucial step in cryptography. The usual strategy is to take classical methods to keep the secret: storage in a safe, transport of the keys by secure means, etc.; it is always difficult to assess its safety, which depends on many human factors.

On the other hand, quantum sharing of keys relies on fundamental physical laws, which are impossible to break: however clever and inventive spies may be, they will not be able to violate the laws of quantum mechanics! The basic idea is that Alice and Bob will create their common key by making quantum measurements on particles in an EPR correlated state; in this way they can generate series of random numbers that can be subsequently used as a secret communication. What happens if Eve tries to intercept the photons, for example by coupling some elaborate optical device to the optical fiber where the photons propagate between Alice and Bob, and then making measurements?

If she wants to operate unnoticed, she clearly cannot just absorb the photons in the measurement; this would change the correlation properties observed by Alice and Bob. The next idea is to try to “clone” photons, in order to make several identical copies of the initial photon; she could then use a few of them to measure their properties, and re-send the last of them on the line so that no-one will notice anything. But, it turns out that “quantum cloning” is fundamentally impossible: within the rules of quantum mechanics, there is absolutely no way in which several particles can be put into the same arbitrary and unknown state $|\varphi\rangle$ as one given particle [108] [109] - see also [110] for a discussion of multiple cloning. In appendix IV we discuss why, if state cloning were possible, it would be possible to apply it to each particle of an EPR pair of correlated particles; then the multiple realization of the states could be used to transmit information on the orientations a and b used by the experimenters. Since such a scheme would not be subject to any minimum time delay, it could also transmit messages at superluminal velocities, and be in contradiction with relativity.

Fortunately for the consistency of physics, the contradiction is avoided by the fact that cloning of single systems is impossible in quantum mechanics!

So, whatever Eve does to get some information will automatically change the properties of the photons at both ends of the optical fibre, and thus be detectable by Alice and Bob, if they carefully compare their data and their correlation rates. Of course, they do not necessarily have a way to prevent Eve's eavesdropping, but at least they know what data can be used as a perfectly safe key. There is actually a whole variety of schemes for quantum cryptography, some based on the use of EPR correlated particles, others not [107]; but this subject is beyond our scope here.

5.4.2 Teleporting a quantum state

The notion of quantum teleportation [111] is also related to quantum non-locality; the idea is to take advantage of the correlations between two entangled particles, which initially are for instance in state (24) (for $N = 2$), in order to reproduce at a distance any arbitrary spin state of a third particle. The scenario is the following: initially, two entangled particles propagate towards two remote regions of space; one of them reaches the laboratory of the first actor, Alice, while the second reaches that of the second actor, Bob; a third particle in quantum state $|\varphi\rangle$ is then provided to Alice in her laboratory; the final purpose of the all the scenario is to put Bob's particle into exactly the same state $|\varphi\rangle$, whatever it is (without, of course, transporting the particle itself). One then says that state $|\varphi\rangle$ has been teleported.

More precisely, what procedure is followed in teleportation? Alice has to resist the temptation of performing any measurement on the particle in state $|\varphi\rangle$ to be teleported; instead, she performs a "combined measurement" that involves at the same time this particle as well as her particle from the entangled pair. In fact, for the teleportation process to work, an essential feature of this measurement is that no distinction between the two particles involved must be established. With photons one may for instance, as in ref. [112], direct the particles onto opposite sides of the same optical beam splitter, and measure on each side how many photons are either reflected or transmitted; this device does not allow one to decide from which initial direction the detected photons came, so that the condition is fulfilled. Then, Alice communicates the result of the measurement to Bob; this is done by some classical method such as telephone, e-mail etc., that is by a method that is not instantaneous but submitted to the limitations related to the finite velocity of light. Finally, Bob applies to his particle an unitary transformation that depends on the classical information he has received; this operation puts it exactly into the same state $|\varphi\rangle$ as the initial state of the third particle, and realizes the "teleportation" of the state. The whole scenario is "mixed" because it involves a combination of transmission of quantum information (through the entangled state) and classical informa-

tion (the phone call from Alice to Bob).

Teleportation may look trivial, or magical, depending how one looks at it. Trivial because the possibility of reproducing at a distance a state from classical information is not in itself a big surprise. Suppose for instance that Alice decided to choose what the teleported state should be, and filtered the spin (she sends particles through a Stern-Gerlach system until she gets a +1 result³⁷); she could then ask Bob by telephone to align his Stern-Gerlach filter in the same direction, and to repeat the experiment until he also observes a +1 result. This might be called a trivial scenario, based only on the transmission of classical information. But teleportation does much more than this! First, the state that is transported is not necessarily chosen by Alice, but can be completely arbitrary. Second, a message with only binary classical information, such as the result of the combined experiment made by Alice in the teleportation scheme, is certainly not sufficient information to reconstruct a quantum state; in fact a quantum state depends on continuous parameters, while results of experiments correspond to discrete information only. Somehow, in the teleportation process, binary information has turned into continuous information! The latter, in classical information theory, would correspond to an infinite number of bits (in the trivial scenario above, sending the complete information on the state with perfect accuracy would require an infinite time).

Let us come back in more detail to these two differences between teleportation and what we called the trivial scenario. Concerning the arbitrary character of the state, of course Alice may also, if she wishes, teleport a known state. For this, beforehand, she could for instance perform a Stern-Gerlach experiment on the third particle in order to filter its spin state. The remarkable point, nevertheless, is that teleportation works exactly as well is she is given a spin in a completely unknown state, by a third partner for instance; in this case, it would be totally impossible for her to know what quantum state has been sent just from the result of the combined experiment. A natural question then arises: if she knows nothing about the state, is it not possible to improve the transmission efficiency by asking her to try and determine the state in a first step, making some trivial single-particle measurement? The answer to the question is no, and for a very general reason: it is impossible to determine the unknown quantum state of a single particle (even if one accepts only an a posteriori determination of a perturbed state); one quantum measurement clearly does not provide sufficient information to reconstruct the whole state; but several measurements do not provide more information, since the first measurement has already changed

³⁷For filtering a spin state, one obviously needs to use a non-destructive method for detection after the Stern-Gerlach magnet. One could for instance imagine a laser detection scheme, designed in such a way that the atom goes through an excited state, and then emits a photon by returning to the same internal ground state (closed optical pumping cycle - this is possible for well chosen atomic transition and laser polarization).

the spin state of the particle. In fact, acquiring the complete information on an unknown spin-1/2 state would require from Alice to start from an infinite number of particles that have been initially prepared into this same state, and to perform measurements on them; this is, once more, because the information given by each measurement is discrete while the quantum state has continuous complex coefficients. Alice cannot either clone the state of the single particle that she is given, in order to make several copies of it and measure them (see preceding section and appendix IV). So, whatever attempt Alice makes to determine the state before teleportation will not help the process.

Concerning the amount of transmitted information, what Bob receives has two components: classical information sent by Alice, with a content that is completely “uncontrolled”, since it is not decided by her, but just describes the random result of an experiment; quantum information contained in the teleported state itself (what we will call a “q-bit” in the next section) and can possibly be controlled. We know that neither Bob nor Alice can determine the teleported state from a single occurrence, but also that Alice can prepare the state to be teleported by a spin filtering operation in a direction that she decides arbitrarily; Bob then receives some controlled information as well. For instance, if the teleportation is repeated many times, by successive measurements on the teleported particles Bob will be able to determine its quantum state with arbitrary accuracy, including the direction that was chosen by Alice; he therefore receives a real message from her (for a discussion of the optimum strategy that Bob should apply, see ref. [113]).

If one wishes to describe teleportation things in a sensational way, one could explain that, even before Bob receives any classical information, he has already received “almost all the information” on the quantum state, in fact all the controllable information since the classical message does not have this property; this “information” has come to him instantaneously, exactly at the time when Alice performed her combined experiment, without any minimum delay that is proportional to the distance covered. The rest of the information, which is the “difference” between a continuous “information” and a discrete one, comes only later and is, of course, subject to the minimum delay associated with relativity. But this is based on an intuitive notion of “difference between quantum/controllable and classical/non-controllable information” that we have not precisely defined; needless to say, this should not be taken as a real violation of the basic principles of relativity!

Finally, has really something been transported in the teleportation scheme, or just information? Not everyone agrees on the answer to this question, but this may be just a discussion on words, so that we will not dwell further on the subject. What is perfectly clear in any case is that the essence of the teleportation process is completely different from any scenario of classical communication between human beings. The relation between quantum teleportation and Bell-type non-locality experiments is discussed in [114]; see

also [115] as well as [116] for a review of recent results.

5.5 Quantum computing and information

Another recent development is quantum computing [117] [118] [119]. Since this is still a rather new field of research, our purpose here cannot be to give a general overview, while new results are constantly appearing in the literature. We will therefore slightly change our style in the present section, and limit ourselves to an introduction of the major ideas, at the level of general quantum mechanics and without any detail; we will provide references for the interested reader who wishes to learn more.

The general idea of quantum computing [120] is to base numerical calculations, not on classical “bits”, which can be only in two discrete states (corresponding to 0 and 1 in usual binary notation), but on quantum bits, or “q-bits”, that is on quantum systems that have access to a two-dimensional space of states; this means that q-bits can not only be in one of the two states $|0\rangle$ and $|1\rangle$, but also in any linear superposition of them. It is clear that a continuum of states is a much “larger” ensemble than two discrete states only; in fact, for classical bits, the dimension of the state space increases linearly with the number of bits (for instance, the state of 3 classical bits defines a vector with 3 components, each equal to 0 or 1); for q-bits, the dimension increases exponentially (this is a property of the tensor product of spaces; for instance, for three q-bits the dimension of space is $2^3 = 8$). If one assumes that a significant number of q-bits is available, one gets access to a space state with an enormous “size”, where many kinds of interference effects can take place. Now, if one could somehow make all branches of the state vector “work in parallel” to perform independent calculations, it is clear that one could perform much faster calculations, at least in theory. This “quantum parallelism” opens up many possibilities; for instance, the notion of unique computational complexity of a given mathematical problem, which puts limits on the efficiency of classical computers, no longer applies in the same way. Indeed, it has been pointed out [121] that the factorization of large numbers into prime factors may become faster than by classical methods, and by enormous factors. Similar enhancements of the speed of computation are expected in the simulation of many-particle quantum systems [122]. For some other problems the gain in speed is only polynomial in theory, still for some others there is no gain at all.

Fundamentally, there are many differences between classical and quantum bits. While classical bits have two reference states that are fixed once and for all, q-bits can use any orthogonal basis in their space of states. Classical bits can be copied at will and ad infinitum, while the no-cloning theorem mentioned in the preceding section (see also appendix IV) applies to q-bits. On the other hand, classical bits can be transmitted only into the forward direction of light cones, while the use of entanglement and telepor-

tation removes this limitation for q-bits. But we have to remember that there is more distance between quantum q-bits and information than there is for their classical bits: in order to transmit and receive useable information from q-bits, one has to specify what kind of measurements are made with them (this is related to the flexibility on space state basis mentioned above). Actually, as all human beings, Alice and Bob can communicate only at a classical level, by adjusting the macroscopic settings of their measurement apparatuses and observing the red and green light flashes associated with the results of measurements. Paraphrasing Bohr (see the end of § 1.2), we could say that "there is no such concept as quantum information; information is inherently classical, but may be transmitted through quantum q-bits"; nevertheless, the whole field is now sometimes called "quantum information theory". For an early proposal of a practical scheme of a quantum computer with cold trapped ions, see ref. [123].

Decoherence is the big enemy of quantum computation, for a simple reason: it constantly tends to destroy the useful coherent superpositions; this sadly reduces the full quantum information to its classical, boolean, component made of diagonal matrix elements only. It is now actually perfectly clear that a "crude" quantum computer based on the naive use of non-redundant q-bits will never work, at least with more than a very small number of them; it has been remarked that this kind of quantum computer would simply be a sort or resurgence of the old analog computers (errors in quantum information form a continuum), in an especially fragile version! But it has also been pointed out that an appropriate use of quantum redundancy may allow one to design efficient error correcting schemes [124] [125]; decoherence can be corrected by using a system containing more q-bits, and by projecting its state into some subspaces in which the correct information about the significant q-bit survives without error [126]; the theoretical schemes involve collective measurements of several q-bits, which give access to some combined information on all them, but none on a single q-bit. It turns out that it is theoretically possible to "purify" quantum states by combining several systems in perturbed entangled states and applying to them local operations, in order to extract a smaller number of systems in non-perturbed states [127]; one sometimes also speaks of "quantum distillation" in this context. This scheme applies in various situations, including quantum computation as well as communication or cryptography [128]. Similarly the notion of "quantum repeaters" [129] has been introduced recently in order to correct for the effect of imperfections and noise in quantum communication. Another very different approach to quantum computation has been proposed, based on a semiclassical concept where q-bits are still used, but communicate only through classical macroscopic signals, which are used to determine the type of measurement performed on the next q-bit [130]; this kind of computer should be much less sensitive to decoherence.

Generally speaking, whether or not it will be possible one day to beat

decoherence in a sufficiently large system for practical quantum computing still remains to be seen. Moreover, although the factorization into prime numbers is an important question (in particular for cryptography), as well as the many-body quantum problem, it would be nice to apply the principles of quantum computation to a broader scope of problems! The question as to whether or not quantum computation will become a practical tool one day remains open to debate [131] [119], but in any case this is an exciting new field of research.

6 Various interpretations

In section 1 we have already mentioned some of the other, “unorthodox” interpretations of quantum mechanics that have been proposed, some of them long ago and almost in parallel with the “orthodox” Copenhagen interpretation. Our purpose was then to discuss, in a historical context, why they are now generally taken more seriously by physicists than they were in the middle of the 20th. century, but not to give any detail; this article would be incomplete without, at least, some introduction to the major alternative interpretations of quantum mechanics that have been proposed over the years, and this is the content of the present section.

It is clearly out of the question to give here an exhaustive discussion of all possible interpretations. This would even probably be an impossible task! The reason is that, while one can certainly distinguish big families among the interpretations, it is also possible to combine them in many ways, with an almost infinite number of nuances. Even the Copenhagen interpretation itself is certainly not a monolithic construction; it can be seen from different points of view and can be declined in various forms. An extreme case was already mentioned in § 2.2: what is sometimes called the “Wigner interpretation” of quantum mechanics, probably because of the title and conclusion of ref. [37] - but views along similar lines were already discussed by London and Bauer in 1939 [132]. In this interpretation, the origin of the state vector reduction should be related to consciousness. For instance, London and Bauer emphasize that state vector reduction restores a pure state from a statistical mixture of the measured sub-system (see § 5.3.2), and “the essential role played by the consciousness of the observer in this transition from a mixture to a pure state”; they then explain this special role by the faculty of introspection of conscious observers. Others prefer to invoke “special properties” of the electrical currents which correspond to perception in a human brain, but how seriously this explanation is put forward is not always entirely clear. In fact, Wigner may have seen the introduction of an influence of consciousness just as an extreme case (exactly as the Schrödinger cat was introduced by Schrödinger), just for illustrating the necessity of a non-linear step in order to predict definite results. In any

event, the merit of the idea is also to show how the logic of the notion of measurement in the Copenhagen interpretation can be pushed to its limits: indeed, how is it possible to ascribe such special properties to the operation of measurement without considering that the human mind also has very special properties?

For obvious reasons of space, here we limit ourselves to a sketchy description of the major families of interpretations. We actually start with what we might call a “minimal interpretation”, a sort of common ground that the vast majority of physicists will consider as a safe starting point. We will then proceed to discuss various families of interpretations: additional variables, non-linear evolution of the state vector, consistent histories, Everett interpretation. All of them tend to change the status of the postulate of the wave packet reduction; some interpretations incorporate it into the normal Schrödinger evolution, others make it a consequence of another physical process that is considered as more fundamental, still others use a formalism where the reduction is hidden or even never takes place. But the general purpose always remains the same: to solve the problems and questions that are associated with the coexistence of two postulates for the evolution of the state vector.

6.1 Common ground; “correlation interpretation”

The method of calculation that we discuss in this section belongs to standard mechanics; it is actually common to almost all interpretations of quantum mechanics and, as a calculation, very few physicists would probably put it into question. On the other hand, when the calculation is seen as an interpretation, it may be considered by some as too technical, and not sufficiently complete conceptually, to be really called an interpretation. But others may feel differently, and we will nevertheless call it this way; we will actually use the words “correlation interpretation”, since all the emphasis is put on the correlations between successive results of experiments.

The point of view in question starts from a simple remark: the Schrödinger equation alone, more precisely its transposition to the “Heisenberg point of view”, allows a relatively straightforward calculation of the probability associated with any sequence of measurements, performed at different times. To see how, let us assume that a measurement³⁸ of a physical quantity associated with operator M is performed at time t_1 , and call m the possible results; this is followed by another measurement of observable N at time t_2 , with possible results n , etc. Initially, we assume that the system is described by a pure state $|\Psi(t_0)\rangle$, but below we generalize to a density operator $\rho(t_0)$. According to the Schrödinger equation, the state vector evolves between time t_0 and time t_1 from $|\Psi(t_0)\rangle$ to $|\Psi(t_1)\rangle$; let us then expand

³⁸Here, we assume that all measurements are ideal; if non-ideal measurements are considered, a more elaborate treatment is needed.

this new state into its components corresponding to the various results that can be obtained at time t_1 :

$$|\Psi(t_1)\rangle = \sum_m |\Psi_m(t_1)\rangle \quad (32)$$

where $|\Psi_m(t_1)\rangle$ is obtained by applying to $|\Psi(t_1)\rangle$ the projector $P_M(m)$ on the subspace corresponding to result m :

$$|\Psi_m(t_1)\rangle = P_M(m) |\Psi(t_1)\rangle \quad (33)$$

Now, just after the first measurement, we can “chop” the state vector into different “slices”, which are each of the terms contained in the sum of (32). In the future, these terms will never give rise to interference effects, since they correspond to different measurement results; actually, each component becomes correlated to an orthogonal state of the environment (the pointer of the measurement apparatus for instance) and a full decoherence will ensure that any interference effect is cancelled.

Each “slice” $|\Psi_m(t_1)\rangle$ of $|\Psi(t_1)\rangle$ can then be considered as independent from the others, and taken as a new initial state of the system under study. From time t_1 to time t_2 , the state in question will then evolve under the effect of the Schrödinger equation and become a state $|\Psi_m(t_2)\rangle$. For the second measurement, the procedure repeats itself; we “slice” again this new state according to:

$$|\Psi_m(t_2)\rangle = \sum_n |\Psi_{m,n}(t_2)\rangle \quad (34)$$

where $|\Psi_{m,n}(t_2)\rangle$ is obtained by the action of the projector $P_N(n)$ on the subspace corresponding to result n :

$$|\Psi_{m,n}(t_2)\rangle = P_N(n) |\Psi_m(t_2)\rangle \quad (35)$$

The evolution of each $|\Psi_m(t_2)\rangle$ will now be considered independently and, if a third measurement is performed at a later time t_3 , generate one more decomposition, and so on. It is easy to check³⁹ that the probability of any given sequence of measurements m, n, p , etc. is nothing but by the square of the norm of the final state vector:

$$\mathcal{P}(m, t_1; n, t_2; p, t_3; \dots) = |\langle \Psi_{m,n,p,\dots,q}(t_q) | \Psi_{m,n,p,\dots,q}(t_q) \rangle|^2 \quad (36)$$

Let us now describe the initial state of the system through a density operator $\rho(t_0)$; it turns out that the same result can be written in a compact way,

³⁹This can be done for instance by successive applications of the postulate of the wave packet reduction and the evaluation of conditional probabilities. Note that we have not restored the norm of any intermediate state vector to 1, as opposed to what one usually does with the wave packet reduction; this takes care of intermediate probabilities and explains the simplicity of result (36).

according to a formula that is sometimes called Wigner formula [38] [133]. For this purpose, we consider the time-dependent version, in the Heisenberg point of view⁴⁰, of all projectors: $\hat{P}_M(m; t)$ corresponds to $P_M(m)$, $\hat{P}_N(n; t)$ to $P_N(n)$, etc. One can then show that the probability for obtaining result m followed by result n is given by⁴¹:

$$\mathcal{P}(m, t_1; n, t_2) = Tr \left\{ \hat{P}_N(n; t_2) \hat{P}_M(m; t_1) \rho(t_0) \hat{P}_M(m; t_1) \hat{P}_N(n; t_2) \right\} \quad (37)$$

(generalizing this formula to more than two measurements, with additional projectors, is straightforward).

Equation (37) can be seen as a consequence of the wave packet reduction postulate of quantum mechanics, since we obtained it in this way. But it is also possible to take it as a starting point, as a postulate in itself: it then provides the probability of any sequence of measurements, in a perfectly unambiguous way, without resorting, either to the wave packet reduction, or even to the Schrödinger equation itself. The latter is actually contained in the Heisenberg evolution of projection operators, but it remains true that a direct calculation of the evolution of $|\Psi\rangle$ is not really necessary. As for the wave packet reduction, it is also contained in a way in the trace operation of (37), but even less explicitly. If one just uses formula (37), no conflict of postulates takes place, no discontinuous jump of any mathematical quantity; why not then give up entirely the other postulates and just use this single formula for all predictions of results?

This is indeed the best solution for some physicists: if one accepts the idea that the purpose of physics is only to correlate the preparation of a physical system, contained mathematically in $\rho(t_0)$, with all possible sequence of results of measurements (by providing their probabilities), it is true that nothing more than (37) is needed. Why then worry about which sequence is realized in a particular experiment? It is sufficient to assume that the behavior of physical systems is fundamentally indeterministic, and that there is no need in physics to do more than just giving rules for the calculation of probabilities. The “correlation interpretation” is therefore a perfectly consistent attitude; on the other hand, it is completely opposed to the line of the EPR reasoning, since it shows no interest whatsoever in questions related to physical reality as something “in itself”. Questions such as: “how should the physical system be described when one first measurement has already been performed, but before the second measurement is decided” should be dismissed as meaningless. Needless to say, the notion of the EPR

⁴⁰Let $U(t, t_0)$ be the unitary operator associated with the evolution of the state vector between time t_0 and time t_1 , in the Schrödinger point of view. If P is any operator, one can obtain its transform $\hat{P}(t)$ in the “Heisenberg point of view” by the unitary transformation: $\hat{P}(t) = U^\dagger(t, t_0) P U(t, t_0)$, where $U^\dagger(t, t_0)$ is the Hermitian conjugate of $U(t, t_0)$; the new operator depends in general of time t , even if this is not the case for the initial operator.

⁴¹Using circular permutation under the trace, one can in fact suppress one of the extreme projectors $\hat{P}_N(n; t_2)$ in formula (37), but not the others.

elements of reality becomes completely irrelevant, at least to physics, a logical situation which automatically solves all potential problems related to Bell, GHZ and Hardy type considerations. The same is true of the emergence of a single result in a single experiment; in a sense, the Schrödinger cat paradox is eliminated by putting it outside of the scope of physics, because no paradox can be expressed in terms of correlations. An interesting feature of this point of view is that the boundary between the measured system and the environment of the measuring devices is flexible; an advantage of this flexibility is that the method is well suited for successive approximations in the treatment of a measurement process, for instance the tracks left by a particle in a bubble chamber as discussed by Bell [34].

In practice, most physicists who favor the correlation interpretation do not feel the need for making it very explicit. Nevertheless, this is not always the case; see for instance the article by Mermin [134], which starts from the statement: “Throughout this essay, I shall treat correlations and probabilities as primitive concepts”. In a similar context, see also a recent “opinion” in *Physics Today* by Fuchs and Peres [31] who emphasize “the internal consistency of the theory without interpretation”. On the other hand, the correlation interpretation is seen by some physicists as minimalistic because it leaves aside, as irrelevant, a few questions that they find important; an example is the notion of physical reality, seen as an entity that should be independent of measurements performed by human beings. Nevertheless, as we have already mentioned, the interpretation can easily be supplemented by others that are more specific. In fact, experience shows that defenders of the correlation point of view, when pressed hard in a discussion to describe their point of view with more accuracy, often express themselves in terms that come very close to the Everett interpretation (see § 6.5); in fact, they may sometimes be proponents of this interpretation without realizing it!

Let us finally mention in passing that formula (37) may be the starting point for many interesting discussions, whether or not it is considered as basic in the interpretation, or just as a convenient formula. Suppose for instance that the first measurement is associated with a degenerate eigenvalue of an operator, in other words that $\hat{P}_M(m; t_1)$ is a projector over a subspace of more than one dimension:

$$\hat{P}_M(m; t_1) = \sum_{i=1}^n |\varphi_i\rangle\langle\varphi_i| \quad (38)$$

(for the sake of simplicity we assume that $t_1 = t_0$, so that no time dependence appears in this expression). Inserting this expression into (37) immediately shows the appearance of interference terms (or crossed terms) $i \neq j$ between the contribution of the various $|\varphi_i\rangle$. Assume, on the other hand, that more information was actually obtained in the first measurement, so that the value of i was also determined, but that this information was lost, or forgotten;

the experimenter ignores which of two or more i results was obtained. Then, what should be calculated is the sum of the probabilities associated with each possible result, that is a single sum over i from which all crossed term $i \neq j$ have disappeared. In the first case, interference terms arise because one has to add probability amplitudes; in the second, they do not because one has to add the probabilities themselves (exclusive events). The contrast between these two situations may be understood in terms of decoherence: in the first case, all states of the system correlate to the same state of the measuring apparatus, which plays here the role of the environment; they do not in the second case, so that by partial trace all interference effect vanish. This remark is useful in the discussion of the close relation between the so called “Zeno paradox in quantum mechanics” [135] and decoherence; it is also basic in the definition of consistency conditions in the point of view of decoherent histories, to which we will come back later (§ 6.4).

6.2 Additional variables

We now leave the range of considerations that are more or less common to all interpretations; from now on, we will introduce in the discussion some elements that clearly do not belong to the orthodox interpretation. We begin with the theories with additional variables, as the De Broglie theory of the pilot wave [136]; the work of Bohm is also known as a major reference in the subject [9] [137]; see also the almost contemporary work of Wiener and Siegel [10]. More generally, with or without explicit reference to additional variables, one can find a number of authors who support the idea that the quantum state vector should be used only for the description of statistical ensembles, not of single events, - see for instance [138] [46].

We have already emphasized that the EPR theorem itself can be seen as an argument in favor of the existence of additional variables (we will come back later to the impact of the Bell and BKS theorems). Theories with such variables can be built mathematically in order to reproduce exactly all predictions of orthodox quantum mechanics; if they give exactly the same probabilities for all possible measurements, it is clear that there is no hope to disprove experimentally orthodox quantum mechanics in favor of these theories, or the opposite. In this sense they are not completely new theories, but rather variations on a known theory. They nevertheless have a real conceptual interest: they can restore not only realism, but also determinism (this is a possibility but not a necessity: one can also build theories with additional variables that remain fundamentally non-deterministic).

6.2.1 General framework

None of the usual ingredients of orthodox quantum mechanics disappears in theories with additional variables. In a sense, they are even reinforced,

since the wave function loses its subtle status (see § 1.2), and becomes an ordinary field with two components (the real part and the imaginary part of the wave function - for simplicity, we assume here that the particle is spinless); these components are for instance similar to the electric and magnetic components of the electromagnetic field⁴². The Schrödinger equation itself remains strictly unchanged. But a completely new ingredient is also introduced: in addition to its wave function field, each particle gets an additional variable λ , which evolves in time according to a new equation. The evolution of λ is actually coupled to the real field, through a sort of “quantum velocity term”⁴³ that depends on the wave function; but, conversely, there is no retroaction of the additional variables onto the wave function. From the beginning, the theory therefore introduces a marked asymmetry between the two mathematical objects that are used to describe a particle; we will see later that they also have very different physical properties.

For anyone who is not familiar with the concept, additional variables may look somewhat mysterious; this may explain why they are often called “hidden”, but this is only a consequence of our much better familiarity with ordinary quantum mechanics! In fact, these variables are less abstract than the wave functions, which in these theories becomes a sort of auxiliary field, even if perfectly real. The additional variables are directly “seen” in a measurement, while the state vector remains invisible; it actually plays a rather indirect role, through its effect on the additional variables. Let us take the example of a particle creating a track in a bubble chamber: on the photograph we directly see the recording of the successive values of an additional variable, which is actually nothing but.. the position of the particle! Who has ever taken a photograph of a wave function?

For a single particle, the additional variable λ may therefore also be denoted as \mathbf{R} since it describes its position; for a many particle system, λ is nothing but a symbol for the set of positions $\mathbf{R}_1, \mathbf{R}_2$ etc. of all the particles. The theory postulates an initial random distribution of these variables that depends on the initial wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots)$ and reproduces exactly the initial distribution of probability for position measurements; using hydrodynamic versions of the Schrödinger equation [139], one can easily show that the evolution under the effect of the “quantum velocity term” ensures that this property continues to be true for any time. This provides a close contact

⁴²The components of the electromagnetic field are vectors while, here, we are dealing with scalar fields; but this is unessential.

⁴³In Bohm’s initial work, a Newton law for the particle acceleration was written in terms of a “quantum potential”. Subsequent versions of Bohmian mechanics discarded the quantum potential in favor of a quantum velocity term providing directly a contribution to the velocity. Both points of view are nevertheless consistent. An unexpected feature of the quantum velocity term is that it depends only on the gradient of the phase of the wave function, not on its modulus. Therefore, vanishingly small wave functions may have a finite influence on the position of the particles, which can be seen as a sort of non-local effect.

with all the predictions of quantum mechanics; it ensures, for instance, that under the effect of the quantum velocity term the position of particles will not move independently of the wave function, but always remain inside it.

At this point, it becomes natural to restore determinism, and to assume that the results of measurements merely reveal the initial pre-existing value of λ , chosen among all possible values in the initial probability distribution. This assumption solves many difficulties, all those related to the Schrödinger cat paradox for instance: depending on the exact initial position of a many-dimension variable λ , which belongs to an enormous configuration space (including the variables associated with the radioactive nucleus as well as all variables associated with the cat), the cat remains alive or dies. But restoring determinism is not compulsory, and non-deterministic versions of additional variables can easily be designed. In any case, the theory will be equivalent to ordinary quantum mechanics; for instance, decoherence will act exactly in the same way, and make it impossible in practice to observe interferences with macroscopic objects in very different states.

To summarize, we have in this context a description of physical reality at two different levels:

(i) one corresponding to the elements associated with the state vector, which can be influenced directly in experiments, since its evolution depends on a Hamiltonian that can be controlled, for instance by applying fields; this level alone is not sufficient to give a complete description of a physical system.

(ii) another corresponding to the additional variables, which cannot be manipulated directly (see appendix V), but obey evolution equations containing the state vector.

The two levels together are necessary and sufficient for a complete description or reality. There is no retroaction of the additional variables onto the state vector, which creates an unusual situation in physics (usually, when two physical quantities are coupled, they mutually influence each other). Amusingly, we are now contemplating another sort of duality, which distinguishes between direct action on physical systems (or preparation) and results of observations performed on them (results of measurements).

A similar line of thought has been developed by Nelson [140], who introduces stochastic motions of point particles in such a way that their statistical behavior reproduces exactly the predictions of the Schrödinger equation. The difference is that the evolution of the wave function is not given by a postulate, but actually derived from other postulates that are considered more fundamental. This leads to a natural derivation of the Schrödinger equation; the formalism is built to lead exactly to the same predictions as orthodox quantum mechanics, so that its interest is mostly conceptual. For the discussion of statistical mixtures in this context, see ref. [141].

6.2.2 Bohmian trajectories

As soon as particles regain a position, they also get a trajectory, so that it becomes natural to study their properties in various situations; actually one then gets a variety of unexpected results. Even for a single particle in free space, because of the effects of its wave function on the evolution of its position, it turns out that the trajectories are not necessarily simple straight lines [142]; in interference experiments, particles may actually follow curved trajectories even in regions of space where they are free, an unusual effect indeed⁴⁴! But this feature is in fact indispensable for the statistics of the positions to reproduce the usual predictions of quantum mechanics (interference fringes) [143]. Bell studied these questions [34] and showed, for instance, that the observation of successive positions of a particle allows one to reconstruct a trajectory that remains physically acceptable.

For systems of two particles or more, the situation becomes even more interesting. Since the Schrödinger equation remains unchanged, the wave functions continues to propagate in the configuration space, while on the other hand the positions propagate in ordinary three dimensional space. The effects of non-locality become especially apparent through the “quantum velocity term”, since the velocity has to be evaluated at a point of configuration space that depends on the positions of both particles; the result for the velocity of particle 1 may then depend explicitly on the position of particle 2. Consider for instance an EPRB experiment of the type described in § 4.1.2 and the evolution of the positions of the two particles when they are far apart. If particle 1 is sent through a Stern-Gerlach analyzer oriented along direction a , the evolution of its Bohmian position will obviously be affected in a way that depends on a (we remarked above that the positions have to follow the quantum wave functions; in this case, it has the choice between two separating wave packets). But this will also change the position $(\mathbf{R}_1, \mathbf{R}_2)$ of the point representing the system in the six dimension configuration space, and therefore change the quantum velocity term for particle 2, in a way that depends explicitly on a . No wonder if such a theory has no difficulty in reproducing the non-local features of quantum mechanics! The advantage of introducing additional variables is, in a sense, to emphasize the effects of non-locality, which often remain relatively hidden in the orthodox formalism (one more reason not to call these variables “hidden”!). Bell for instance wrote “it is a merit of the Broglie-Bohm interpretation to bring this (non-locality) out so explicitly that it can not be ignored” - in fact, historically, he came to his famous inequalities precisely through this channel.

An interesting illustration of this fact can be found in the study of

⁴⁴Another unusual effect takes place for a particle with spin: the spin direction associated with the position of the particle may sometimes spontaneously flip its direction, without any external coupling [144].

Bohmian trajectories in a two-particle interference experiment, or in a similar case studied in reference [144]. The authors of this reference study a situation which involves an interference experiment supplemented by electromagnetic cavities, which can store the energy of photons and be used as a “Welcher Weg” device (a device that tells the experimenter which hole the particle went through in an interference experiment). A second particle probes the state of the field inside the cavity, and when leaving it takes a trajectory that depends on this field. These authors show that, in some events, a particle can leave a photon in a cavity and influence a second particle, while the trajectory of the latter never crosses the cavity; from this they conclude that the Bohmian trajectories are “surrealistic”. Of course, considering that trajectories are surrealistic or not is somewhat a matter of taste. What is clear, however, is that a firm believer in the Bohmian interpretation will not consider this thought experiment as a valid argument against this interpretation - at best he/she will see it as a valid attack against some truncated form of Bohmian theory. One should not mix up orthodox and Bohmian theories, but always keep in mind that, in the latter theory, the wave function has a totally different character: it becomes a real classical field, as real as a laser field for instance. As expressed by Bell [145]: “No one can understand this theory until he is willing to think of Ψ as a real objective field rather than just a probability amplitude”. Therefore, a “particle” always involves a combination of both a position and the associated field, which can not be dissociated; there is no reason whatsoever why the latter could not also influence its surrounding. It would thus be a mistake to assume that influences should take place in the vicinity of the trajectory only.

In this context, the way out of the paradox is then simple: just to say that the real field associated to the first particle⁴⁵ interacted with the electromagnetic field in the cavity, leaving a photon in it; later this photon acted on the trajectory of the second particle. In other words, the effect is a crossed field-trajectory effect, and in these terms it is even perfectly local! One could even add that, even if for some reason one decided to just consider the trajectories of the two particles, the fact that they can influence each other, even if they never come close to each other creates no problem in itself; it is just an illustration of the explicit character of non-locality in the Bohm theory - see the quotation by Bell above, as well as the discussion of this thought experiment by Griffiths [146]. So, we simply have one more example of the fact that quantum phenomena are indeed local in

⁴⁵One sometimes introduces the notion of the “empty part of the wave function” to characterize the wave packet which does not contain a trajectory of the particle, for instance in one arm of a Mach Zehnder interferometer. In the present case, this empty part would deposit something (a photon?) in the cavity that, later, would influence the trajectory of a second particle - in other words we would have an indirect influence of the empty part on the second particle.

configuration space, but not necessarily in ordinary space.

This thought experiment nevertheless raises interesting questions, such as: if in this example a particle can influence events outside of its own Bohmian trajectory, what is then the physical meaning of this trajectory in general? Suppose that, in a cloud chamber for instance, a particle could leave a track that does not coincide at all with the trajectory of the Bohmian position; in what sense then could this variable be called “position”? For the moment, that this strange situation can indeed occur has not been shown (the example treated in [144] is very special and presumably not a good model for a cloud chamber), but the question clearly requests more precise investigation. Another difficulty of theories with additional variables is also illustrated by this thought experiment: the necessity for including fields (in this case the photons in the cavities). Quantum mechanics is used to describe a large variety of fields, from the usual electromagnetic field (quantum electrodynamics) to quarks for instance, and this is truly essential for a physical description of the world; at least for the moment, the complete description of all these fields has not been developed within theories with additional variables, although attempts in this direction have been made.

6.3 Modified (non-linear) Schrödinger dynamics

Another way to resolve the coexistence problem between the two postulates of quantum mechanics is to change the Schrödinger equation itself: one assumes that the equation of evolution of the wave function contains, in addition to the usual Hamiltonian terms, non-linear (and possibly stochastic) terms, which will also affect the state vector [4] [11] [13] [147] [148]. These terms may be designed so that their effects remain extremely small in all situations involving microscopic objects only (atoms, molecules, etc.); this will immediately ensure that all the enormous amount of successful predictions of quantum mechanics is capitalized. On the other hand, for macroscopic superpositions involving for instance pointers of measurement apparatuses, the new terms may mimic the effects of wave packet reduction, by selecting one branch of the superposition and cancelling all the others. Clearly, one should avoid both extremes: either perturb the Schrödinger equation too much, and make interference effects disappear while they are still needed (for instance, possible recombination of the two beams at the exit of a Stern-Gerlach magnet); or too little, and not ensure the complete disappearance of Schrödinger cats! This result is obtained if the perturbation term becomes efficient when (but not before) any microscopic system becomes strongly correlated to a macroscopic environment, which ensures that significant decoherence has already taken place; we then know that the recovery of interference effects is impossible in practice anyway. If carefully designed, the process then reproduces the effect of the postulate of the wave function collapse, which no longer appears as an independent postulate, but

as a consequence of the “normal” evolution of the wave function.

6.3.1 Various forms of the theory

There are actually various versions of theories with modified Schrödinger dynamics. Some versions request the introduction of additional variables into the theory, while others do not. The approach proposed in 1966 by Bohm and Bub [4] belongs to the first category, since these authors incorporate in their theory additional variables previously considered by Wiener and Siegel [10]; these variables are actually contained in a “dual vector”, similar to the usual state vector $|\Psi\rangle$, but obeying an entirely different equation of motion -in fact, both vectors evolve with coupled equations. What is then obtained is a sort of combination of theories with additional variables and modified dynamics. For some “normal” distribution of the new variables, the prediction of usual quantum mechanics are recovered; but it is also possible to assume the existence “dispersion free” distributions that lead to non-orthodox predictions. An example of models that are free of additional variables is given by the work of Pearle [11], published ten years later, in which nothing is added to the usual conceptual frame of standard quantum mechanics. The theory is based on a modified dynamics for the modulus and phases of the quantum amplitudes, which get appropriate equations of evolution; the result is that, depending on the initial values of the phases before a measurement, all probability amplitudes but one go to zero during a measurement. Because, when a microscopic system is sent towards a macroscopic apparatus, the initial phases are impossible to control with perfect mathematical accuracy, an apparent randomness in the results of experiments is predicted; the equations are designed so that this randomness exactly matches the usual quantum predictions. In both theories, the reduction of the state vector becomes a dynamical process which, as any dynamical process, has a finite time duration; for a discussion of this question, see [149], which remarks that the theory of ref. [4] introduces an infinite time for complete reduction.

Another line of thought was developed from considerations that were initially not directly related to wave function collapse, but to continuous observations and measurements in quantum mechanics [150] [151]. This was the starting point for the work of Ghirardi et al. [13], who introduce a random and sudden process of “spontaneous localization” with an arbitrary frequency (coupling constant), which resembles the effect of approximate measurements in quantum mechanics. The constant is adjusted so that, for macroscopic systems (and for them only), the occurrence of superposition of far-away states is destroyed by the additional process; the compatibility between the dynamics of microscopic and macroscopic systems is ensured, as well as the disappearance of macroscopic coherent superpositions (transformation of coherent superpositions into statistical mixtures). This approach

solves problems that were identified in previous work [11], for instance the “preferred basis problem”, since the basis is that of localized states; the relation to the quantum theory of measurement is examined in detail in [152]. In this model, for individual systems⁴⁶ the localization processes are sudden (they are sometimes called “hitting processes”), which makes them completely different from the usual Schrödinger dynamics. Nevertheless, later work [153] showed that it is possible to design theories involving only continuous evolution that retain the attractive features of the model. For instance, the discontinuous Markov processes in Hilbert space reduce, in an appropriate limit, to a continuous spontaneous localization, which may result in a new version of non-linear Schrödinger dynamics [154] called continuous spontaneous localization (CSL); another achievement of [154] is a full compatibility with the usual notion of identical particles in quantum mechanics. See also [147] for an earlier version of modified Schrödinger dynamics with very similar equations of evolution.

A similar line was followed by Diosi [148], who also started initially from the treatment of continuous measurements [155] by the introduction of stochastic processes (“quantum Wiener processes” [10]) that are added to the usual deterministic Schrödinger dynamics. This author then introduced a treatment of the collapse of the wave function from an universal law of density localization [156], with a strength that is proportional to the gravitational constant, resulting in a parameter free unification of micro- and macro-dynamics. Nevertheless, this approach was found to create severe problems at short distances by the authors of [157], who then proposed a modification of the theory that solves them, but at the price of re-introducing a constant with dimension (a length).

Generally speaking, beyond their fundamental purpose (an unification of all kinds of physical evolution, including wave function reduction), two general features of these theories should of be emphasized. The first is that new constants appear, which may in a sense look like ad hoc constants, but actually have an important conceptual role: they define the limit between the microscopic and macroscopic world (or between reversible and irreversible evolution); the corresponding border is no longer ill-defined, as opposed to the situation for instance in the Copenhagen interpretation. The second (related) feature is that these theories are more predictive. They are actually the only ones which propose a real physical mechanism for the emergence of a single result in a single experiment, which is of course attractive from a physical point of view. At the same time, and precisely

⁴⁶For ensemble of systems, the discontinuities are averaged, and one recovers continuous equations of evolution for the density operator. Since most of the discussion of [13] is given in terms of density operators/matrices, and of the appearance of statistical mixtures (decoherence), one may get the (incorrect) impression that individual realizations are not considered in this work; but this is in fact not the case and “hitting processes” are indeed introduced at a fundamental level.

because they are more predictive, these theories become more vulnerable to falsification, and one has to carefully design the mechanism in a way that satisfies many constraints. For instance, we have already mentioned that, in the initial Bohm-Bub theory, a complete collapse of the wave function is never obtained in any finite time. The same feature actually exists in CSL: there is always what is called a “tail” and, even when most of the wave function goes to the component corresponding to one single outcome of an experiment, there always remain a tiny component on the others (extremely small and continuously going down in size). The existence of this component is not considered as problematic by the proponents of the CSL theory, as illustrated by the contributions of Pearle and Ghirardi in [158]. In the context of possible conflicts with experiments, see also the discussion of [157] concerning incompatibilities of another form of the theory with the well-known properties of microscopic objects, as well as [159] for a critical discussion of another version of non-linear dynamics. A similar case is provided by the generalization of quantum mechanics proposed by Weinberg [160], which this author introduced as an illustration of a non-linearity that is incompatible with available experimental data; see also [161] for an application of the same theory to quantum optics and [162] for a proof of the incompatibility of this theory relativity, due to the prediction of superluminal communication (the proof is specific of the Weinberg form of the non-linear theory and does not apply to the other forms mentioned above).

6.3.2 Physical predictions

Whatever specific form of the theory is preferred, similar physical descriptions are obtained. For instance, when a particle crosses a bubble chamber, the new terms creates the appearance (at a macroscopic level) of a particle trajectory; they also select one of the wave packets at the measurement output of a Stern-Gerlach analyzer (and eliminate the other), but not before these packets become correlated to orthogonal states of the environment (e.g. detectors). Of course, any process of localization of the wave function tends to operate in the space of positions rather than in the space of momenta, which reduces to some extent the usual symmetry between positions and momenta in quantum mechanics. This is actually not a problem, but a convenient feature: one can easily convince oneself that, in practice, what is measured in all experiments is basically the positions of particles or objects (pointers, etc.), while momenta are only indirectly measured. Generally speaking, it is a different spatial localization that produces wave packet collapse.

How is an EPRB experiment described in this point of view? In the case of Bohmian trajectories, we emphasized the role of the “quantum velocity term”, which has a value defined in configuration space and not in ordinary space; here, what is essential is the role of the added non-linear localization

term in the Schrödinger equation, which also acts in the 6 dimensional configuration space. This term is designed so that, when correlation with the environment takes place, one of the components in the corresponding basis (“basis of decoherence”) is selected. Nothing special then occurs as long as particle 1 propagates within a Stern-Gerlach analyzer, since it is microscopic and can perfectly well go through superpositions of far away states; but as soon as it hits a detector at the output of the magnet, the system develops correlations with the particles contained in the detector, the amplifier, etc. so that a macroscopic level is reached and the localization term becomes effective. Here, we see that it is the a dependence of the spatial localization (in other words, the basis of decoherence) that introduces an overall effect on the two-particle state vector; it provides particle 2 with, not only a privileged spin-state basis, but also a reduction of its spin state to one single component (when particle 1 hits the detector). Since this point of view emphasizes the role of the detectors and not of the analyzers, it is clearly closer to the usual interpretation, in terms of wave packet reduction, than the Bohmian interpretation. Nevertheless it also puts into light the role of non-locality in an explicit way, as this interpretation does.

What about the Schrödinger cat and similar paradoxes? If the added non-linear term has all the required properties and mimic the wave packet reduction, they are easily solved. For instance, a broken poison bottle must have at least some parts that have a different spatial localization (in configuration space) than an unbroken bottle; otherwise it would have all the same physical properties. It is then clear that the modified dynamics will resolve the components long before it reaches the cat, so that the emergence of a single possibility is ensured. For a recent discussion of the effects of the modified dynamics on “all or nothing coherent states” (§ 5.3.1) in the context of quantum optics, and of the effects on perception in terms of the “relative state of the brain” (§6.5), see ref. [163].

The program can be seen as a sort of revival of the initial hopes of Schrödinger, where all relevant physics was contained in the wave function and its progressive evolution (see the end of §1.1.2); this is especially true, of course, of the versions of non-linear dynamics that are continuous (even if fluctuating extra quantities may be introduced), and not so much of versions including “hits” that are too reminiscent of the wave packet reduction. Here, the state vector directly describes the physical reality, in contrast with our discussion of §1.2; we have a new sort of wave mechanics, where the notion of point particles is given up in favor of tiny wave packets. The theory is different from theories with additional variables, because the notion of precise position in configuration space never appears. As we have seen, another important difference is that these theories with modified dynamics are really new theories: they may, in some circumstances, lead to predictions that differ from those of orthodox quantum mechanics, so that experimental tests might be possible. We should emphasize that, in this point of view, the

wave function can still not be considered as an ordinary field: it continues to propagate in a high dimension configuration space instead of the usual three dimension space.

A mild version of these theories is found in a variant where the Schrödinger equation remains exactly the same, but where stochastic terms are introduced as a purely computational tool, and without any fundamental purpose, for the calculation of the evolution of a partial trace density matrix describing a subsystem [164] [165] [166]; in other words, a master equation for a density operator is replaced by an average over several state vectors submitted to a random perturbation, which may in some circumstances turn out to save computing time very efficiently. Another line of thought that can be related to some extent to modified Schrödinger dynamics is the “transactional interpretation” of quantum mechanics [167], where a quantum event is described by the exchange of advanced and retarded waves; as in modified non-linear Schrödinger dynamics, these waves are then interpreted as real, and non-locality is made explicit.

6.4 History interpretation

The interpretation of “consistent histories” is also sometimes called “decoherent history interpretation”, or just “history interpretation” as we prefer to call it here (because the notion of consistency is essential at the level of families of histories, rather than at the level of individual histories). It proposes a logical framework that allows the discussion of the evolution of a closed quantum system, without reference to measurements. The general idea was introduced and developed by Griffiths [15] but it has also been used, and sometimes adapted, by other authors [168] [169] [170]. Since this interpretation is the most recent among those that we discuss in this article, we will examine it in somewhat more detail than the others. We will nevertheless remain within the limits of a non-specialized introduction; the reader interested in more precise information on the subject should go to the references that are provided - see also a recent article in *Physics Today* [171] and the references contained.

6.4.1 Histories, families of histories

Consider any orthogonal projector P on a subspace \mathcal{F} of the space of states of a system; it has two eigenvalues, $+1$ corresponding to all the states belonging to \mathcal{F} , and 0 corresponding to all states that are orthogonal to \mathcal{F} (they belong to the supplementary sub-space, which is associated with the projector $Q = 1 - P$). One can associate a measurement process with P : if the result of the measurement is $+1$, the state of the system belongs to \mathcal{F} ; if it is zero, it is orthogonal to \mathcal{F} . Assume now that this measurement is made at time

t_1 on a system that is initially (at time t_0) described by a density operator $\rho(t_0)$; the probability for finding the state of the system in \mathcal{F} at time t_1 is then given by formula (37), which in this case simplifies into:

$$\mathcal{P}(\mathcal{F}, t_1) = \text{Tr} \left\{ \hat{P}(t_1) \rho(t_0) \hat{P}(t_1) \right\} \quad (39)$$

This result can obviously be generalized to several subspaces $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots$, and several measurement times t_1, t_2, t_3, \dots (we assume $t_1 < t_2 < t_3 < \dots$). The probability that the state of the system belongs to \mathcal{F}_1 at time t_1 , then to \mathcal{F}_2 at time t_2 , then to \mathcal{F}_3 at time t_3 , etc. is, according to the Wigner formula:

$$\mathcal{P}(\mathcal{F}_1, t_1; \mathcal{F}_2, t_2; \mathcal{F}_3, t_3, \dots) = \text{Tr} \left\{ \dots \hat{P}_3(t_3) \hat{P}_2(t_2) \hat{P}_1(t_1) \rho(t_0) \hat{P}_1(t_1) \hat{P}_2(t_2) \hat{P}_3(t_3) \dots \right\} \quad (40)$$

where, as above, the $\hat{P}_i(t_i)$ are the projectors over subspaces $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in the Heisenberg point of view. We can now associate an ‘‘history’’ of the system with this equation: an history \mathcal{H} is defined by a series of arbitrary times t_i , each of them associated with an orthogonal projector P_i over any subspace; its probability is given by (40) which, for simplicity, we will write as $\mathcal{P}(\mathcal{H})$. In other words, an history is the selection of a particular path, or branch, for the state vector in a Von Neumann chain, defined mathematically by a series of projectors. Needless to say, there is an enormous number of different histories, which can have all sorts of properties; some of them are accurate because they contain a large number of times associated with projectors over small subspaces \mathcal{F} 's; others remain very vague because they contain a few times only with projectors over large subspaces \mathcal{F} 's (one can even decide that \mathcal{F} is the entire states of spaces, so that no information at all is contained in the history at the corresponding time).

There are in fact so many histories that it useful to group them into families, or sets, of histories. A family is defined again by an arbitrary series of times t_1, t_2, t_3, \dots , but now we associate to each of these times t_i an ensemble of orthogonal projectors $P_{i,j}$ that, when summed, restore the whole initial space of states. For each time we then have, instead of one single projector, a series of orthogonal projectors that provide a decomposition of the unity operator:

$$\sum_j P_{i,j} = 1 \quad (41)$$

This gives the system a choice, so to say, among many projectors for each time t_i , and therefore a choice among many histories of the same family. It is actually easy to see from (41) and (40) that the sum of probabilities of all histories of a given family is equal to one:

$$\sum_{\text{histories of a family}} \mathcal{P}(\mathcal{H}) = 1 \quad (42)$$

which we interpret as the fact that the system will always follow one, and only one, of them.

A family can actually also be built from a single history, the simplest way to incorporate the history into a family is to associate, at each time t_i ($i = 1, 2, \dots, N$), in addition to the projector P_i , the supplementary projector $Q_i = 1 - P_i$; the family then contains 2^N individual histories. Needless to say, there are many other ways to complement to single family with “more accurate” histories than those containing the Q ’s; this can be done by decomposing each Q into many individual projectors, the only limit being the dimension of the total space of states.

6.4.2 Consistent families

All this looks very simple, but in general it is actually too simple to ensure a satisfactory logical consistency in the reasonings. Having chosen a given family, it is very natural to also enclose in the family all those histories that can be built by replacing any pair or projectors, or actually any group of projectors, by their sum; this is because the sum of two orthogonal projectors is again a projector (onto a subspace that is the direct sum of the initial subspaces). The difference introduced by this operation is that, now, at each time, the events are no longer necessarily exclusive⁴⁷; the histories incorporate a hierarchy in their descriptive accuracy, including even cases where the projector at a given time is just the projector over the whole space of states (no information at all on the system at this time).

Consider the simplest case where two projectors only, occurring at time t_i , have been grouped into one single projector to build a new history. The two “parent” histories correspond to two exclusive possibilities (they contain orthogonal projectors), so that their probabilities add independently in the sum (42). What about the daughter history? It is exclusive of neither of its parents and, in terms of the physical properties of the system, it contains less information at time t_i : the system may have either of the properties associated to the parents. But a general theorem in probability theory states that the probability associated to an event than can be realized by either of two exclusive events is the sum of the individual probabilities; one then expects that the probability of the daughter history should be the sum of the parent probabilities. On the other hand, inspection of (40) shows that this is not necessarily the case; since any projector, $\hat{P}_2(t_2)$ for instance, appears twice in the formula, replacing it by a sum of projectors introduces four terms: two terms that give the sum of probabilities, as expected, but also two crossed terms (or “interference terms”) between the parent histories, so that the probability of the daughter history is in general different from the sums of the parent probabilities. These crossed terms are actually very

⁴⁷For these non exclusive families, relation (42) no longer holds since it would involve double counting of possibilities.

similar to the right hand side of (40), but the trace always contains at some time t_i one projector $\widehat{P}_{i,j}(t_i)$ on the left of $\rho(t_0)$ and an orthogonal projector $\widehat{P}_{i,k}(t_i)$ on the right. This difficulty was to be expected: we know that quantum mechanics is linear at the level of probability amplitudes, not probabilities themselves; interferences occur because the state vector at time t_i , in the daughter story, may belong to one of the subspaces associated with the parents, but may also be any linear combination of such states. As a consequence, a linearity condition for probabilities is not trivial.

One way to restore the additivity of probabilities is to impose the condition:

$$\begin{aligned} & \text{Tr} \left\{ \dots \widehat{P}_{3,j_3}(t_3) \widehat{P}_{2,j_2}(t_2) \widehat{P}_{1,j_1}(t_1) \rho(t_0) \widehat{P}_{1,j'_1}(t_1) \widehat{P}_{2,j'_2}(t_2) \widehat{P}_{3,j'_3}(t_3) \dots \right\} \\ & \propto \delta_{j_1,j'_1} \times \delta_{j_2,j'_2} \times \delta_{j_3,j'_3} \times \dots \end{aligned} \quad (43)$$

Because of the presence of the product of δ 's in the right hand side, the left hand side of (43) vanishes as soon as a least one pair of the indices (j_1, j'_1) , (j_2, j'_2) , (j_3, j'_3) , etc. contains different values; if they are all equal, the trace merely gives the probability $\mathcal{P}(\mathcal{H})$ associated with the particular history of the family. What is important for the rest of the discussion is the notion of consistent family: if condition (43) is fulfilled for all projectors of a given family of histories, we will say that this family is logically consistent, or consistent for short. Condition (43) is basic in the history interpretation of quantum mechanics; it is sometimes expressed in a weaker form, as the cancellation of the real part only of the left hand side; this, as well as other points related to this condition, is briefly discussed in Appendix VI. We now discuss how consistent families can be used as an interpretation of quantum mechanics.

6.4.3 Quantum evolution of an isolated system

Let us consider an isolated system and suppose that a consistent family of histories has been chosen to describe it; any consistent family may be selected but, as soon as the choice is made, it cannot be modified and all the other families are excluded (we discuss later what happens if one attempts to describe the same system with more than one family). This unique choice provides us with a well-defined logical frame, and with a series of possible histories that are accessible to the system and give information at all intermediate times t_1, t_2, \dots . Which history will actually occur in a given realization of the physical system is not known in advance: we postulate the existence of some fundamentally random process of Nature that selects one single history among all those of the family. The corresponding probability $\mathcal{P}(\mathcal{H})$ is given by the right hand side of (40); since this formula belongs to standard quantum mechanics, this postulate ensures that the standard predictions of the theory are automatically recovered. For each realization, the system will then possess at each time t_i all physical properties associated to the particular projectors $P_{i,j}$ that occur in the selected history. This

provides a description of the evolution of its physical properties that can be significantly more accurate than that given by its state vector; in fact, the smaller the subspaces associated to the projectors $P_{i,j}$'s, the more accuracy is gained (obviously, no information is gained if all $P_{i,j}$'s are projectors over the whole space of states, but this corresponds to a trivial case of little interest). For instance, if the system is a particle and if the projector is a projector over some region of space, we will say that the particle is in this region at the corresponding time, even if the whole Schrödinger wave function extends over a much larger region. Or, if a photon strikes a beam splitter, or enters a Mach-Zehnder interferometer, some histories of the system may include information on which trajectory is chosen by the photon, while standard quantum mechanics considers that the particle takes all of them at the same time. Since histories contain several different times, one may even attempt to reconstruct an approximate trajectory for the particle, even in cases where this is completely out of the question in standard quantum mechanics (for instance for a wave function that is a spherical wave); but of course one must always check that the projectors that are introduced for this purpose remain compatible with the consistency of a family.

In general, the physical information contained in the histories is not necessarily about position only: a projector can also project over a range of eigenstates of the momentum operator, include mixed information on position and momentum (subject, of course, to Heisenberg relations, as always in quantum mechanics), information on spin, etc.. There is actually a huge flexibility on the choice of projectors; for each choice, the physical properties that may be ascribed to the system are all those that are shared by all states of the projection subspace, but not by any orthogonal state. A frequent choice is to assume that, at a particular time t_i , all $P_{i,j}$'s are the projectors over the eigenstates of some Hermitian operator H : the first operator $P_{i,j=1}$ is the projector over all the eigenstates of H corresponding to the eigenvalue h_1 , the second $P_{i,j=2}$ the corresponding projector for the eigenvalue h_2 , etc. In this case, all histories of the family will include an exact information about the value of the physical quantity associated at time t_i to H (for instance the energy if H is the Hamiltonian). Let us nevertheless caution the reader once more that we are not free to choose any operator H_i at any time t_i : in general, there is no reason why the consistency conditions should be satisfied by a family built in this way.

Using histories, we obtain a description of the properties of the system in itself, without any reference to measurements, conscious observers, etc.. This does not mean that measurements are excluded; they can be treated merely as particular cases, by incorporating the corresponding physical devices in the system under study. Moreover, one attributes properties to the system at different times; this is in contrast with the orthodox interpretation; where a measurement does not necessarily reveal any pre-existing property of the physical system, and projects it into a new state that may

be totally independent of the initial state. It is easy to show that the whole formalism of consistent families is invariant under time reversal, in other words that it makes no difference between the past and the future (instead of the initial density operator $\rho(t_0)$, one may use the final density operator $\rho(t_N)$ and still use the same quantum formalism [172]) - for more details, and even an intrinsic definition of consistency that involves no density operator at all, see §III of ref. [173]. In addition, one can develop a relation between consistent families and semi-classical descriptions of a physical system; see ref. [169] for a discussion of how classical equations can be recovered for a quantum system provided sufficient coarse graining is included (in order to ensure, not only decoherence between the various histories of the family, but also what these authors call “inertia” to recover classical predictability). See also chap. 16 of [170] for a discussion of how classical determinism is restored, in a weak version that ensures perfect correlations between the values of quasi-classical observables at different times (or course, there is no question of fundamental determinism in this context). The history point of view undoubtedly has many attractive features, and seems to be particularly clear and easy to use, at least as long as one limits oneself to one single consistent family of histories.

How does the history interpretation deals with the existence of several consistent families? They are all a priori equally valid, but they will obviously lead to totally different descriptions of the evolution of the same physical system; this is actually the delicate aspect of the interpretation (we will come back to it in the next subsection). The answer of the history interpretation to the question is perfectly clear: different consistent families are to be considered as mutually exclusive (except, of course, in very particular cases where the two families can be embedded into a single large consistent family); all families may be used in a logical reasoning, but never combined together. In other words: the physicist is free to choose any point of view in order to describe the evolution of the system and to ascribe properties to the system; in a second independent step, another consistent family may also be chosen in order to develop other logical considerations within this different frame; but it would be totally meaningless (logically inconsistent) to combine considerations arising from the two frames. This a very important fundamental rule that must be constantly kept in mind when one uses this interpretation. We refer the reader to [173] for a detailed and systematic discussion of how to reason consistently in the presence of disparate families, and to [174] for simple examples of incompatible families of histories (photon hitting a beam splitter, §II) and the discussion of quantum incompatibility (§V); various classical analogies are offered for this incompatibility, including a two-dimensional representation of a three-dimensional object by a draftsman, who can choose many points of view to make a drawing, but can certainly not take several at the same time - otherwise the projection would become inconsistent.

6.4.4 Comparison with other interpretations

In the history interpretation, as we have already seen, there is no need to invoke conscious observers, measurement apparatuses, etc.; the system has properties in itself, as in the non-orthodox interpretations that we discussed before (considering that the correlation interpretation is orthodox). A striking feature of the history interpretation, when compared to the others, is the enormous flexibility that exists for the selection of the point of view (family) that can be chosen for describing the system, since all the times t_1, t_2, \dots are arbitrary (actually their number is also arbitrary) and, for each of them, many different projectors P may be introduced. One may even wonder if the interpretation is sufficiently specific, and if this very large number of families of histories is not a problem. This question will come naturally in a comparison between the history interpretation to the other interpretations that we have already discussed.

First, what is the exact relation between the history interpretation and the orthodox theory? The relation is certainly very close, but several concepts are expressed a more precise way. For instance, complementarity stands in the Copenhagen interpretation as a general, almost philosophical, principle. In the history interpretation, it is related to mathematical conditions, such as consistency conditions; also, every projector can not be more precise than the projector over a single quantum state $|\varphi\rangle$, which is itself obviously subject to the uncertainty relations because of the very structure of the space of states. Of course, considerations on incompatible measurement devices may still be made but, as the Bohrian distinction between the macroscopic and microscopic worlds, they lose some of their fundamental character. In the same vein, the history interpretation allows a quantum theory of the universe (compare for instance with quotation *v* at the end of § 2); we do not have to worry about dividing the universe into observed systems and observers. The bigger difference between the orthodox and the history interpretations is probably the way they describe the time evolution of a physical system. In the usual interpretation, we have two different postulates for the evolution of a single entity, the state vector, which may sometimes create conflicts; in the history interpretation, the continuous Schrödinger evolution and the random evolution of the system among histories are put at very different levels, so that the conflict is much less violent. Actually, in the history point of view, the Schrödinger evolution plays a role only at the level of the initial definition of consistent families (through the evolution operators that appear in the Heisenberg operators) and in the calculation of the probability $\mathcal{P}(\mathcal{H})$; the real time evolution takes place between the times t_i and t_{i+1} and is purely stochastic. In a sense, there is a kind of inversion of priorities, since it is now the non-determinist evolution that becomes the major source of evolution, while in the orthodox point of view it is rather the deterministic evolution of an isolated system. Nevertheless, and despite these differences,

the decoherent history interpretation remains very much in the spirit of the orthodox interpretation; indeed, it has been described as an “extension of the Copenhagen interpretation”, or as “a way to emphasize the internal logical consistency of the notion of complementarity”. On the other hand, Gell-Mann takes a more general point of view on the history interpretation which makes the Copenhagen interpretation just “a special case of a more general interpretation in terms of the decoherent histories of the universe. The Copenhagen interpretation is too special to be fundamental, ...” [175].

What about the “correlation interpretation”? In a sense, this minimal interpretation is contained in both the orthodox interpretation (from which some elements such as the reduction of the state vector have been removed) and in the history interpretation. Physicists favoring the correlation interpretation would probably argue that adding a physical discussion in terms of histories to their mathematical calculation of probabilities does not add much to their point of view: they are happy with the calculation of correlations and do not feel the need for making statements on the evolution of the properties of the system itself. Moreover, they might add that they wish to insert whatever projectors correspond to a series of measurements in (37), and not worry about consistency conditions: in the history interpretation, for arbitrary sequences of measurements, one would get inconsistent families for the isolated physical system, and one has to include the measurement apparatuses to restore consistency. We have already remarked in § 6.1 that the correlation interpretation allows a large flexibility concerning the boundary between the measured system and the environment. For these physicists, the history description appears probably more as an interesting possibility than as a necessity; but there is no contradiction either.

Are there also similarities with theories with additional variables? To some extent, yes. Within a given family, there are many histories corresponding to the same Schrödinger evolution and, for each history, we have seen that more information on the evolution of physical reality is available than through the state vector (or wave function) only. Under these conditions, the state vector can be seen as a non-complete description of reality, and one may even argue that the histories themselves constitute additional variables (but they would then be family dependent, and therefore not EPR elements of reality, as we discuss later). In a sense, histories provide a kind of intermediate view between an infinitely precise Bohmian trajectory for a position and a very delocalized wave function. In the Bohm theory, the wave function pilots the position of the particles; in the decoherent history interpretation, the propagation of the wave function pilots rather the definition of histories (through a consistency condition) as well as a calculation of probabilities, but not the evolution between times t_i and t_{i+1} , which is supposed to be fundamentally random. Now, of course, if one wished, one could make the two sorts of theories even more similar by assuming the existence of a well defined point in the space of histories; this point would

then be defined as moving in a completely different space from the Bohm theory: instead of the configuration space, it would move in the space defined by the family, and thus be defined as family dependent. In this way, the history interpretation could be made deterministic if, for some reason, this was considered useful. On many other aspects, the theories with additional variables are very different from the history interpretation and we can probably conclude this comparison by stating that they belong to rather different point of view on quantum mechanics.

Finally, what is the comparison with theories incorporating additional non-linear terms in the Schrödinger evolution? In a sense, they correspond to a completely opposite strategy: they introduce into one single equation the continuous evolution of the state vector as well as a non-linear deterministic mechanism simulating the wave packet reduction when needed; the history interpretation puts on different levels the continuous Schrödinger evolution and a fundamentally random selection of history selection by the system. One might venture to say that the modified non-linear dynamics approach is an extension of the purely wave program of Schrödinger, while the history interpretation is a modern version of the ideas put forward by Bohr. Another important difference is that a theory with modified dynamics is not strictly equivalent to usual quantum mechanics, and could lead to experimental tests, while the history interpretation is built to reproduce exactly the same predictions in all cases - even if it can sometimes provide a convenient point of view that allows to grasp its content more conveniently [130].

6.4.5 A profusion of points of view; discussion

We finally come back to a discussion of the impact of the profusion of possible points of view, which are provided by all the families that satisfy the consistency condition. We have already remarked that there is, by far, no single way in this interpretation to describe the evolution of properties of a physical system - for instance all the complementary descriptions of the Copenhagen interpretation appear at the same level. This is indeed a large flexibility, much larger than in classical physics, and much larger than in the Bohmian theory for instance. Is the “no combination of points of view” fundamental rule really sufficient to ensure that the theory is completely satisfactory? The answer to this question is not so clear for several reasons. First, for macroscopic systems, one would like an ideal theory to naturally introduce a restriction to sets corresponding to quasi-classical histories; unfortunately, the number of consistent sets is in fact much too large to have this property [176]. This is the reason why more restrictive criteria for mathematically identifying the relevant sets are (or have been) proposed, but no complete solution or consensus has yet been found; the detailed physical consequences of consistency conditions are still being explored, and actually provide an interesting subject of research. Moreover, the paradoxes that we

have discussed above are not all solved by the history interpretation. Some of them are, for instance the Wigner friend paradox, to the extent where no reference to observers is made in this interpretation. But some others are not really solved, and the interpretation just leads to a reformulation in a different formalism and vocabulary. Let us for instance take the Schrödinger cat paradox, which initially arose from the absence of any ingredient in the Schrödinger equation for the emergence of single macroscopic result - in other words, for excluding impossible macroscopic superpositions of an isolated, non-observed, system. In the history interpretation, the paradox transposes in terms of choice of families of histories: the problem is that there is no way to eliminate the families of histories where the cat is at the same time dead and alive; actually, most families that are mathematically acceptable through the consistency condition contain projectors on macroscopic superpositions, and nevertheless have exactly the same status as the families that do not. One would much prefer to have a “super-consistency” rule that would eliminate these superpositions; this would really solve the problem, but such a rule does not exist for the moment. At this stage, one can then do two things: either consider that the choice of sensible histories and reasonable points of view is a matter of good sense - a case in which and one returns to the usual situation in the traditional interpretation, where the application of the postulate of wave packet is also left to the good taste of the physicist; or invoke decoherence and coupling to the external world in order to eliminate all these unwanted families - a case in which one returns to the usual situation where, conceptually, it is impossible to ascribe reasonable physical properties to a closed system without refereeing to the external world and interactions with it⁴⁸, which opens again the door to the Wigner friend paradox, etc.

Finally one may note that, in the decoherent history interpretation, there is no attempt to follow “in real time” the evolution of the physical system; one speaks only of histories that are seen as complete, “closed in time”, almost as histories of the past in a sense. Basic questions that were initially at the origin of the introduction of the wave packet postulate, such as “how to describe the physical reality of a spin that has already undergone a first measurement but not yet a second”, are not easily answered. In fact, the consistency condition of the whole history depends on the future choice of the observable that will be measured, which does not make the discussion simpler than in the traditional interpretation, maybe even more complicated

⁴⁸For instance, one sometimes invokes the practical impossibility to build an apparatus that would distinguish between a macroscopic superposition and the orthogonal superposition; this would justify the elimination of the corresponding histories from those that should be used in the description of reality. Such an argument reintroduces the notion of measurement apparatus and observers in order to select histories, in contradiction with the initial motivations of this point of view - see Rosenfeld’s citation in §2. Moreover, this immediately opens again the door to Wigner friend type paradoxes, etc.

since its very logical frame is now under discussion. What about a series of measurements which may be, or may not be, continued in the future, depending on a future decision? As for the EPR correlation experiments, they can be re-analyzed within the history interpretation formalism [177] (see also [101] for a discussion of the Hardy impossibilities and the notion of “consistent contrafactuality”); nevertheless, at a fundamental level, the EPR reasoning still has to be dismissed for exactly the same reason that Bohr invoked already long ago: it introduces the EPR notion of “elements of reality”, or counterfactual arguments, that are not more valid within the history interpretation than in the Copenhagen interpretation (see for instance §V of [177] or the first letter in [175]). We are then brought back to almost the same old debate, with no fundamentally new element. We have nevertheless already remarked that, like the correlation interpretation, the history interpretation may be supplemented by other ingredients, such as the Everett interpretation⁴⁹ or, at the other extreme, EPR or deterministic ingredients, a case in which the discussion would of course become different.

For a more detailed discussion of this interpretation, see the references given at the beginning of this section; for a discussion of the relation with decoherence, the notion of “preferred (pointer) bases”, and classical predictability, see [176]; for a critique of the decoherent history interpretation, see for instance [178], where it is argued among others that consistency conditions are not sufficient to predict the persistence of quasi-classicality, even at large scales in the Universe; see also [179] which claims that they are not sufficient either for a derivation of the validity of the Copenhagen interpretation in the future; but see also the reply to this critique by Griffiths in [174]. Finally, another reference is a recent article in *Physics Today* [16] that contains a discussion of the history interpretation in terms that stimulated interesting reactions from the proponents of the interpretation [175].

6.5 Everett interpretation

A now famous point of view is that proposed by Everett, who named it “relative state interpretation” - but in its various forms it is sometimes also called “many-worlds interpretation”, or “branching universe interpretation” (the word “branching” refers here to the state vector of the universe). In this interpretation, any possible contradiction between the two evolution postulates is cancelled by a simple but efficient method: the second postulate is merely suppressed!

In the Everett interpretation [180], the Schrödinger equation is taken even more seriously than in the orthodox interpretation. Instead of trying to explain how successive sequences of well-defined measurement results are

⁴⁹Nevertheless, since the Everett interpretation completely suppresses from the beginning any specific notion of measurement, measuring apparatus, etc., the usefulness of completing it with the history interpretation is not obvious.

obtained, one merely considers that single results never emerge: all possibilities are in fact realized at the same time! The Von Neumann chain is never broken, and its tree is left free to develop its branch ad infinitum. The basic remark of this interpretation is that, for a composite system of correlated subsystems (observed system, measurement apparatus, and observer, all considered after a measurement), “there does not exist anything like a single state for one subsystem....one can arbitrarily choose a state for one subsystem and be led to the relative state for the remainder” - this is actually just a description of quantum entanglement, a well-known concept. But, now, the novelty is that the observer is considered as a purely physical system, to be treated within the theory exactly on the same footing as the rest of the environment. It can then be modelled by an automatically functioning machine, coupled to the recording devices and registering past sensory data, as well as its own machine configurations. This leads Everett to the idea that “current sensory data, as well as machine configuration, is immediately recorded in the memory, so that all the actions of the machine at a given instant can be considered as functions of the memory contents only” ..; similarly, all relevant experience that the observer keeps from the past is also contained in this memory. From this Everett concludes that “there is no single state of the observer; ..with each succeeding observation (or interaction), the observer state branches into a number of different states... All branches exist simultaneously in the superposition after any sequence of observations”. Under these conditions, the emergence of well-defined results from experiments is not considered as a reality, but just as a delusion of the mind of the observer. What the physical system does, together with the environment, is to constantly ramify its state vector into all branches corresponding to all measurement results, without ever selecting one of these branches. The observer is also part of this ramification process, that nevertheless has properties which prevent him/her to bring to his/her mind the perception of several of them at the same time. Indeed, each “component of the observer” remains completely unaware of all the others, as well as of the state vectors that are associated to them (hence the name “relative state interpretation”). The delusion of the emergence of a single result in any experiment then appears as a consequence of the limitations of the human mind: in fact, the process that we call “quantum measurement” never takes place!

How is an EPRB experiment seen in this point of view? In the Bohmian interpretation we emphasized the role of Stern-Gerlach analyzers, in the nonlinear evolution interpretation that of the detectors and decoherence; here we have to emphasize the role of the correlations with the external world on the mind of the two human observers. The state vector will actually develop its Von Neumann chain through the analyzers and the detectors and, at some point, include these observers whose brain will become part of the superposition. For each choice of the settings a and b , four branches

of the state vector will coexist, containing observers whose mind is aware of the result associated with each branch. So, the choice of a has a distant influence on the mind of the second observer, through the definition of the relevant basis for the Von Neumann chain, and non-locality is obtained as a result.

It is sometimes said that “what is most difficult in the Everett interpretation is to understand exactly what one does not understand”. Indeed, it may look simple and attractive at first sight, but turns out to be as difficult to defend as to attack (see nevertheless §3 of ref. [181], where the author considers the theory as ambiguous because dynamical stability conditions are not considered). The question is, to some extent, what one should expect from a physical theory, and what it should explain. Does it have to explain in detail how we perceive results of experiments, and if so of what nature should such an explanation be? What is clear, anyway, is that the whole point of view is exactly opposite to that of the proponents of the additional variables: the emphasis is put, not on the physical properties of the systems themselves, but on the effects that they produce on our minds. Notions such as perception (ref. [180] speaks of “trajectory of the memory configuration”) and psychology become part of the debate. But it remains true that the Everett interpretation solves beautifully all difficulties related to Bohrian dichotomies, and makes the theory at the same time simpler and more pleasant esthetically. Since the human population of earth is made of billions of individuals, and presumably since each of them is busy making quantum measurements all the time without even knowing it, should we see the state vector of the universe as constantly branching at a really fantastic rate?

CONCLUSION

Quantum mechanics is, with relativity, the essence of the big conceptual revolution of the physics of the 20th century. Now, do we really understand quantum mechanics? It is probably safe to say that we understand its machinery pretty well; in other words, we know how to use its formalism to make predictions in an extremely large number of situations, even in cases that may be very intricate. Heinrich Hertz, who played such a crucial role in the understanding of electromagnetic waves in the 19th century (Hertzian waves), remarked that, sometimes, the equations in physics are “more intelligent than the person who invented them” [182]. The remark certainly applies to the equations of quantum mechanics, in particular to the Schrödinger equation, or to the superposition principle: they contain probably much more substance than any of their inventors thought, for instance in terms of unexpected types of correlations, entanglement, etc. It is astonishing to see that, in all known cases, the equations have always predicted exactly the correct results, even when they looked completely counter-intuitive. Conceptually, the situation is less clear. One major issue is whether or not the

present form theory of quantum mechanics is complete. If it is, it will never be possible in the future to give a more precise description of the physical properties of a single particle than its wave function (or of two particles, for instance in an EPR type experiment); this is the position of the proponents of the Copenhagen interpretation. If it is not, future generations may be able to do better and to introduce some kind of description that is more accurate.

We have shown why the EPR argument is similar to Gregor Mendel's reasoning, which led him from observations performed between 1854 and 1863 to the discovery of specific factors, the genes (the word appeared only later, in 1909), which turned out to be associated with microscopic objects hidden inside the plants that he studied. In both cases, one infers the existence of microscopic "elements of reality" from the results of macroscopic observations. Mendel could derive rules obeyed by the genes, when they combine in a new generation of plants, but at his time it was totally impossible to have any precise idea of what they really could be at a microscopic level (or actually even if they were microscopic objects, or macroscopic but too small to be seen with the techniques available at that time). It took almost a century before O.T. Avery and coll. (1944) showed that the objects in question were contained in DNA molecules; later (1953), F. Crick and J. Watson illustrated how subtle the microscopic structure of the object actually was, since genes corresponded to subtle arrangement of nucleic bases hidden inside the double helix of DNA molecules. We now know that, in a sense, rather than simple microscopic objects, the genes are arrangements of objects, and that all the biological machinery that reads them is certainly far beyond anything that could be conceived at Mendel's time. Similarly, if quantum mechanics is one day supplemented with additional variables, these variables will not be some trivial extension of the other variables that we already have in physics, but variables of a very different nature. But, of course, this is only a possibility, since the histories of biology and physics are not necessarily parallel! Anyway, the discussion of additional variables leads to interesting questions, which we have tried to illustrate in this article by a brief description of several possible interpretations of quantum mechanics that have been or are still proposed; some introduce additional variables that indeed have very special properties, others do not, but in any case the theory contains at some stage features that are reminiscent of these difficulties.

A natural comparison is with special relativity, since neither quantum mechanics nor relativity is intuitive; indeed, experience shows that both, initially, require a lot of thought from each of us before they become intellectually acceptable. But the similarity stops here: while it is true that, the more one thinks about relativity, the more understandable it becomes (at some point, one even gets the feeling that relativity is actually a logical necessity!), one can hardly say the same thing about quantum mechanics.

Nevertheless, among all intellectual constructions of the human mind, quantum mechanics may be the most successful of all theories since, despite all efforts of physicists to find its limits of validity (as they do for all physical theories), and many sorts of speculation, no one for the moment has yet been able to obtain clear evidence that they even exist. Future will tell us if this is the case; surprises are always possible!

APPENDICES

I. An attempt for constructing a “separable” quantum theory (non-deterministic but local theory)

We come back to the discussion of § 3.2 but now give up botany; in this appendix we consider a physicist who has completely assimilated the rules of quantum mechanics concerning non-determinism, but who is sceptical about the essential character of non-locality in this theory (or non-separability; for a detailed discussion of the meaning of these terms, see for instance [21] [34]). So, this physicist thinks that, if measurements are performed in remote regions of space, it is more natural to apply the rules of quantum mechanics separately in these two regions. In other words, in order to calculate the probability of any measurement result, he/she will apply the rules of quantum mechanics, in a way that is perfectly correct locally; the method assumes that it is possible to reason separately in the two regions of space, and therefore ignores the non-separable character of quantum events (quantum events may actually involve both space regions at the same time). Let us take an extreme case, where the two measurement take place in two different galaxies: our physicist would be prepared to apply quantum mechanics to the scale of a galaxy, but not at an intergalactic scale!

How will he/she then treat the measurement process that takes place in the first galaxy? It is very natural to assume that the spin that it contains is described by a state vector (or by a density operator, it makes no difference for our reasoning here) that may be used to apply the orthodox formula for obtaining the probabilities of each possible result. If our experimenter is a good scientist, he/she will realize at once that it is not a good idea to assume that the two-spin system is described by a tensor product of states (or of density operators); this would never lead to any correlation between the results of measurements performed in the two galaxies. Therefore, in order to introduce correlations, he/she will assume that the states in question (or the density operators) are random mathematical objects, which fluctuate under the effect of the conditions of emission of the particles (for instance). The method is clear: for any possible condition of the emission, one performs an orthodox quantum calculation in each region of space, and then takes an average value over the conditions in question. After all, this is nothing but

the universal method for calculating correlations in all the rest of physics! We note in passing that this approach takes into account the indeterministic character of quantum mechanics, but introduces a notion of space separability in the line of the EPR reasoning. Our physicist may for instance assume that the two measurement events are separated by a space-like interval in the sense of relativity, so that no causal relation can relate them in any circumstance; this seem to fully justify an independent calculation of both phenomena.

Even if this is elementary, and for the sake of clarity, let us give the details of this calculation. The fluctuating random variable that introduces the correlations is called λ , and the density operator of the first spin $\rho_1(\lambda)$; for a direction of measurement defined by the unit vector a , the eigenstate of the measurement corresponding to result $+1$ is noted $|+a\rangle$. The probability for obtaining result $+$ if the first measurement is made along direction a is then written as:

$$\mathcal{P}_+(a, \lambda) = \langle +a | \rho_1(\lambda) | +a \rangle \quad (44)$$

In the same way, one writes the probability for the result -1 in the form:

$$\mathcal{P}_-(a, \lambda) = \langle -a | \rho_1(\lambda) | -a \rangle \quad (45)$$

If, instead of direction a , another different direction a' is chosen, the calculations remain the same and lead to two functions $\mathcal{P}_\pm(a', \lambda)$. As for measurements performed in the second region of space, they provide two functions $\mathcal{P}_\pm(b, \lambda)$ and $\mathcal{P}_\pm(b', \lambda)$.

We now calculate the number which appears in the Bell theorem (BCHSH inequality), namely the linear combination, as in (6), of four average values of products of results associated with the couples of orientations (a, b) , (a, b') , (a', b) , (a', b') . Since we have assumed that results are always ± 1 , the average value depends only on the differences:

$$A(\lambda) = \mathcal{P}_+(a, \lambda) - \mathcal{P}_-(a, \lambda) \quad (46)$$

or:

$$A'(\lambda) = \mathcal{P}_+(a', \lambda) - \mathcal{P}_-(a', \lambda) \quad (47)$$

(with similar notation for the measurements performed in the other region of space) and can be written as the average value over λ of expression:

$$A(\lambda)B(\lambda) + A(\lambda)B'(\lambda) - A'(\lambda)B(\lambda) + A'(\lambda)B'(\lambda) \quad (48)$$

We are now almost back to the calculation of section 4.1.2, with a little difference nevertheless: the A 's and B 's are now defined as probability differences so that their values are not necessarily ± 1 . It is nonetheless easy to see that they are all between $+1$ and -1 , whatever the value of λ is. Let us

for a moment consider λ , A and A' as fixed, keeping only B and B' as variables; in the space of these variables, expression (48) corresponds to a plane surface which, at the four corners of the square $B = \pm 1$, $B' = \pm 1$, takes values $\pm 2A$ or $\pm 2A'$, which are between ± 2 ; at the center of the square, the plane goes through the origin. By linear interpolation, it is clear that, within the inside of the square, the function given by (48) also remains bounded between ± 2 ; finally, its average value has the same property. Once more we find that the Bell theorem holds in a large variety of contexts!

Since we know that quantum mechanics as well as experiments violate the Bell inequality, one may wonder what went wrong in the approach of our physicist; after all, his/her reasoning is based on the use of the usual formalism of quantum mechanics. What caused the error was the insistence of treating the measurements as separable events, while orthodox quantum mechanics requires us to consider the whole two-spin system as a single, non-separable, system; in this system, no attempt should be made to distinguish subsystems. The only correct reasoning uses only state vectors/density operators that describe this whole system in one mathematical object. This example illustrates how it is really separability and/or locality which are at stake in a violation of the Bell inequalities, not determinism.

It is actually instructive, as a point of comparison, to make the calculation of standard quantum mechanics as similar as possible to the reasoning that led to the inequality (48). For this purpose, we notice that any density operator ρ of the whole system belongs to a space that is the tensor product of the corresponding spaces for individual systems; therefore ρ can always be expanded as:

$$\rho = \sum_{n,p} c_{n,p} [\rho_n(1) \otimes \rho_p(2)] \quad (49)$$

From this, one can obtain the probability of obtaining two results $+1$ along directions a and b as:

$$\mathcal{P}_{++}(a, b) = \sum_{n,p} c_{n,p} \langle +/a | \rho_n | +/a \rangle \langle +/b | \rho_p | +/b \rangle \quad (50)$$

(probabilities corresponding to the other combinations of results are obtained in the same way). The right hand side of this equation is not completely different from the sum over λ that was used above; actually it is very similar, since the sum over the indices n and p plays the same role as the sum over the different values of λ . In fact, if all $c_{n,p}$'s were real positive numbers, and if all operators ρ_n and ρ_p were positive (or semi-positive) operators, nothing would prevent us from doing exactly the same reasoning again and deriving the Bell inequality; in other words, any combined system that is a statistical mixture (which implies positive coefficients) of uncorrelated states satisfies the Bell inequalities. But, in general, the positivity conditions are not fulfilled, and this is precisely why the quantum mechanical results can violate the inequalities.

II. Maximal probability for a Hardy state.

In this appendix we give more details on the calculations of § 4.2; the two-particle state corresponding to the measurement considered in (i) is the tensor product of ket (9) by its correspondent for the second spin:

$$\cos^2 \theta |+, + \rangle + \sin \theta \cos \theta [|+, - \rangle + |-, + \rangle] + \sin^2 \theta |-, - \rangle \quad (51)$$

which has the following scalar product with ket (14):

$$\cos^2 \theta \sin \theta - 2 \sin \theta \cos^2 \theta = -\sin \theta \cos^2 \theta \quad (52)$$

The requested probability is obtained by dividing the square of this expression by the square of the norm of the state vector:

$$\mathcal{P} = \frac{\sin^2 \theta \cos^4 \theta}{2 \cos^2 \theta + \sin^2 \theta} = \frac{\sin^2 \theta (1 - \sin^2 \theta)^2}{2 - \sin^2 \theta} \quad (53)$$

A plot of this function shows that it has a maximum of about 0.09.

III. Proof of relations (17) and (18).

Let us start with the ket:

$$|\Psi \rangle = |+, +, + \rangle + \eta |-, -, - \rangle \quad (54)$$

where:

$$\eta = \pm 1 \quad (55)$$

We wish to calculate the effect of the product operator $\sigma_{1x}\sigma_{2y}\sigma_{3y}$ on this ket. Since every operator in the product commutes with the two others, the order in which they are applied is irrelevant; let us then begin with the operator associated with the first spin:

$$\begin{aligned} \sigma_{1+} |\Psi \rangle &= 2\eta |+, -, - \rangle \\ \sigma_{1-} |\Psi \rangle &= 2 |-, +, + \rangle \end{aligned} \quad (56)$$

which provides:

$$\sigma_{1x} |\Psi \rangle = |\Psi' \rangle = \eta |+, -, - \rangle + |-, +, + \rangle \quad (57)$$

For the second spin:

$$\begin{aligned} \sigma_{2+} |\Psi' \rangle &= 2\eta |+, +, - \rangle \\ \sigma_{2-} |\Psi' \rangle &= 2 |-, -, + \rangle \end{aligned} \quad (58)$$

so that:

$$\sigma_{2y} |\Psi' \rangle = |\Psi'' \rangle = \frac{1}{i} (\eta |+, +, - \rangle - |-, -, + \rangle) \quad (59)$$

Finally, the third spin gives:

$$\begin{aligned}\sigma_{3+} | \Psi'' \rangle &= -2i\eta | +, +, + \rangle \\ \sigma_{3-} | \Psi'' \rangle &= +2i | -, -, - \rangle\end{aligned}\quad (60)$$

which leads to:

$$\sigma_{3y} | \Psi'' \rangle = -\eta | +, +, + \rangle - | -, -, - \rangle = -\eta | \Psi \rangle \quad (61)$$

(since $\eta^2 = 1$). Indeed, we find that $| \Psi \rangle$ is an eigenstate of the product of the three spin operators $\sigma_{1x}\sigma_{2y}\sigma_{3y}$, with eigenvalue $-\eta$. By symmetry, it is obvious that the same is true for the product operators $\sigma_{1y}\sigma_{2x}\sigma_{3y}$ and $\sigma_{1y}\sigma_{2y}\sigma_{3x}$.

Let us now calculate the effect of operator $\sigma_{1x}\sigma_{2x}\sigma_{3x}$ on $| \Psi \rangle$; from (58) we get:

$$\sigma_{2x} | \Psi' \rangle = | \Psi''' \rangle = (\eta | +, +, - \rangle + | -, -, + \rangle) \quad (62)$$

so that:

$$\begin{aligned}\sigma_{3+} | \Psi''' \rangle &= 2\eta | +, +, + \rangle \\ \sigma_{3-} | \Psi''' \rangle &= 2 | -, -, - \rangle\end{aligned}\quad (63)$$

and, finally:

$$\sigma_{3x} | \Psi''' \rangle = \eta | +, +, + \rangle + | -, -, - \rangle = \eta | \Psi \rangle \quad (64)$$

The change of sign between (61) and (64) may easily be understood in terms of simple properties of the Pauli spin operators (anticommutation and square equal to one).

IV. Impossibility of superluminal communication and of cloning quantum states.

In EPR schemes, applying the reduction postulate projects the second particle instantaneously onto an eigenstate corresponding to the same quantization axis as the first measurement. If it were possible to determine this state completely, superluminal communication would become accessible: from this state, the second experimenter could calculate the direction of the quantization axis to which it corresponds, and rapidly know what direction was chosen by the first experimenter⁵⁰, without any special effect

⁵⁰Note that what is envisaged here is communication through the choice of the settings of the measurement apparatuses; this makes sense since the settings are chosen at will by the experimenters; on the other hand, the results of the experiments are not controlled, but random, so that they cannot be directly used as signals.

of the distance, for instance even if the experimenters are in two different remote galaxies. This, obviously, could be used as a sort of telegraph, completely free of any relativistic minimum delay (proportional to the distance covered). The impossibility for superluminal communications therefore relies on the impossibility of a complete determination of a quantum state from a single realization of this state. Such a realization allows only one single measurement, which (almost always) perturbs the state, so that a second measurement on the same state is not feasible; there is not, and by far, sufficient information in the first measurement for a full determination of the quantum state - see discussion in § 5.4.

Now, suppose for a moment that a perfect “cloning” of quantum states could be performed - more precisely the multiple reproduction (with many particles) of the unknown state of a single particle⁵¹. Applying the cloning process to the second particle of an EPR pair, one could then make a large number of perfect copies of its state; in a second step, one could perform a series of measurements on each of these copies, and progressively determine the state in question with arbitrary accuracy. In this way, the possibility for superluminal communication would be restored! But, in reality, quantum mechanics does not allow either for such a perfect reproduction of quantum states [108] [109]; for instance, if one envisages using stimulated emission in order to clone the state of polarization of one single photon into many copies, the presence of spontaneous emission introduces noise in the process and prevents perfect copying. A discussion of multi-particle cloning is given in [110].

This, nevertheless, does not completely solve the general question: even without cloning quantum states, that is only with the information that is available in one single measurement in each region of space, it is not so obvious that the instantaneous reduction of the wave packet cannot be used for superluminal communication. After all, it is possible to repeat the experiment many times, with many independent pairs of correlated particles, and to try to extract some information from the statistical properties of the results of all measurements. The EPR correlations are very special and exhibit such completely unexpected properties (e.g. violations of the Bell inequalities)! Why not imagine that, by using or generalizing EPR schemes (more than two systems, delocalized systems, etc.), one could invent schemes where superluminal communication becomes possible? Here we show why such schemes do not exist; we will sketch the general impossibility proof in the case of two particles (or two regions of space), but the generalization to more systems in several different regions of space is straightforward.

⁵¹The “cloning” operation is not to be confused with the preparation of a series of particles into a same known quantum state: this operation can be performed by sending many spin 1/2 half particles through the same Stern-Gerlach magnet, or many photons through the same polarizing filter. What is theoretically impossible is to perfectly duplicate an initially unknown (and arbitrary) state.

Suppose that, initially, the two remote observers already possess a collection of pairs of correlated particles, which have propagated to their remote galaxies before the experiment starts. Each pair is in an arbitrary state of quantum entanglement, and we describe it with a density operator ρ in a completely general way. The first observer then chooses a setting a or, more generally, any local observable O_A to measure; the second observer is equally free to choose any local observable O_B , and may use as many particles as necessary to measure the frequency of occurrence of each result (i.e. probabilities); the question is whether the second observer can extract some information on O_A from any statistical property of the observed results. The impossibility proof relies on the fact that all operators (observables) corresponding to one of the two sub-systems always commute with all operators corresponding to the other; consequently, for any choice of the operators, it is always possible to construct a common eigenbasis $\{|\varphi_k, \theta_j\rangle\}$ in the space of states of the two-particle system, where the $|\varphi_k\rangle$'s are the eigenstates of O_A and the $|\theta_j\rangle$'s are the eigenstates of O_B . We can then calculate the probability of sequences of measurement where the first operator obtains result A_m (corresponding, if this eigenvalue is degenerate, to some range D_m for the index k) and the second result B_n (corresponding to range D_n for index j). But, what we are interested in is slightly different: the probability that the second observer will obtain each result B_n after a measurement performed by the other observer, independently of the result A_m , since there is no way to have access to this result in the second galaxy; our purpose is to prove that this probability is independent of the choice of the operator O_A .

Mathematically, extracting the probabilities concerning the second observer only amounts to summing over all possible results A_m , with the appropriate weights (probabilities); this is a classical problem, which leads to the notion of “partial trace” ρ_B over the variables of the sub-system A . This operator acts only in the space of states of system B and is defined by its matrix elements:

$$\langle \theta_i | \rho_B | \theta_j \rangle = \sum_k \langle \varphi_k, \theta_i | \rho | \varphi_k, \theta_j \rangle \quad (65)$$

It contains all information that the second experimenter needs for making predictions, exactly as from any ordinary density operator for an isolated system; for instance, the probability of observing result B_n is simply:

$$\mathcal{P}(B_n) = Tr \left\{ \sum_{i \in D_n} |\theta_i\rangle \langle \theta_i | \rho_B \right\} \quad (66)$$

Equations (65) and (66) can be derived in different ways. One can for instance use formula (37), if it has been proved before. Otherwise, one can proceed in steps: one first expands ρ in terms of projectors onto its own

eigenstates $|\Psi_l\rangle$, with positive eigenvalues; one then applies the wave packet reduction postulate to each $|\Psi_l\rangle$ separately in order to get the probability of any sequence of results; one finally performs the sum over l as well as the appropriate sum over indices k (unknown result) and j (if the observed eigenvalue is degenerate) in order to obtain the “reduced probabilities” - by these words we mean the probabilities relevant to the second observer, just after the other has performed a measurement of O_A , but before it has been possible to communicate the result to the second by some classical channel. This calculation provides the above expressions.

From formula (65), one might get the impression that the partial trace depends on the choice of the basis $\{|\varphi_k\rangle\}$, so that there is some dependence of operator ρ_B on the choice of O_A . This is a false impression: in fact, a simple algebra shows that the sum contained in the partial trace is completely independent of the basis chosen in the traced space of states; it does not even matter if the first experimenter has performed any experiment or not. Therefore, the second experimenter receives exactly the same information, completely independently of the decisions made by the first experimenter; no superluminal communication is possible.

Finally, one could object that it is not indispensable to have one system located in one region of space, the other in the second region, as we have assumed until now; each of them could perfectly well be delocalized in a superposition of states in different locations. Does the proof hold in this case? Yes, it does, after some modification. In this case, one should now associate the letters A and B , as well as operators O_A and O_B , not to sub-systems as before, but to measurements performed in each region of space. Each relevant operator can then be put between two projectors onto states that are localized either in the first (projector P_A), or the second (projector P_B), region of space. Since P_A and P_B are orthogonal, it is then simple to show that all operators with index A commute with all operators with index B (this is similar, in field theory, to the commutation of field operators that are outside mutual light cones); this remains true even if they act in the space of states of the same particle. We are now dealing with a generalization of the notion of partial trace, which is no longer related to the existence of different sub-systems (it may actually apply to one particle only), but to two different sets of operators acting in the same space of states. If all operators of one set commute with all operators of the second set, the notion of partial trace can indeed be transposed, and it turns out that the final result is independent of the operator that was chosen in the first set in order to calculate the trace. This allows one to prove that the information available in one region of space is completely independent of the kind of measurement performed in the other. Indeed, quantum mechanics is not contradictory with relativity!

Using the hydrodynamic equations associated with the evolution of the wave function, in order to guide the evolution of the additional variables (positions), may look like a very natural idea. In other fields of physics, it is known that the hydrodynamic equations can be obtained by taking averages of microscopic quantities over positions and velocities of point like particles; there is some analogy between the guiding term and the force term in a Landau type kinetic equations, where each particle is subject to an average force proportional to the gradient of the density for instance. Nevertheless, here we are dealing with a single particle, so that the guiding term can not be associated with interactions between particles. Moreover, we also know from the beginning that rather unusual properties must be contained in the guiding equations, at least if the idea is to exactly reproduce the predictions of usual quantum mechanics: the Bell theorem states that the additional variables have to evolve non-locally in ordinary three dimension space (on the other hand, in the configuration space of the system, they may evolve locally, exactly as for the state vector). In other words, the additional variables must be able to influence each other at an arbitrary distance in real space. Indeed, in the Bohmian equation of motion of the additional variables, the velocity of a particle contains an explicit dependence on its own position, as expected, but also a dependence on the positions of all the other particles (assuming that the particles are entangled). This is not a problem in itself: as mentioned in the main text, one can consider that making non-locality completely explicit in the equations is actually an advantage of Bohmian mechanics.

But one also has to be careful when this non-local term is included in the equations of motion: since relativity is based on the idea that it is totally impossible to send a message at a velocity exceeding the velocity of light, one must avoid features in the theory that would create conflicts with this principle. We must distinguish two cases, depending whether we consider influences on the additional variables that are direct (one modifies them “by hand”, in a completely arbitrary way, as for instance the position of a billiard ball), or indirect (applying external fields changes the Hamiltonian of the system, therefore modifies the evolution of the wave function so that, in turn, the evolution of the additional variables is affected). In the latter case, one can check that the non-local Bohmian term creates no problem: it cannot be used to transmit instantaneous information through the additional variables. This is a general result, which holds simply because the statistical predictions of Bohmian theory are equivalent to usual quantum mechanics, which itself does not allow superluminal communication. But assume for instance that we could manipulate directly the additional variable attached to a particle belonging to an EPR correlated pair, in a completely arbitrary

way (even at a microscopic scale), and without changing the wave function; then, the “quantum velocity term” acting on the additional variables of the other particle would instantaneously be affected, and so would be its subsequent position in space; since that particle may be in principle at an arbitrary distance, one could use this property to send messages at a velocity exceeding the velocity of light. The conclusion is that such manipulations should be considered as impossible: the only possible source of evolution of the additional variables has to be the wave function dependent term.

If the additional variables cannot be directly manipulated at a microscopic scale, can we then somehow filter them in a range of values, as one does for the state vector when the Oz component is filtered in a Stern-Gerlach apparatus? Suppose for instance that we could, for a particle in an eigenstate of the Oz component of its spin, select the values of the additional variable that will correspond to a result $+1$ in a future measurement of the Ox component; were such a selection possible with the help of any physical device, the theory with additional variables would obviously no longer be completely equivalent to standard quantum mechanics (this is because, within orthodox theory, if a spin $1/2$ particle is initially selected into the $+1$ spin state by an Oz oriented Stern-Gerlach apparatus, it becomes completely impossible to make any prediction on the deviation observed later in an Ox oriented Stern-Gerlach apparatus). Theories such as that developed in [4] include this as a possibility; indeed, if it is ever demonstrated experimentally, there will be very good reasons to abandon standard quantum theory in favor of theories with additional variables! Of course, we cannot predict the future and conceptual revolutions are always possible, but for the moment it may seem safer to provide the additional variable theories with features that make them equivalent to orthodox theory. In this perspective, it becomes necessary to assume that the additional variables can neither be manipulated directly nor filtered, as opposed to the state vector. In other words, the additional variables describe an objective reality, but at a different level from the reality of the field of the wave function, since only the latter can be influenced directly by human decisions.

VI. Constructing consistent families of histories

This appendix provides a discussion of the consistency condition (43). First, we should mention that other conditions have been proposed and used in the literature; in the initial article on histories [15], a weaker condition involving only the cancellation of the real part of the left hand side of (43) was introduced. For simplicity, here we limit ourselves to the stronger condition (43), which is a sufficient but not necessary condition to the weaker form; it turns out that, as noted in ref. [178], it seems more useful in this context to introduce selectivity than generality in the definition of consistent histories.

At first sight, a natural question that comes to mind is whether or not it is easy, or even possible at all, to fulfil exactly the large number of conditions contained in (43); actually, it has been proposed by Gell-Mann and Hartle to give a fundamental role to families that satisfy consistency conditions in only an approximate way [169], but here we leave aside this possibility and consider only exact consistency conditions. Let us assume for instance that the system under study is a particle propagating in free space; the various projectors may then define ranges of positions for the particle, playing a role similar to diaphragms or spatial filters in optics that confine an optical beam in the transverse direction. Then the consistency condition will appear as similar to a non-interference condition for the Huyghens wavelets that are radiated by the inner surface of each diaphragm. But we know that diffraction is unavoidable in the propagation of light; even if it can be a very small effect when the wavelength is sufficiently short and the diaphragms sufficiently broad, it is never strictly zero. Can we then satisfy the non-interference conditions exactly? The answer is not obvious. It turns out to be yes, but it is necessary to exploit the enormous flexibility that we have in the choice of subspaces and projectors in a large space of states, and not to limit ourselves to projectors over well-defined positions only. To understand why, we now briefly sketch one possible systematic method to construct consistent families of histories.

The simplest method is to guide the construction on the structure of (43), and to introduce the eigenstates $|\varphi_n^0\rangle$ of the density operator $\rho(t_0)$ (an Hermitian operator can always be diagonalized); let us then define the operators $\widehat{P}_{1,j_1}(t_1)$ as:

$$\widehat{P}_{1,n}(t_1) = |\varphi_n^0\rangle\langle\varphi_n^0| \quad (67)$$

which is equivalent to assuming that their Schrödinger counterparts $P_{1,j}$ are the projectors over the states that have evolved from the $|\varphi_n^0\rangle$'s from time t_0 to time t_1 . Because $\rho(t_0)$ is of course diagonal in its own basis, this choice already ensures the presence of a factor $\delta_{j_1,j_1'}$ in the right hand side of (43). Now, we can also assume that the P_{2,j_2} 's are defined as the projectors over the states that have evolved from the $|\varphi_n^0\rangle$'s from time t_0 to time t_2 , so that a relation similar to (67) is again obtained; this will ensure, not only the presence of factors $\delta_{j_2,j_2'}$ in the right hand side of (43), but actually also the appearance of a delta function δ_{j_1,j_2} . The procedure can be repeated as many times as needed, and in this way a consistent family is built.

It is nevertheless a very special family, for several reasons. The first is that each projector corresponds to a subspace of dimension 1 only, which corresponds to histories that are “maximally accurate”; the second is that most histories of the family have zero probability: in fact, only those with $j_1 = j_2 = j_3 = \dots$ are possible, which means that the only randomness occurs at time t_1 , and that all subspaces at later times are then perfectly

determined. The description that we obtain is, in a sense, trivial: initially, the system is in one of the eigenstates that are contained in $\rho(t_0)$, and then evolves deterministically from this initial state.

But it is possible to make the family less singular by grouping together, for each time t_i , several projectors into one single projector; different associations of projectors may be used at different times. In this way, the description of the evolution of the state within this family becomes less accurate, but also less trivial since projectors at different times are no longer associated pair by pair. On the other hand, it is possible to see that this grouping of projectors has not destroyed the consistent character of the family; of course, other methods for constructing consistent families are also possible.

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References

- [1] N. Bohr, “The Solvay meetings and the development of quantum mechanics”, Essays 1958-62 on atomic physics and human knowledge, Vintage, New York (1966); Atomic physics and the Description of Nature”, Cambridge University Press, 1930.
- [2] J. von Neumann, “Mathematical Foundations of Quantum Mechanics”, Princeton University Press (1955).
- [3] J.S. Bell, “On the problem of hidden variables in quantum mechanics”, Rev. Mod. Phys. **38**, 447-452 (1966); reprinted in “Quantum theory of measurement”, J.A. Wheeler and W.H. Zurek eds, Princeton Univ. Press (1983), 396-402.
- [4] D. Bohm and J. Bub, “A proposed solution of the measurement problem in quantum mechanics by a hidden variable theory”, Rev. Mod. Phys. **38**, 453-469 (1966).
- [5] D. Bohm and J. Bub, “A refutation of the proof by Jauch and Piron that hidden variables can be excluded in quantum mechanics”, Rev. Mod. Phys. **38**, 470-475 (1966).
- [6] N.D. Mermin, “Hidden variables and the two theorems of John Bell”, Rev. Mod. Phys. **65**, 803-815 (1993); in particular see § III.
- [7] J.S. Bell, “Speakable and unspeakable in quantum mechanics”, Cambridge University Press (1987); this lovely little book contains the complete set of J. Bell’s articles on quantum mechanics.
- [8] A. Shimony, Am. J. Phys. **31**, 755-773 (1963) “Role of the observer in quantum theory”.
- [9] D. Bohm, “A suggested interpretation of the quantum theory in terms of “hidden” variables”, Phys. Rev. **85**, 166-179 and 180-193 (1952); by the same author, see also “Quantum Theory”, Constable, London (1954), although this book does not discuss theories with additional variables.
- [10] N. Wiener and A. Siegel, “A new form for the statistical postulate of quantum mechanics”, Phys. Rev. **91**, 1551-1560 (1953); A. Siegel and N. Wiener, ”Theory of measurement in differential space quantum theory”, Phys. Rev. **101**, 429-432 (1956).
- [11] P. Pearle, “Reduction of the state vector by a non-linear Schrödinger equation”, Phys. Rev. **D13**, 857-868 (1976).

- [12] P. Pearle, "Toward explaining why events occur", *Int. J. Theor. Phys.* **18**, 489-518 (1979)
- [13] G.C. Ghirardi, A. Rimini, T. Weber, T. "Unified dynamics for microscopic and macroscopic systems", *Phys. Rev. D* **34**, 470-491 (1986); "Disentanglement of quantum wave functions", *Phys. Rev. D* **36**, 3287-3289 (1987).
- [14] B.S. de Witt, "Quantum mechanics and reality", *Physics Today* **23**, 30-35 (September 1970).
- [15] R.B. Griffiths, "Consistent histories and the interpretation of quantum mechanics", *Journ. Stat. Phys.* **36**, 219-272 (1984).
- [16] S. Goldstein, "Quantum Theory without observers", *Physics Today* **51**, 42-46 (March 1998) and 38-41 (April 1998).
- [17] "Quantum mechanics debate", *Physics Today* **24**, 36-44 (April 1971); "Still more quantum mechanics", *Physics Today* **24**, 11-15 (Oct. 1971).
- [18] B.S. de Witt and R.N. Graham, "Resource letter IQM-1 on the interpretation of quantum mechanics", *Am. J. Phys.* **39**, 724-738 (1971).
- [19] M. Jammer, "The conceptual development of quantum mechanics", Mc. Graw Hill (1966); second edition (1989). The same author has written another interesting book on a similar subject: "The Philosophy of quantum mechanics", Wiley (1974).
- [20] O. Darrigol, "From c-numbers to q-numbers; the classical analogy in the history of quantum theory", Univ. of California Press (1992).
- [21] B. d'Espagnat, "Conceptual Foundations of Quantum Mechanics, Benjamin, New York (1971).
- [22] B. d'Espagnat, "Veiled Reality; an Analysis of present day quantum mechanics concepts", Addison Wesley (1995); "Le réel voilé, analyse des concepts quantiques", Fayard, Paris (1994); "Une incertaine réalité, la connaissance et la durée", Gauthier-Villars, Paris (1985); "A la recherche du réel", Gauthier Villars Bordas, Paris (1979).
- [23] A. Pais, "Einstein and the quantum theory", *Rev. Mod. Phys.* **51**, 863-911 (1979).
- [24] O. Darrigol, "Strangeness and soundness in Louis de Broglie's early works", *Physis* **30**, 303-372 (1993).
- [25] E.A. Cornell and C.E. Wieman, "The Bose-Einstein condensate", *Scientific American* **278**, March 1998, 26-31.

- [26] W. Heisenberg, “The Physical Principles of the Quantum Theory”, University of Chicago Press (1930).
- [27] P.A.M. Dirac, “The Principles of Quantum Mechanics”, Oxford University Press (1958).
- [28] H.P. Stapp, “S-matrix interpretation of quantum theory”, *Phys. Rev.* **D3**, 1303-1320 (1971).
- [29] A. Peres, “What is a state vector?”, *Am. J. Phys.* **52**, 644-650 (1984).
- [30] B.G. Englert, M.O. Scully and H. Walther, “Quantum erasure in double-slit interferometers with which-way detectors”, *Am. J. Phys.* **67**, 325-329 (1999); see the first few lines of § IV.
- [31] C.A. Fuchs and A. Peres, “Quantum theory needs no ‘interpretation’”, *Physics Today* **53**, March 2000, 70-71; see also various reactions to this text in the letters of the September 2000 issue.
- [32] C. Chevalley, “Niels Bohr’s words and the Atlantis of Kantianism”, pages 33-57 in “Niels Bohr and contemporary philosophy”, J. Faye and H. Folse eds., Dordrecht Kluwer (1994).
- [33] J.S. Bell, “Bertlmann’s socks and the nature of reality”, *J. Physique* **C2**, 41-62 (1981).
- [34] J.S. Bell, “Quantum mechanics for cosmologists”, C. Isham, R. Penrose and D. Sciama eds, in *Quantum Gravity* **2**, 611-637, Clarendon Press (1981); page 117 of [7].
- [35] L. Rosenfeld, “The measuring process in quantum mechanics”, *Suppl. Prog. Theor. Phys.*, extra number 222 “Commemoration of the thirtieth anniversary of the meson theory by Dr. H. Yukawa”(1965).
- [36] M. Brune, E. Hagley, J. Dreyer, W. Maître, A. Maali, C. Wunderlich, J.M. Raimond et S. Haroche, “Observing the progressive decoherence of the ”meter” in a quantum measurement” *Phys. Rev. Lett.* **77**, 4887-4890 (1996).
- [37] E.P. Wigner, “Remarks on the mind-body question” in “The scientist speculates”, 284-302, I.J. Good ed. Heinemann, London (1961); reprinted in E.P. Wigner, “Symmetries and reflections”, Indiana University Press (1967), 171-184.
- [38] E.P. Wigner, “The problem of measurement”, *Am. J. Phys.* **31**, 6-15 (1963); reprinted in “Symmetries and Reflections”, Indiana University Press, page 153-170; or again in “Quantum theory of measurement”, J.A. Wheeler and W.H. Zurek eds, Princeton University Press (1983), 324-341.

- [39] E. Schrödinger, “Die gegenwärtige Situation in der Quantenmechanik”, *Naturwissenschaften* **23**, 807-812, 823-828, 844-849 (1935). English translation: “The present situation in quantum mechanics”, *Proc. Am. Phil. Soc.* **124**, 323-38 (1980); or pages 152-167 in “Quantum theory of measurement”, J.A. Wheeler and W.H. Zurek eds, Princeton University Press (1983).
- [40] E. Schrödinger, *Proc. Cambridge Phil. Soc.* **31**, 555 (1935) and **32**, 446 (1936).
- [41] A. Einstein, B. Podolsky and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?”, *Phys. Rev.* **47**, 777-780 (1935); or in “Quantum theory of measurement”, J.A. Wheeler and W.H. Zurek eds, Princeton University Press (1983), 138-141.
- [42] M. Born ed., “The Einstein-Born letters” (1916 - 1955), MacMillan, London (1971).
- [43] N. Bohr, “Quantum mechanics and physical reality”, *Nature* **136**, 65 (1935); “Can quantum-mechanical description of physical reality be considered complete?”, *Phys. Rev.* **48**, 696-702 (1935).
- [44] D. Mermin, “Is the moon there when nobody looks ? Reality and the quantum theory”, *Physics Today* **38**, 38-47 (April 1985).
- [45] D. Home and F. Selleri, “Bell’s theorem and EPR paradox”, *Rivista del Nuov. Cim.* **14**, 1-95 (1991).
- [46] P. Pearle, “Alternative to the orthodox interpretation of quantum theory”, *Am. J. Phys.* **35**, 742-753 (1967).
- [47] J.S. Bell, “On the Einstein-Podolsky-Rosen paradox”, *Physics*, **I**, 195-200 (1964).
- [48] F. Laloë, “Les surprenantes prédictions de la mécanique quantique”, *La Recherche n° 182* (novembre 1986), 1358-1367.
- [49] F. Laloë, “Cadre général de la mécanique quantique; les objections de Einstein, Podolsky et Rosen”, *J. Physique colloques* **C-2**, 1-40 (1981). See also the following articles, especially that by J Bell on Bertlmann socks, which is a classics!
- [50] D. Bohm, “Quantum Theory”, Prentice Hall (1951).
- [51] P. Eberhard, “Bell’s theorem and the different concepts of locality”, *Nuov. Cim.* **B46**, 392-419 (1978).

- [52] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, “Proposed experiment to test local hidden-variables theories”, *Phys. Rev. Lett.* **23**, 880-884 (1969).
- [53] A. Peres, “Unperformed experiments have no results”, *Am. J. Phys.* **46**, 745-747 (1978).
- [54] J.A. Wheeler, “Niels Bohr in Today’s Words” in *Quantum Theory and Measurement*, J.A. Wheeler and W.H. Zurek editors, Princeton University Press (1983).
- [55] E.P. Wigner, “On hidden variables and quantum mechanical probabilities”, *Am. J. Phys.* **38**, 1005-1009 (1970).
- [56] S.J. Freedman and J.F. Clauser, “Experimental test of local hidden variable theories”, *Phys. Rev. Lett.* **28**, 938-941 (1972); S.J. Freedman, thesis, University of California, Berkeley; J.F. Clauser, “Experimental investigations of a polarization correlation anomaly”, *Phys. Rev. Lett.* **36**, 1223 (1976).
- [57] E.S. Fry and R.C. Thompson, “Experimental test of local hidden variable theories”, *Phys. Rev. Lett.* **37**, 465-468 (1976).
- [58] M. Laméhi-Rachti and W. Mittig, “Quantum mechanics and hidden variables: a test of Bell’s inequality by the measurement of spin correlation in low energy proton-proton scattering”, *Phys. Rev.* **D14**, 2543-2555 (1976).
- [59] A. Aspect, P. Grangier et G. Roger, “Experimental tests of realistic local theories via Bell’s theorem”, *Phys. Rev. Lett.* **47**, 460-463 (1981); “Experimental realization of Einstein-Podolsky-Bohm gedankenexperiment : a new violation of Bell’s inequalities”, **49**, 91-94 (1982).
- [60] W. Perrie, A.J. Duncan, H.J. Beyer and H. Kleinpoppen, “Polarization correlations of the two photons emitted by metastable atomic deuterium: a test of Bell’s inequality”, *Phys. Rev. Lett.* **54**, 1790-1793 (1985).
- [61] T.E. Kiess, Y.E. Shih, A.V. Sergienko, and C.O. Alley, “Einstein-Podolsky-Rosen-Bohm experiments using light quanta produced by type-II parametric conversion”, *Phys. Rev. Lett.* **71**, 3893-3897 (1993).
- [62] J.S. Bell, contribution to “Foundations of quantum mechanics”, *Proceedings of the International School of Physics ”Enrico Fermi”*, course II, page 171, Academic, New York (1971).
- [63] Jarrett, ”On the Physical Significance of the Locality conditions in the Bell Arguments”, *Nóus* **18**, 569-589 (1984).

- [64] L.E. Ballentine and J.P. Jarrett, “Bell’s theorem: does quantum mechanics contradict relativity?”, *Am. J. Phys.* **55**, 696-701 (1987).
- [65] A. Shimony, “Events and processes in the quantum world”, pages 182-203 in “Quantum concepts in space and time”, edited by R. Penrose and C.J. Isham, Oxford University Press (1986).
- [66] J.D. Franson, “Bell inequality for position and time”, *Phys. Rev. Lett.* **62**, 2205-2208 (1989).
- [67] J. Brendel, E. Mohler and W. Martienssen, “Experimental test of Bell’s inequality for energy and time”, *Eur. Phys. Lett.* **20**, 575-580 (1993).
- [68] N. Gisin, “Bell’s inequality holds for all non-product states”, *Phys. Lett.* **A154**, 201-202 (1991).
- [69] S. Popescu and D. Rohrlich, “Generic quantum non locality”, *Phys. Lett.* **A166**, 293-297 (1992).
- [70] L. Hardy, “Quantum mechanics, local realistic theories, and Lorentz invariant realistic theories”, *Phys. Rev. Lett.* **68**, 2981-2984 (1992).
- [71] N.D. Mermin, “What’s wrong with this temptation ?”, *Physics Today* **47**, June 1994, page 9-11; “Quantum mysteries refined”, *Am. J. Phys.* **62**, 880-887 (1994).
- [72] D. Boschi, S. Branca, F. De Martini, L. Hardy, “Ladder proof of non-locality without inequalities: theoretical and experimental results”, *Phys. Rev. Lett.* **79**, 2755-2758 (1997).
- [73] S. Goldstein, “Nonlocality without inequalities for almost all entangled states for two particles”, *Phys. Rev. Lett.* **72**, 1951-1954 (1994).
- [74] D.M. Greenberger, M.A. Horne and A. Zeilinger, “Bell’s theorem, Quantum Theory, and Conceptions of the Universe”, M. Kafatos ed.; Kluwer, p. 69-72, 1989; this reference is not always easy to find, but one can also read the following article, published one year later.
- [75] D.M. Greenberger, M.A. Horne, A. Shimony, A. Zeilinger, “Bell’s theorem without inequalities”, *Am. J. Phys.* **58**, 1131-1143 (1990).
- [76] F. Lalöe, “Correlating more than two particles in quantum mechanics”, *Current Science* **68** 1026-1035 (1995).
- [77] N.D. Mermin, “Quantum mysteries revisited”, *Am. J. Phys.* **58**, 731-733 (1990).

- [78] R. Laflamme, E. Knill, W.H. Zurek, P. Catasi and S.V.S Mariappan, “NMR GHZ”, quant-physics/9709025 and Phil. Trans. Roy. Soc. Lond. **A356**, 1941-1948 (1998)
- [79] Seth Lloyd, “Microscopic analogs of the Greenberger-Horne-Zeilinger experiment”, Phys. Rev. **A57**, R1473-1476 (1998).
- [80] U. Sakaguchi, H. Ozawa, C. Amano and T. Fokumi, “Microscopic analogs of the Greenberger-Horne-Zeilinger experiment on an NMR quantum computer”, Phys. Rev. **60**, 1906-1911 (1999).
- [81] Jian-Wei Pan, Dik Bouwmeester, H. Weinfurter and A. Zeilinger, “Experimental entanglement swapping : entangling photons that never interacted”, Phys. Rev. Lett. **80**, 3891-3894 (1998).
- [82] Jian-Wei Pan, Dik Bouwmeester, H. Weinfurter and A. Zeilinger and M. Daniell, “Observation of three-photon Greenberger-Horne-Zeilinger entanglement”, Phys. Rev. Lett. **82**, 1345-1349 (1999).
- [83] N.D. Mermin, “Extreme quantum entanglement in a superposition of macroscopically distinct states”, Phys. Rev. Lett. **65**, 1838-1841 (1990).
- [84] S. Kochen and E.P. Specker, “The problem of hidden variables in quantum mechanics”, J. Math. Mech. **17**, 59-87 (1967).
- [85] F. Belinfante, “Survey of hidden variables theories”, Pergamon Press (1973).
- [86] A. Peres, “Incompatible results of quantum measurements”, Phys. Lett. **151**, 107-108 (1990).
- [87] J.S. Bell, oral presentation to the EGAS conference in Paris, July 1979 (published in an abridged version in the next reference).
- [88] J.S. Bell, “Atomic cascade photons and quantum mechanical non-locality”, Comments on atomic and molecular physics **9**, 121 (1980); CERN preprint TH.2053 and TH 2252.
- [89] P.G. Kwiat, P.H. Eberhard, A. M. Steinberg and R. Chiao, “Proposal for a loophole-free Bell inequality experiment”, Phys. Rev. **A49**, 3209-3220 (1994).
- [90] E.S. Fry, T. Walther and S. Li, “Proposal for a loophole-free test of the Bell inequalities”, Phys. Rev. **A52**, 4381-4395 (1995).
- [91] A. Aspect, J. Dalibard and G. Roger, “Experimental tests of Bell’s inequalities using time varying analyzers”, Phys. Rev. Lett. **49**, 1804-1807 (1982).

- [92] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter and A. Zeilinger, “Violation of Bell’s inequality under strict Einstein locality conditions”, *Phys. Rev. Lett.* **81**, 5039-5043 (1998).
- [93] J.S. Bell, “Free variables and local causality”, 100-104 in [7].
- [94] H.P. Stapp, “Whiteheadian approach to quantum theory and the generalized Bell’s theorem”, *Foundations of Physics* **9**, 1-25 (1979); “Bell’s theorem and the foundations of quantum physics”, *Am. J. Phys.* **53**, 306-317 (1985).
- [95] H.P. Stapp, “Nonlocal character of quantum theory”, *Am. J. Phys.* **65**, 300-304 (1997); “Meaning of counterfactual statements in quantum physics”, *Am. J. Phys.* **66**, 924-926 (1998).
- [96] D. Mermin, “Nonlocal character of quantum theory?”, *Am. J. Phys.* **66**, 920-924 (1998).
- [97] M. Redhead, “Incompleteness, Nonlocality and Realism”, chap. 4, Clarendon Press (1988).
- [98] J.F. Clauser and A. Shimony, “Bell’s theorem : experimental tests and implications”, *Rep. Progr. Phys.* **41**, 1881-1927 (1978).
- [99] B. d’Espagnat, “Nonseparability and the tentative descriptions of reality”, *Phys. Rep.* **110**, 201-264 (1984).
- [100] B. d’Espagnat, “Reality and the physicist”, Cambridge University Press (1989).
- [101] R.B. Griffiths, “Consistent quantum counterfactuals”, *Phys. Rev. A* **60**, R5-R8 (1999).
- [102] J.J Bollinger, W.M. Itano, D.J. Wineland, “Optimal frequency measurements with maximally correlated states”, *Phys. Rev. A* **54**, R4649-4652 (1996).
- [103] E. Hagley, X. Maître, G. Nogues, C. Wunderlich, M. Brune, J.M. Raimond and S. Haroche, “Generation of EPR pairs of atoms”, *Phys. Rev. Lett.* **79**, 1-5 (1997).
- [104] K. Molmer and A. Sorensen, “Multiple particle entanglement of hot trapped ions”, *Phys. Rev. Lett.* **82**, 1835-1838 (1999).
- [105] C.A. Sackett, D. Klepinski, B.E King, C. Langer, V. Meyer, C.J. Myatt, M. Rowe, O.A. Turchette, W.M. Itano, D.J Wineland and C. Monroe, “Experimental entanglement of four particles”, *Nature* **404**, 256-259 (2000).

- [106] A. Ekert, “Quantum cryptography based on Bell’s theorem”, *Phys. Rev.* **67**, 661-663 (1991).
- [107] C. H. Bennett, G. Brassard, N.D. Mermin, “Quantum cryptography without Bell’s theorem”, *Phys. Rev. Lett.* **68**, 557-559 (1992)
- [108] W.K. Wootters and W.H. Zurek, “A single quantum cannot be cloned”, *Nature* **299**, 802-803 (1982).
- [109] D. Dieks, “Communication by EPR devices”, *Phys. Lett.* **A92**, 271-272 (1982).
- [110] N. Gisin and S. Massar, “Optimal quantum cloning machines”, *Phys. Rev. Lett.* **79**, 2153-2156 (1997).
- [111] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.L. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels”, *Phys. Rev. Lett.* **70**, 1895-1898 (1993).
- [112] D. Bouwmeester, J.W. Pan, K. Mattle, M. Eibl, H. Weinfurter and A. Zeilinger, “Experimental quantum teleportation”, *Nature* **390**, 575-579 (1997).
- [113] S. Massar and S. Popescu, “Optimal extraction of information from finite quantum ensembles”, *Phys. Rev. Lett.* **74**, 1259-1263 (1995).
- [114] S. Popescu, “Bell’s inequalities versus teleportation : what is nonlocality ?”, *Phys. Rev. Lett.* **72**, 797-800 (1994); quant phys.9501020.
- [115] T. Sudbury, “The fastest way from A to B”, *Nature* **390**, 551-552 (1997).
- [116] G.P. Collins, “Quantum teleportation channels opened in Rome and Innsbruck”, *Physics Today* **51**, 18-21 (February 1998).
- [117] C.H. Bennett, “Quantum information and computation”, *Phys. Today* **48**, 24-30 (October 1995).
- [118] D.P. DiVincenzo, “Quantum computation”, *Science* **270**, 255-261 (October 1995).
- [119] C.H. Bennett and D.P. DiVincenzo, “Quantum information and computation”, *Science* **404**, 247-255 (2000).
- [120] D. Deutsch, “Quantum theory, the Church-Turing principle and the universal quantum computer”, *Proc . Roy. Soc.* **A400**, 97-117 (1985).

- [121] P. Shor, Proceedings of the 55th annual symposium on the foundations of Computer Science, IEEE Computer Society Press, p. 124-133, Los Alamitos, California (1994).
- [122] D.S. Abrams and S. Lloyd, "Simulation of many-body Fermi systems on a universal quantum computer", Phys. Rev. Lett. **79**, 2586-2589 (1997).
- [123] J.I. Cirac and P. Zoller, "Quantum computation with cold trapped ions", Phys. Rev. Lett. **74**, 4091-4094 (1995).
- [124] P. Shor, "Scheme for reducing decoherence in quantum computer memory", Phys. Rev. **A52**, 2493-2496 (1995).
- [125] A.M. Steane, "Error correcting codes in quantum theory", Phys. Rev. Lett. **77**, 793-796 (1996).
- [126] J. Preskill, "Battling decoherence: the fault-tolerant quantum computer", Phys. Today **52**, 24-30 (June 1999).
- [127] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin and W.K. Woiters, "Purification of noisy entanglement and faithful teleportation via noisy channels", Phys. Rev. Lett. **76**, 722-725 (1996).
- [128] C.H. Bennett, D.P. DiVicenzo, J.A. Smolin and W.A. Woiters, "Mixed-state entanglement and quantum error correction", Phys. Rev. **A54**, 3824-3851 (1996).
- [129] H.J. Briegel, W. Dür, J.I. Cirac and P. Zoller, "Quantum repeaters: the role of imperfect local operations in quantum communication", Phys. Rev. Lett. **81**, 5932-5935 (1998).
- [130] R.B. Griffiths and Chi-Sheng Niu, "Semiclassical Fourier transform for quantum computation", Phys. Rev. Lett. **76**, 3228-3231 (1996).
- [131] S. Haroche and J.M. Raimond, "Quantum computing: dream or nightmare?", Phys. Today **49**, 51-52 (August 1996).
- [132] F. London and E. Bauer, "La théorie de l'observation en mécanique quantique", no 775 of Actualités scientifiques et industrielles, exposés de physique générale; Hermann, Paris (1939); translated into English as "The theory of observation in quantum mechanics" in "Quantum theory of measurement", J.A. Wheeler and W.H. Zurek eds, Princeton University Press, pages 217-259 (1983); see in particular §11, but also 13 and 14.

- [133] E.P. Wigner, “Interpretation of quantum mechanics”, lectures given in 1976 at Princeton University; later published in “Quantum theory of measurement”, J.A. Wheeler and W.H. Zurek eds. , Princeton University Press (1983), 260-314; see also Wigner’s contribution in “Foundations of quantum mechanics”, Proc. Enrico Fermi Int. Summer School, B. d’Espagnat editor, Academic Press (1971).
- [134] N.D. Mermin, “What is quantum mechanics trying to tell us?”, Am. J. Phys. **66**, 753-767 (1998).
- [135] B. Misra and E.C.G. Sudarshan, “The Zeno’s paradox in quantum theory”, J. Math. Phys. (NY) **18**, 756-763 (1977).
- [136] L. de Broglie, “Tentative d’interprétation causale et non-linéaire de la mécanique ondulatoire”, Gauthier-Villars, Paris (1956).
- [137] D. Bohm, “Proof that probability density approaches $|\Psi|^2$ in causal interpretation of quantum theory”, Phys. Rev. **108**, 1070-1076 (1957).
- [138] L.E. Ballentine, “The statistical interpretation of quantum mechanics”, Rev. Mod Phys. **42**, 358-381 (1970).
- [139] E. Madelung; “Quantentheorie in hydrodynamische Form”, Z. Phys. **40**, 322-326 (1927).
- [140] E. Nelson, “Derivation of the Schrödinger equation from Newtonian mechanics”, Phys. Rev. **150**, 1079-1085 (1966).
- [141] R. Werner, “A generalization of stochastic mechanics and its relation to quantum mechanics”, Phys. Rev. **D34**, 463-469 (1986).
- [142] C. Philippidis, C. Dewdney and B.J. Hiley, “Quantum interference and the quantum potential”, Nuov. Cim. **52B**, 15-23 (1979).
- [143] J.S. Bell, “De Broglie-Bohm, delayed choice, double slit experiment, and density matrix”, International Journal of Quantum Chemistry, symposium **14**, 155-159 (1980).
- [144] B.G. Englert, M.O. Scully, G. Süssmann and H. Walther, “Surrealistic Bohm trajectories”, Z. Naturforschung **47a**, 1175-1186 (1992).
- [145] J.S. Bell, §4 of [34] or page 128 of [7].
- [146] R.B. Griffiths, “Bohmian mechanics and consistent histories”, Phys. Lett. **A 261**, 227-234 (1999).
- [147] N. Gisin, “Quantum measurements and stochastic processes”, Phys. Rev. Lett. **52**, 1657- 1660 (1984); Helv. Phys. Acta, “Stochastic quantum dynamics and relativity”, **62**, 363-371 (1989).

- [148] L. Diosi, “Quantum stochastic processes as models for state vector reduction” *J. of Physics* **A21**, 2885-2898 (1988).
- [149] P. Pearle, “On the time it takes a state vector to reduce”, *J. Stat. Phys.* **41**, 719-727 (1985).
- [150] A. Barchielli, L. Lanz and G.M. Prosperi, “A model for the macroscopic description and continual observations in quantum mechanics”, *Nuov. Cim.* **42 B**, 79-121 (1982).
- [151] A. Barchielli, “Continual measurements for quantum open systems”, *Nuov. Cim.* **74 B**, 113-138(1983); “measurement theory and stochastic differential equations in quantum mechanics”, *Phys. Rev.* **A 34**, 1642-1648 (1986).
- [152] F. Benatti, G.C. Ghirardi, A. Rimini and T. Weber, “Quantum mechanics with spontaneous localization and the quantum theory of measurement”, *Nuov. Cim.* **100 B**, 27-41 (1987).
- [153] P. Pearle, “Combining stochastic dynamical state-vector reduction with spontaneous localization”, *Phys. rev.* **A 39**, 2277-2289 (1989).
- [154] G.C. Ghirardi, P. Pearle and A. Rimini, “Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles”, *Phys. Rev.* **A 42**, 78-89 (1990).
- [155] L. Diosi, “Continuous quantum measurement and Ito formalism”, *Phys. Lett.* **129 A**, 419-423 (1988).
- [156] L. Diosi “Models for universal reduction of macroscopic quantum fluctuations”, *Phys. Rev.* **A 40**, 1165-1174 (1989).
- [157] G. Ghirardi, R. Grassi and A. Rimini, “Continuous-spontaneous-reduction model involving gravity”, *Phys. Rev.* **A 42**, 1057-1064 (1990).
- [158] “Experimental Metaphysics, Quantum Mechanical Studies for Abner Shimony”, *Festschrift volumes 1 and 2*, Robert S. Cohen, Michael Horne, John J. Stachel eds, *Boston studies in the philosophy of science*, vol. 193 et 194, Kluwer academic publishers (1997).
- [159] F. Benatti, G.C. Ghirardi, A. Rimini and T. Weber, “Operations involving momentum variables in non-hamiltonian evolution equations”, *Nuov. Cim.* **101 B**, 333-355 (1988).
- [160] S. Weinberg, “Precision tests of quantum mechanics”, *PRL* **62**, 485-488 (1989).

- [161] K. Wodkiewicz and M.O. Scully, “Weinberg’s nonlinear wave mechanics”, *Phys. Rev. A* **42**, 5111-5116 (1990).
- [162] N. Gisin, “Weinberg’s non-linear quantum mechanics and supraluminal communications”, *Phys. Lett.* **143**, 1-2 (1990).
- [163] G.C. Ghirardi, “Quantum superpositions and definite perceptions: envisaging new feasible tests”, *Phys. Lett. A* **262**, 1-14 (1999).
- [164] N. Gisin and I. Percival, “The quantum-state diffusion model applied to open systems”, *J. Phys. A* **25**, 5677-5691 (1992); “Quantum state diffusion, localization and quantum dispersion entropy”, **26**, 2233-2243 (1993); “The quantum state diffusion picture of physical processes”, **26**, 2245-2260 (1993).
- [165] I. Percival, “Quantum State Diffusion”, Cambridge University Press (1998).
- [166] M.B. Plenio and P. Knight, “The quantum-jump approach to dissipative dynamics in quantum optics”, *Rev. Mod. Phys.* **70**, 101-141 (1998).
- [167] J.G. Cramer, “The transactional interpretation of quantum mechanics”, *Rev. Mod. Phys.* **58**, 647-687 (1986); in an appendix, this article contains a review of the various interpretations of quantum mechanics.
- [168] R. Omnès, “Logical reformulation of quantum mechanics”, *Journ. Stat. Phys.* **53**, “I: Foundations”, 893-932; “II: Interferences and the EPR experiments”, 933-955; “III: Classical limit and irreversibility”, 957-975 (1988).
- [169] M. Gell-Mann and J. Hartle, “Classical equations for quantum systems”, *Phys. Rev. D***47**, 3345-3382 (1993).
- [170] R. Omnès “The Interpretation of Quantum Mechanics”, Princeton University Press (1994); “Understanding quantum mechanics”, Princeton University Press (1999).
- [171] R.B. Griffiths and R. Omnès, “Consistent histories and quantum measurements”, *Physics Today* **52**, 26-31 (August 1999).
- [172] Y. Aharonov, P.G. Bergmann and J. L. Lebowitz, “Time symmetry in the quantum process of measurement”, *Phys. Rev. B* **134**, 1410-1416 (1964).
- [173] R.B. Griffiths, “Consistent histories and quantum reasoning”, *Phys. Rev. A* **54**, 2759-2774 (1996).

- [174] R.B. Griffiths, “Choice of consistent family, and quantum incompatibility”, *Phys. Rev. A* **57**, 1604-1618 (1998).
- [175] *Physics Today*, “Observant readers take the measure of novel approaches to quantum theory ; some get Bohmed”, **52**, 11-15 and 89-92 (February 1999).
- [176] W.H. Zurek, “Preferred states, predictability, classicality and the environment-induced decoherence”, *Progr. Theor. Phys.* **89**, 281-312 (1993).
- [177] R.B. Griffiths, “Correlations in separated quantum systems: a consistent history analysis of the EPR problem”, *Am. J. Phys.* **55**, 11-17 (1987).
- [178] F. Dowker and A. Kent, “Properties of consistent histories”, *Phys. Rev. Lett.* **75**, 3038-3041 (1995); “On the consistent histories approach to quantum mechanics”, *J. Stat. Phys.* **82**, 1575-1646 (1996).
- [179] A. Kent, “Quasiclassical dynamics in a closed quantum system”, *Phys. Rev. A* **54**, 4670-4675 (1996).
- [180] H. Everett III, “Relative state formulation of quantum mechanics”, *Rev. Mod. Phys.* **29**, 454-462 (1957); reprinted in “Quantum theory and measurement”, J.A. Wheeler and W.H. Zurek eds., 315-323, Princeton University Press (1983).
- [181] H.D. Zeh, “On the interpretation of measurement in quantum theory”, *Found. Physics I*, 69-76 (1970).
- [182] H. Hertz, *Miscellaneous papers*, translated from first German edition (1895) by D.E. Jones and J.A. Schott (London: MacMillan, 1896), vol. **1**, p. 318.

Partial separability and entanglement criteria for multiqubit quantum states

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We explore the subtle relationships between partial separability and entanglement of subsystems in multiqubit quantum states and give experimentally accessible conditions that distinguish between various classes and levels of partial separability in a hierarchical order. These conditions take the form of bounds on the correlations of locally orthogonal observables. Violations of such inequalities give strong sufficient criteria for various forms of partial inseparability and multiqubit entanglement. The strength of these criteria is illustrated by showing that they are stronger than several other well-known entanglement criteria (the fidelity criterion, violation of Mermin-type separability inequalities, the Laskowski-Żukowski criterion, and the Dür-Cirac criterion) and also by showing their great noise robustness for a variety of multiqubit states, including N -qubit Greenberger-Horne-Zeilinger states and Dicke states. Furthermore, for $N \geq 3$ they can detect bound entangled states. For all these states, the required number of measurement settings for implementation of the entanglement criteria is shown to be only $N+1$. If one chooses the familiar Pauli matrices as single-qubit observables, the inequalities take the form of bounds on the antidiagonal matrix elements of a state in terms of its diagonal matrix elements.

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I. INTRODUCTION

The problem of characterizing entanglement for multipartite quantum systems has recently drawn much attention. An important issue in this problem is that, apart from the extreme cases of full separability and full entanglement of all particles in the system, one also has to face the intermediate cases in which only some particles in the system are entangled and others not. The latter states are usually called “partially separable” or, more precisely, “ k -separable” when they take the form of a mixture of states that factorize when the N -partite system is partitioned into k subsystems ($k \leq N$) [1–4]. In this paper we will focus on multiqubit systems only. We propose a classification of partially separable states for such systems, slightly extending the classification introduced by Dür and Cirac [2]. This classification consists of a hierarchy of levels corresponding to the k -separable states for $k=1, \dots, N$, and within each level various classes are distinguished by specifying under which partitions of the system the state is separable or not.

Several experimentally accessible conditions to characterize k -separable multiqubit states have already been proposed, e.g., by Laskowski and Żukowski [5], Mermin-type separability inequalities [1,6–10], or in terms of entanglement witnesses [11]. However, these conditions do not distinguish the various classes within the levels. Separability conditions that do distinguish some of these classes in the hierarchy were developed by Dür and Cirac. Here we present separability conditions that extend and strengthen all the conditions just mentioned.

These conditions take the form of sets of inequalities that bound the correlations for standard Bell-type experiments (involving at each site measurement of two orthogonal spin

observables). They form a hierarchy with bounds that decrease by a factor of 4 for each level k in the partial separability hierarchy. For the classes within a given level, the inequalities give state-dependent bounds, differing for each class. Violations of the inequalities provide strong sufficient criteria for various forms of inseparability and multiqubit entanglement.

We demonstrate the strength of these conditions in two ways: First, by showing that they imply several other general separability conditions, namely the fidelity criterion [12–14], the partial separability conditions just mentioned, i.e., the Laskowski-Żukowski condition (with a strict improvement for $k=2, N$), the Dür-Cirac condition, and the Mermin-type separability inequalities. We also show that the latter are equivalent to the Laskowski-Żukowski condition.

Second, we compare the conditions to other state-specific multiqubit entanglement criteria [11,15,16] both for their white noise robustness and for the number of measurement settings required in their implementation. In particular, we show (i) detection of bound entanglement for $N \geq 3$ with noise robustness for detecting the bound entangled states of Ref. [3] that goes to 1 for large N (i.e., maximal noise robustness), (ii) detection of the four qubit Dicke state with noise robustness 0.84 and 0.36 for detecting it as entangled and fully entangled, respectively, (iii) great noise and decoherence robustness [17,18] in detecting entanglement of the N -qubit Greenberger-Horne-Zeilinger (GHZ) state where for colored noise and for decoherence due to dephasing the robustness for detecting full entanglement goes to 1 for large N , and lastly, (iv) better white noise robustness than the stabilizer witness criteria of Ref. [11] for detecting the N -qubit GHZ states. In all these cases it is shown that only $N+1$ settings are needed.

Choosing the familiar Pauli matrices as the local orthogonal observables yields a convenient matrix element representation of the partial separability conditions. In this representation, the inequalities give specific bounds on the antidiagonal matrix elements in terms of the diagonal ones.

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Further, some comments will be made along the way on how these results relate to the original purpose [19] of Bell-type inequalities to test local hidden-variable (LHV) models against quantum mechanics. Most notably, when the number of parties is increased, there is not only an exponentially increasing factor that separates the correlations allowed in maximally entangled states in comparison to those of local hidden-variable theories, but, surprisingly, also an exponentially increasing factor between the correlations allowed by LHV models and those allowed by nonentangled qubit states.

This paper is structured as follows. In Sec. II we define the relevant partial separability notions and extend the hierarchic partial separability classification of Ref. [2]. There we also introduce the notions of k -separable entanglement and of m -partite entanglement in order to investigate the relation between partial separability and multipartite entanglement. We then discuss four known partial separability conditions discussed above. In Sec. III we derive partial separability conditions for N qubits in terms of locally orthogonal observables. They provide the desired necessary conditions for the full hierarchic separability classification. In Sec. IV the experimental strength of these criteria is discussed. We end in Sec. V with a discussion of the results obtained.

II. PARTIAL SEPARABILITY AND MULTIPARTITE ENTANGLEMENT

In this section we introduce terminology and definitions to be used in later sections. We define the notions of k -separability, α_k -separability, k -separable entanglement, and m -partite entanglement and use these notions to capture aspects of the separability and entanglement structure in multipartite states. We review the separability hierarchy introduced by Dür and Cirac [2] and extend their classification. We also discuss four partial separability conditions known in the literature. These conditions will be strengthened in Sec. III.

A. Partial separability and the separability hierarchy

Consider an N -qubit system with Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$. Let $\alpha_k = (S_1, \dots, S_k)$ denote a partition of $\{1, \dots, N\}$ into k disjoint nonempty subsets ($k \leq N$). Such a partition corresponds to a division of the system into k distinct subsystems, also called a k -partite split [2]. A quantum state ρ of this N -qubit system is k -separable under a specific k -partite split α_k [1–4] if and only if it is fully separable in terms of the k subsystems in this split, i.e., if and only if

$$\rho = \sum_i p_i \otimes_{n=1}^k \rho_i^{S_n}, \quad p_i \geq 0, \quad \sum_i p_i = 1, \quad (1)$$

where ρ^{S_n} is a state of subsystem corresponding to S_n in the split α_k . We denote such states as $\rho \in \mathcal{D}_N^{\alpha_k}$ and also call them α_k -separable, for short. Clearly, $\mathcal{D}_N^{\alpha_k}$ is a convex set. A state of the N -qubit system outside this set is called α_k -inseparable.

More generally, a state ρ is called k -separable [5,20–23] (denoted as $\rho \in \mathcal{D}_N^{k\text{-sep}}$) if and only if there exists a convex decomposition

$$\rho = \sum_j p_j \otimes_{n=1}^k \rho_n^{S_n^{(j)}}, \quad p_j \geq 0, \quad \sum_j p_j = 1, \quad (2)$$

where each state $\otimes_{n=1}^k \rho_n^{S_n^{(j)}}$ is a tensor product of k density matrices of the subsystems corresponding to some such partition $\alpha_k^{(j)}$, i.e., it factorizes under this split $\alpha_k^{(j)}$. In this definition, the partition may vary for each j , as long as it is a k -partite split, i.e., contains k disjoint nonempty sets. Clearly $\mathcal{D}_N^{k\text{-sep}}$ is also convex; it is the convex hull of the union of all $\mathcal{D}_N^{\alpha_k}$ for fixed values of k and N . States that are not k -separable will be called k -inseparable. Note that a k -separable state need not be α_k -separable for any particular split α_k [24]; and even the converse implication need not hold: If a state is biseparable under every bipartition, it does not have to be fully separable, as shown by the three-partite examples in Ref. [25]. Similar observations (using different terminology) were presented in Refs. [20,21], but below we will present a more systematic investigation.

The notion of k -separability naturally induces a hierarchic ordering of the N -qubit states. Indeed, the sequence of sets $\mathcal{D}_N^{k\text{-sep}}$ is nested: $\mathcal{D}_N^{k\text{-sep}} \subset \mathcal{D}_N^{(k-1)\text{-sep}} \subset \dots \subset \mathcal{D}_N^{1\text{-sep}}$. In other words, k -separability implies ℓ -separability for all $\ell \leq k$. We call a k -separable state that is not $(k+1)$ -separable “ k -separable entangled.” Thus each N -qubit state can be characterized by the level k for which it is k -separable entangled, and these levels provide a hierarchical ranking: at one extreme end are the 1-separable entangled states which are fully entangled (e.g., the GHZ states), at the other end are the N -separable or fully separable states (e.g., product states or the “white noise state” $1/2^N$).

Often, it is interesting to know how many qubits are entangled in a k -separable entangled state. However, this question does not have a unique answer. For example, take $N=4$ and $k=2$ (biseparability). In this case two types of states may occur in the decomposition (2), namely $\rho^{\{ij\}} \otimes \rho^{\{kl\}}$ and $\rho^{\{i\}} \otimes \rho^{\{jkl\}}$ ($i, j, k, l=1, 2, 3, 4$). A 2-separable entangled four-partite state might thus be two- or three-partite entangled.

In general, an N -qubit state ρ will be called “ m -partite entangled” if and only if a decomposition of the state such as in Eq. (2) exists such that each subset $S^{(i)}$ contains at most m parties, but no such decomposition is possible when all the k subsets are required to contain less than m parties [13]. [In Refs. [20,21] this is called “not producible by $(m-1)$ -partite entanglement.”] It follows that a k -separable entangled state is also m -partite entangled, with $[N/k] \leq m \leq N-k+1$. Here $[N/k]$ denotes the smallest integer which is not less than N/k . Thus a state that is k -separably entangled ($k < N$) is at least $[N/k]$ -partite entangled and might be up to $(N-k+1)$ -partite entangled. Therefore conditions that distinguish k -separability from $(k+1)$ -separability also provide conditions for m -partite entanglement, but generally allowing a wide range of values of m . For example, for $N=100$ and $k=2$, m might lie anywhere between 50 and 99.

Of course, a much tighter conclusion about m -partite entanglement can be drawn if we know exactly under which splits the state is separable. This is why the notion of α_k -separability is helpful, since it provides these finer distinctions. For example, suppose that a 100-qubit state is separable under the bipartite split ($\{1\}, \{2, \dots, 100\}$) but under

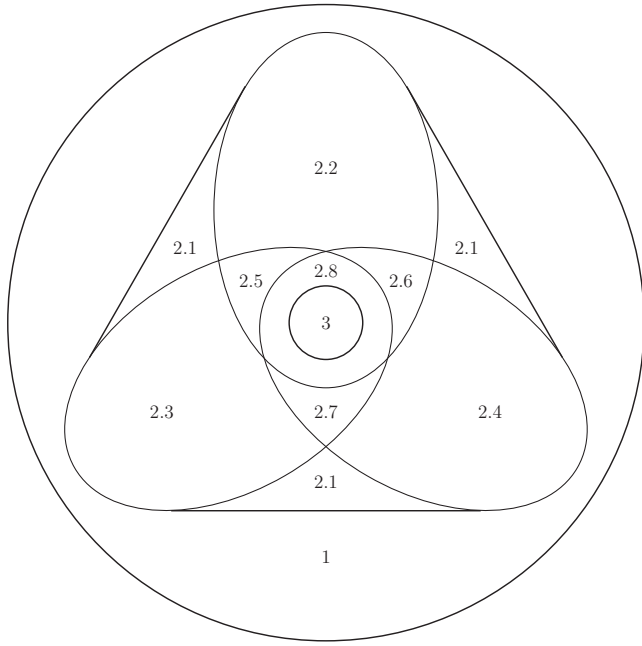


FIG. 1. Schematic representation of the ten partial separability classes of three-qubit states.

no other bipartite split. This state would then be 2-separable (biseparable) but now we could also infer that $m=99$. On the other hand, if the state were only separable under the split $(\{1, \dots, 50\}, \{51, \dots, 100\})$, it would still be biseparable, but only m -partite entangled for $m=50$.

Dür and Cirac [2] provided such a fine-grained classification of N -qubit states by considering their separability or inseparability under all k -partite splits. Let us introduce this classification (with a slight extension) by means of the example of three qubits, labeled as a, b, c .

Class 3. Starting with the lowest level $k=3$, there is only one 3-partite split, $a-b-c$, and consequently only one class to be distinguished at this level, i.e., \mathcal{D}_3^{a-b-c} . This set coincides with $\mathcal{D}_3^{3\text{-sep}}$.

Classes 2.1–2.8. Next, at level $k=2$, there are three bipartite splits: $a-(bc)$, $b-(ac)$, and $c-(ab)$ which define the sets $\mathcal{D}_3^{a-(bc)}$, $\mathcal{D}_3^{b-(ac)}$, and $\mathcal{D}_3^{c-(ab)}$. One can further distinguish classes defined by all logical combinations of separability and inseparability under these splits, i.e., all the set-theoretical intersections and complements shown in Fig. 1. This leads to classes 2.2–2.8. Dür and Cirac showed that all these classes are nonempty. To these, we add one more class 2.1: the set of biseparable states that are not separable under any split. As we have seen, this set is nonempty too.

Class 1. Finally, at level $k=1$ there is again only one (trivial) split (abc) , and thus only one class, consisting of all the fully entangled states, i.e., $\mathcal{D}_3^{1\text{-sep}} \setminus \mathcal{D}_3^{2\text{-sep}}$.

We feel that the above extension is desirable since otherwise the Dür-Cirac classification would not distinguish between class 2.1 and class 1. However, states in class 2.1 are simply convex combinations of states that are biseparable under different bipartite splits. Such states can be realized by mixing the biseparable states and are conceptually different from the fully inseparable states of class 1.

This three-partite example serves to illustrate how the Dür-Cirac separability classification works for general N .

Level k ($1 \leq k \leq N$) of the separability hierarchy consists of all k -separable entangled states. Each level is further divided into distinct classes by considering all logically possible combinations of separability and inseparability under the various k -partite splits. The number of such classes increases rapidly with N , and therefore we will not attempt to list them. In general, all such classes may be nonempty. As an extension of the Dür-Cirac classification, we distinguish at each level $1 < k < N$ one further class, consisting of k -separable entangled states that are not separable under any k -partite split.

In order to find relations between these classes, the notion of a *contained split* is useful [2]. A k -partite split α_k is contained in a l -partite split α_l , denoted as $\alpha_k < \alpha_l$, if α_l can be obtained from α_k by joining some of the subsets of α_k . The relation $<$ defines a partial order between splits at different levels. This partial order is helpful because α_k -separability implies α_l -separability of all splits α_l containing α_k . We will use this implication below to obtain conditions for separability of a k -partite split at level k from such conditions on all $(k-1)$ -partite splits at level $k-1$ this k -partite split is contained in. Conditions at a lower level thus imply conditions at a higher level.

The multipartite entanglement properties of k -separable or α_k -separable states are subtle, as can be seen from the following examples.

(i) Mixing states does not conserve m -partite entanglement. Take $N=3$, then mixing the 2-partite entangled 2-separable states $|0\rangle \otimes (|00\rangle + |11\rangle) / \sqrt{2}$ and $|0\rangle \otimes (|00\rangle - |11\rangle) / \sqrt{2}$ with equal weights gives a 3-separable state $(|000\rangle\langle 000| + |011\rangle\langle 011|) / 2$.

(ii) An N -partite state can be m -partite entangled ($m < N$) even if it has no m -partite subsystem whose (reduced) state is m -partite entangled [13,20]. Such states are said to have irreducible m -partite entanglement [26]. Thus a state of which some reduced state is m -partite entangled is itself at least m -partite entangled, but the converse need not be true.

(iii) Consider a biseparable entangled state that is only separable under the bipartite split $(\{1\}, \{2, \dots, N\})$. One cannot infer that the subsystem $\{2, \dots, N\}$ is $(N-1)$ -partite entangled. A counterexample is the three-qubit state $\rho = (|0\rangle\langle 0| \otimes P_-^{(bc)} + |1\rangle\langle 1| \otimes P_+^{(bc)}) / 2$ which is biseparable only under the partition $a-(bc)$, and thus bipartite entangled, but has no bipartite subsystem whose reduced state is entangled. Here $P_+^{(bc)}$ and $P_-^{(bc)}$ denote projectors on the Bell states $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ for parties b and c , respectively.

(iv) A state that is inseparable under all splits but which is not fully inseparable (i.e., $\rho \in \mathcal{D}_N^{k\text{-sep}}$ with $k > 1$ and $\rho \notin \cup_{\alpha_k} \mathcal{D}_N^{\alpha_k}$, $\forall \alpha_k, k$) might still have all forms of m -partite entanglement apart from full entanglement, i.e., it could be m -partite entangled with $2 \leq m \leq N-1$. Thus the state could even have m -partite entanglement as low as 2-partite entanglement, although it is inseparable under all splits. For example, Tóth and Gühne [21] consider a mixture of two N -partite states where each of them is $(\lceil N/2 \rceil)$ -separable according to different splits. This mixed state is by construction $(\lceil N/2 \rceil)$ -separable, not biseparable under any split, yet only 2-partite entangled. See also the example in [24] which is

$(N-1)$ -separable and only 2-partite entangled.

(v) Lastly, N -partite fully entangled states exist where no m -partite reduced state is entangled (such as N -qubit GHZ state) and also where all m -partite reduced states are entangled (such as the N -qubit W states) [27].

These examples serve to emphasize that one should be very cautious in inferring the existence of entanglement in subsystems of a larger system which is known to be m -partite entangled or k -separable entangled for some specific value of m and k .

B. Separability conditions

We now review four separability conditions for qubits, which will all be strengthened in the next section. These are necessary conditions for states to be k -separable, 2-separable, and α_k -separable, respectively.

(1) Laskowski and Żukowski [5] showed that for any k -separable N -qubit state ρ the antidiagonal matrix elements (denoted by $\rho_{j, \bar{j}}$, where $\bar{j}=d+1-j$, $d=2^N$) must satisfy

$$\max_j |\rho_{j, \bar{j}}| \leq \left(\frac{1}{2}\right)^k, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}. \quad (3)$$

This condition can be easily proven by the observation that for any density matrix to be physically meaningful its antidiagonal matrix elements must not exceed $1/2$. Therefore antidiagonal elements of a product of k density matrices cannot be greater than $(1/2)^k$. By convexity, this results then holds all k -separable states. Note that this condition is not basis dependent.

It follows from Eq. (3) that if the antidiagonal matrix elements of state ρ obey

$$\left(\frac{1}{2}\right)^k \geq \max_j |\rho_{j, \bar{j}}| > \left(\frac{1}{2}\right)^{k+1}, \quad (4)$$

then ρ is at most k -separable, i.e., k -separable entangled, and thus at least m -partite entangled, with $m \geq \lceil N/k \rceil$.

The partial separability condition (3) does not yet explicitly refer to directly experimentally accessible quantities. However, in the next section we will rewrite this condition in terms of expectation values of local observables and show that they are equivalent to Mermin-type separability inequalities.

(2) Mermin-type separability inequalities [1,6,8–10]. Consider the familiar Clauser-Horne-Shimony-Holt operator for two qubits (labeled as a and b) which is defined by

$$M^{(2)} := X_a \otimes X_b + X_a \otimes Y_b + Y_a \otimes X_b - Y_a \otimes Y_b. \quad (5)$$

Here, X_a and Y_a denote two spin observables on the Hilbert spaces \mathcal{H}_a and \mathcal{H}_b of qubit a and b . The so-called Mermin operator [28] is a generalization of this operator to N qubits [labeled as (a, b, \dots, n)], defined by the recursive relation

$$M^{(N)} := \frac{1}{2} M^{(N-1)} \otimes (X_n + Y_n) + \frac{1}{2} M'^{(N-1)} \otimes (X_n - Y_n), \quad (6)$$

where M' is the same operator as M but with all X 's and Y 's interchanged.

In the special case where, for each qubit, the spin observables X and Y are orthogonal, i.e., $\{X_i, Y_i\}=0$ for $i \in \{a, \dots, n\}$, Nagata *et al.* [1] obtained the following k -separability conditions:

$$\langle M^{(N)} \rangle^2 + \langle M'^{(N)} \rangle^2 \leq 2^{(N+3)} \left(\frac{1}{4}\right)^k, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}. \quad (7)$$

As just mentioned, the next section will show that these inequalities are equivalent to the Laskowski-Żukowski inequalities. The quadratic inequalities (7) also imply the following sharp linear Mermin-type inequality for k -separability:

$$|\langle M^{(N)} \rangle| \leq 2^{(N+3)/2} \left(\frac{1}{2}\right)^k, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}. \quad (8)$$

For $k=N$ inequality (8) reproduces a result obtained by Roy [10].

(3) The fidelity $F(\rho)$ of a N -qubit state ρ with respect to the generalized N -qubit GHZ state $|\Psi_{\text{GHZ}, \alpha}^N\rangle := (|0\rangle^{\otimes N} + e^{i\alpha}|1\rangle^{\otimes N})/\sqrt{2}$ ($\alpha \in \mathbb{R}$) is defined as

$$F(\rho) := \max_{\alpha} \langle \Psi_{\text{GHZ}, \alpha}^N | \rho | \Psi_{\text{GHZ}, \alpha}^N \rangle = \frac{1}{2} (\rho_{1,1} + \rho_{d,d}) + |\rho_{1,d}|, \quad (9)$$

The fidelity condition [12–14] (also known as the projection-based witness [11]) says that for all biseparable ρ ,

$$F(\rho) \leq 1/2, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}. \quad (10)$$

In other words, $F(\rho) > 1/2$ is a sufficient condition for full N -partite entanglement. An equivalent formulation of Eq. (10) is

$$2|\rho_{1,d}| \leq \sum_{j \neq 1,d} \rho_{j,j}, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}. \quad (11)$$

Of course, analogous conditions may be obtained by replacing $|\Psi_{\text{GHZ}, \alpha}^N\rangle$ in the definition (9) by any other maximally entangled state [14,29]. Exploiting this feature, one can reformulate Eq. (11) in a basis-independent form:

$$2 \max_j |\rho_{j, \bar{j}}| \leq \sum_{i \neq j, \bar{j}} \rho_{i,i}, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}. \quad (12)$$

Note that in contrast to the Laskowski-Żukowski condition and the Mermin-type separability inequalities, the fidelity condition does not distinguish biseparability and other forms of k -separability. Indeed, a fully separable state (e.g., $|0\rangle^{\otimes N}$) can already attain the value $F(\rho)=1/2$. Thus the fidelity condition only distinguishes full inseparability (i.e., $k=1$) from other types of separability ($k \geq 2$). However, as will be shown in the next section, violation of the fidelity condition yields a stronger test for full entanglement than violation of the Laskowski-Żukowski condition.

(4) The Dür-Cirac depolarization method [2,4] gives necessary conditions for partial separability under specific bipartite splits. It uses a two-step procedure in which a general state ρ is first depolarized to become a member of a special family of states, called ρ_N , after which this depolarized state is tested for α_2 -separability under a bipartite split α_2 . If the

depolarized state ρ_N is not separable under α_2 , then neither is the original state ρ , but not necessarily vice versa since the depolarization process can decrease inseparability.

The special family of states ρ_N is given by

$$\rho_N = \lambda_0^+ |\psi_0^+\rangle\langle\psi_0^+| + \lambda_0^- |\psi_0^-\rangle\langle\psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\psi_j^+\rangle\langle\psi_j^+| + |\psi_j^-\rangle\langle\psi_j^-|), \quad (13)$$

with the so-called orthonormal GHZ-basis $|\psi_j^\pm\rangle = \frac{1}{\sqrt{2}}(|j0\rangle \pm |j'1\rangle)$, where $j=j_1j_2\cdots j_{N-1}$ is in binary notation (i.e., a string of $N-1$ bits), and j' means a bit-flip of j : $j' = j'_1j'_2\cdots j'_{N-1}$, with $j'_i = 1, 0$ if $j_i = 0, 1$. The depolarization process does not alter the values of $\lambda_0^\pm = \langle\psi_0^\pm|\rho|\psi_0^\pm\rangle$ and of $\lambda_j = (\langle\psi_j^+|\rho|\psi_j^+\rangle + \langle\psi_j^-|\rho|\psi_j^-\rangle)/2$ of the original state ρ . The values of $j=j_1j_2\cdots j_{N-1}$ can be used to label the various bipartite splits by stipulating that $j=j_1j_2\cdots j_{N-1}$, $j_n=0, (1)$ corresponds to the n th qubit belonging (not belonging) to the same subset as the last qubit. For example, the splits $a-(bc)$, $b-(ac)$, $c-(ab)$ have labels $j=10, 01, 11$, respectively.

The Dür-Cirac condition [2] says that a state ρ is separable under a specific bipartite split j if

$$|\lambda_0^+ - \lambda_0^-| \leq 2\lambda_j \Leftrightarrow 2|\rho_{l,d}| \leq \rho_{l,l} + \rho_{\bar{l},\bar{l}}, \quad \forall \rho \in \mathcal{D}_N^j, \quad \bar{l} = d + 1 - l. \quad (14)$$

For the states (13) this condition is in fact necessary and sufficient. In the right-hand side of the second inequality of Eq. (14) l is determined from j using $\text{Tr}[\rho|\psi_j^+\rangle\langle\psi_j^+| + |\psi_j^-\rangle\langle\psi_j^-|] = \rho_{l,l} + \rho_{\bar{l},\bar{l}}$.

Separability conditions for multipartite splits are constructed from the conditions (14) by means of the partial order $<$ of containment. As mentioned above, if a state is α_k -separable, then it is also α_2 -separable for all bipartite splits $\alpha_k < \alpha_2$. Therefore the conjunction of all α_2 -separability conditions must hold for such a state.

Note that if $|\lambda_0^+ - \lambda_0^-| > 2 \max_j \lambda_j$, the state is inseparable under all bipartite splits, but this does not imply that it is fully inseparable (cf. [24]). Indeed, this feature also exists for states of the form (13) as the following example shows. Take the following two members of the family (13) for $N=3$: for ρ_3^i we choose $\lambda_0^+ = 1/2$, $\lambda_0^- = 0$, $\lambda_{01} = 0$, $\lambda_{10} = 1/4$, $\lambda_{11} = 0$, and for ρ_3^{ii} : $\lambda_0^+ = 1/2$, $\lambda_0^- = 0$, $\lambda_{01} = 0$, $\lambda_{10} = 0$, $\lambda_{11} = 1/4$. It follows from condition (14) that ρ_3^i is separable under split $a-(bc)$ and inseparable under other splits, while ρ_3^{ii} is separable under the split $c-(ab)$ and inseparable under any other split. Now form a convex mixture of these two states: $\tilde{\rho}_3 = \alpha\rho_3^i + \beta\rho_3^{ii}$ with $\alpha + \beta = 1$ and $\alpha, \beta \in (0, 1)$. This state $\tilde{\rho}_3$ is still of the form (13), so that we can again apply condition (14) to conclude that $\tilde{\rho}_3$ is not separable under any bipartite split, yet biseparable by construction.

In the next section we give necessary conditions for k -separability and α_k -separability that are stronger than the Laskowski-Żukowski condition (for $k=2, N$), the fidelity condition, and the Dür-Cirac condition.

III. DERIVING PARTIAL SEPARABILITY CONDITIONS

This section presents separability conditions for all levels and classes in the separability hierarchy of N -qubit states. We start with the case of $N=2$, which has been treated more extensively in [30]. We next move on to the slightly more complicated case of three qubits, for which explicit separability conditions are given for each of the ten classes in the separability hierarchy which were depicted in Fig. 1. Finally, the case of N qubits is treated by a straightforward generalization.

A. Two-qubit case: Setting the stage

For two-qubit systems the separability hierarchy is very simple: there is only one possible split, and consequently just one class at each of the two levels $k=1$ and 2, i.e., states are either inseparable (entangled) or separable. Consider a system composed of a pair of qubits in the familiar setting of two distant sites, each receiving one of the two qubits, and where, at each site, a measurement of either of two spin observables is made. We will focus on the special case that these local spin observables are mutually orthogonal. Let $(X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)})$ denote three orthogonal spin observables on qubit a , and $(X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)})$ on qubit b . (The superscript 1 denotes that we are dealing with single-qubit operators.) A familiar choice for the orthogonal triples $\{X^{(1)}, Y^{(1)}, Z^{(1)}\}$ are the Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$; but note that the choice of the two sets need not coincide. We further define $I_{a,b}^{(1)} := 1$. For all single-qubit pure states $|\psi\rangle$ we have

$$\langle X_j^{(1)} \rangle_\psi^2 + \langle Y_j^{(1)} \rangle_\psi^2 + \langle Z_j^{(1)} \rangle_\psi^2 = \langle I_j^{(1)} \rangle_\psi^2, \quad j = a, b, \quad (15)$$

and for mixed states ρ

$$\langle X_j^{(1)} \rangle^2 + \langle Y_j^{(1)} \rangle^2 + \langle Z_j^{(1)} \rangle^2 \leq \langle I_j^{(1)} \rangle^2, \quad j = a, b. \quad (16)$$

We write $X_a X_b$ or even XX , etc. as shorthand for $X_a \otimes X_b$ and $\langle XX \rangle := \text{Tr}[\rho X_a \otimes X_b]$ for the expectation value in a general state ρ , and $\langle XX \rangle_\Psi := \langle \Psi | X_a \otimes X_b | \Psi \rangle$ for the expectation in a pure state $|\Psi\rangle$.

So, let two triples of locally orthogonal observables $\{X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)}\}$ and $\{X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)}\}$ be given, where a, b label the different qubits. We introduce two sets of four two-qubit operators on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, labeled by the subscript $x=0, 1$:

$$\begin{aligned} X_0^{(2)} &:= \frac{1}{2}(X^{(1)}X^{(1)} - Y^{(1)}Y^{(1)}), & X_1^{(2)} &:= \frac{1}{2}(X^{(1)}X^{(1)} + Y^{(1)}Y^{(1)}), \\ Y_0^{(2)} &:= \frac{1}{2}(Y^{(1)}X^{(1)} + X^{(1)}Y^{(1)}), & Y_1^{(2)} &:= \frac{1}{2}(Y^{(1)}X^{(1)} - X^{(1)}Y^{(1)}), \\ Z_0^{(2)} &:= \frac{1}{2}(Z^{(1)}I^{(1)} + I^{(1)}Z^{(1)}), & Z_1^{(2)} &:= \frac{1}{2}(Z^{(1)}I^{(1)} - I^{(1)}Z^{(1)}), \\ I_0^{(2)} &:= \frac{1}{2}(I^{(1)}I^{(1)} + Z^{(1)}Z^{(1)}), & I_1^{(2)} &:= \frac{1}{2}(I^{(1)}I^{(1)} - Z^{(1)}Z^{(1)}). \end{aligned} \quad (17)$$

Here, the superscript label indicates that we are dealing with two-qubit operators. Later on, $X_x^{(2)}$ will sometimes be notated

as $X_{x,ab}^{(2)}$, and similarly for $Y_x^{(2)}$, $Z_x^{(2)}$, and $I_x^{(2)}$. This more extensive labeling will prove convenient for the multiqubit generalization. Note that $(X_x^{(2)})^2 = (Y_x^{(2)})^2 = (Z_x^{(2)})^2 = (I_x^{(2)})^2 = I_x^{(2)}$ for $x=0,1$, and that all eight operators mutually anticommute. Furthermore, if the orientations of the two triples are the same, these two sets form representations of the generalized Pauli group, i.e., they have the same commutation relations as the Pauli matrices on C^2 , i.e., $[X_x^{(2)}, Y_x^{(2)}] = 2iZ_x^{(2)}$, etc. and

$$\langle X_x^{(2)} \rangle^2 + \langle Y_x^{(2)} \rangle^2 + \langle Z_x^{(2)} \rangle^2 \leq \langle I_x^{(2)} \rangle^2, \quad x \in \{0,1\}, \quad (18)$$

with equality only for pure states.

Assume for the moment that the two-qubit state is pure and separable. We may thus write $\rho = |\Psi\rangle\langle\Psi|$, where $|\Psi\rangle = |\psi\rangle|\phi\rangle$, to obtain

$$\begin{aligned} \langle X_0^{(2)} \rangle_\Psi^2 + \langle Y_0^{(2)} \rangle_\Psi^2 &= \langle X_1^{(2)} \rangle_\Psi^2 + \langle Y_1^{(2)} \rangle_\Psi^2 \\ &= \frac{1}{4} (\langle X_a^{(1)} \rangle_\psi^2 + \langle Y_a^{(1)} \rangle_\psi^2) (\langle X_b^{(1)} \rangle_\phi^2 + \langle Y_b^{(1)} \rangle_\phi^2) \\ &= \frac{1}{4} (\langle I_a^{(1)} \rangle - \langle Z_a^{(1)} \rangle_\psi) (\langle I_b^{(1)} \rangle - \langle Z_b^{(1)} \rangle_\phi) \\ &= \langle I_0^{(2)} \rangle_\Psi^2 - \langle Z_0^{(2)} \rangle_\Psi^2 = \langle I_1^{(2)} \rangle_\Psi^2 - \langle Z_1^{(2)} \rangle_\Psi^2. \end{aligned} \quad (19)$$

This result for pure separable states can be extended to any mixed separable state $\rho \in \mathcal{D}_2^{\text{sep}}$ by noting that the density operator of any such state is a convex combination of the density operators for pure product states, i.e., $\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$, with $|\Psi_j\rangle = |\psi_j\rangle|\phi_j\rangle$, $p_j \geq 0$, and $\sum_j p_j = 1$. We may thus write for such states:

$$\begin{aligned} &\sqrt{\langle X_x^{(2)} \rangle^2 + \langle Y_x^{(2)} \rangle^2} \\ &\leq \sum_j p_j \sqrt{\langle X_x^{(2)} \rangle_j^2 + \langle Y_x^{(2)} \rangle_j^2} \\ &= \sum_j p_j \sqrt{\langle I_y^{(2)} \rangle_j^2 - \langle Z_y^{(2)} \rangle_j^2} \\ &\leq \sqrt{\langle I_y^{(2)} \rangle^2 - \langle Z_y^{(2)} \rangle^2}, \quad \forall \rho \in \mathcal{D}_2^{\text{sep}}, \quad x, y = 0, 1. \end{aligned} \quad (20)$$

Here, $\langle \cdot \rangle_j$ denotes an expectation value in the state $|\Psi_j\rangle$. The first inequality follows because $\sqrt{\langle X_x^{(2)} \rangle^2 + \langle Y_x^{(2)} \rangle^2}$ are convex functions of ρ for all x and the second because $\sqrt{\langle I_y^{(2)} \rangle^2 - \langle Z_y^{(2)} \rangle^2}$ are concave in ρ for all y . As shown in [30] the right-hand side of this inequality is bounded by 1/2, which follows by considering the equalities of Eq. (19). However, for entangled states [e.g., for the Bell states $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$] the left-hand side can attain the value of 1. Hence inequality (20) provides a nontrivial bound for separable states, and thus a criterion for testing entanglement.

In other words, for all separable 2-qubit states one has

$$\begin{aligned} &\max_{x \in \{0,1\}} \langle X_x^{(2)} \rangle^2 + \langle Y_x^{(2)} \rangle^2 \\ &\leq \min_{x \in \{0,1\}} \langle I_x^{(2)} \rangle^2 - \langle Z_x^{(2)} \rangle^2 \leq \frac{1}{4}, \quad \forall \rho \in \mathcal{D}_2^{\text{sep}}. \end{aligned} \quad (21)$$

In fact, the validity of the inequalities (21) for all orthogonal triples $\{X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)}\}$ and $\{X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)}\}$ provides a necessary and sufficient condition for separability for two-qubit states, pure or mixed. (See [30] for a proof.)

Note that, depending on whether the orientation of the triples of local orthogonal observables is the same or not, the inequalities on the left-hand side of Eq. (21) (leaving out the upperbound 1/4) may be simplified. If we choose the orientations for both parties to be the same, then the interesting separability inequalities in Eq. (21) are $\langle X_0^{(2)} \rangle^2 + \langle Y_0^{(2)} \rangle^2 \leq \langle I_1^{(2)} \rangle^2 - \langle Z_1^{(2)} \rangle^2$ and $\langle X_1^{(2)} \rangle^2 + \langle Y_1^{(2)} \rangle^2 \leq \langle I_0^{(2)} \rangle^2 - \langle Z_0^{(2)} \rangle^2$, whereas the other inequalities in Eq. (21) become trivially true [cf. Eq. (18)]. Choosing the orientations to be different reverses this verdict.

To conclude this section we give an explicit form of the separability inequalities (21) by choosing the Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ for both triples $\{X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)}\}$ and $\{X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)}\}$. This choice enables us to write the inequalities (21) in terms of the density matrix elements on the standard z -basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, labeled here as $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. This choice of observables yields $\langle X_0^{(2)} \rangle = 2 \operatorname{Re} \rho_{1,4}$, $\langle Y_0^{(2)} \rangle = -2 \operatorname{Im} \rho_{1,4}$, $\langle I_0^{(2)} \rangle = \rho_{1,1} + \rho_{4,4}$, $\langle Z_0^{(2)} \rangle = \rho_{1,1} - \rho_{4,4}$, $\langle X_1^{(2)} \rangle = 2 \operatorname{Re} \rho_{2,3}$, $\langle Y_1^{(2)} \rangle = -2 \operatorname{Im} \rho_{2,3}$, $\langle I_1^{(2)} \rangle = \rho_{2,2} + \rho_{3,3}$, $\langle Z_1^{(2)} \rangle = \rho_{2,2} - \rho_{3,3}$. So, in this choice, we can write Eq. (21) as

$$\max\{|\rho_{1,4}|^2, |\rho_{2,3}|^2\} \leq \min\{\rho_{1,1}\rho_{4,4}, \rho_{2,2}\rho_{3,3}\} \leq \frac{1}{16}, \quad \rho \in \mathcal{D}_2^{\text{sep}}. \quad (22)$$

In the form (22), it is easy to compare the result to the separability conditions reviewed in Sec. II B. Assume for simplicity that $|\rho_{1,4}|$ is the largest of all the antidiagonal elements $|\rho_{ij}|$. Then, for $\rho \in \mathcal{D}_2^{\text{sep}}$, and using $\langle M^{(2)} \rangle^2 + \langle M'^{(2)} \rangle^2 = 8(\langle X_0^{(2)} \rangle^2 + \langle Y_0^{(2)} \rangle^2)$ the Mermin-type separability inequality (7) becomes $|\rho_{1,4}|^2 \leq 1/16$, which is equivalent to the Laskowski-Żukowski condition $|\rho_{1,4}| \leq 1/4$; the fidelity/Dür-Cirac conditions read: $2|\rho_{1,4}| \leq \rho_{2,2} + \rho_{3,3}$; and the condition (22): $|\rho_{1,4}|^2 \leq \rho_{2,2}\rho_{3,3}$. Using the trivial inequality $(\sqrt{\rho_{2,2}} - \sqrt{\rho_{3,3}})^2 \geq 0 \Leftrightarrow 2\sqrt{\rho_{2,2}\rho_{3,3}} \leq \rho_{2,2} + \rho_{3,3}$, we can then write the following chain of inequalities:

$$4|\rho_{1,4}| - (\rho_{1,1} + \rho_{4,4}) \stackrel{A}{\leq} 2|\rho_{1,4}| \stackrel{\text{sep}}{\leq} 2\sqrt{\rho_{2,2}\rho_{3,3}} \stackrel{A}{\leq} \rho_{2,2} + \rho_{3,3}, \quad (23)$$

where we used the symbols $\stackrel{A}{\leq}$ and $\stackrel{\text{sep}}{\leq}$ to denote inequalities that hold for all states, and for the separability condition (22), respectively.

The Laskowski-Żukowski condition is then recovered by comparing the first and fourth expressions in this chain, the fidelity/Dür-Cirac conditions by comparing the second and fourth expression, and a new condition—not previously mentioned—can be obtained by comparing the first and third term, whereas condition (22), i.e., the comparison between

the second and third expression in Eq. (23), is the strongest inequality in this chain, and thus implies and strengthens all of these other conditions.

B. Three-qubit case

We now derive separability conditions that distinguish the ten classes in the three-qubit classification of Sec. II A by generalizing the method of Sec. III A. To begin with, define four sets of three-qubit observables from the two-qubit operators (17).

$$X_0^{(3)} := \frac{1}{2}(X^{(1)}X_0^{(2)} - Y^{(1)}Y_0^{(2)}), \quad X_1^{(3)} := \frac{1}{2}(X^{(1)}X_0^{(2)} + Y^{(1)}Y_0^{(2)}),$$

$$Y_0^{(3)} := \frac{1}{2}(Y^{(1)}X_0^{(2)} + X^{(1)}Y_0^{(2)}), \quad Y_1^{(3)} := \frac{1}{2}(Y^{(1)}X_0^{(2)} - X^{(1)}Y_0^{(2)}),$$

$$Z_0^{(3)} := \frac{1}{2}(Z^{(1)}I_0^{(2)} + I^{(1)}Z_0^{(2)}), \quad Z_1^{(3)} := \frac{1}{2}(Z^{(1)}I_0^{(2)} - I^{(1)}Z_0^{(2)}),$$

$$I_0^{(3)} := \frac{1}{2}(I^{(1)}I_0^{(2)} + Z^{(1)}Z_0^{(2)}), \quad I_1^{(3)} := \frac{1}{2}(I^{(1)}I_0^{(2)} - Z^{(1)}Z_0^{(2)}),$$

$$X_2^{(3)} := \frac{1}{2}(X^{(1)}X_1^{(2)} - Y^{(1)}Y_1^{(2)}), \quad X_3^{(3)} := \frac{1}{2}(X^{(1)}X_1^{(2)} + Y^{(1)}Y_1^{(2)}),$$

$$Y_2^{(3)} := \frac{1}{2}(Y^{(1)}X_1^{(2)} + X^{(1)}Y_1^{(2)}), \quad Y_3^{(3)} := \frac{1}{2}(Y^{(1)}X_1^{(2)} - X^{(1)}Y_1^{(2)}),$$

$$Z_2^{(3)} := \frac{1}{2}(Z^{(1)}I_1^{(2)} + I^{(1)}Z_1^{(2)}), \quad Z_3^{(3)} := \frac{1}{2}(Z^{(1)}I_1^{(2)} - I^{(1)}Z_1^{(2)}),$$

$$I_2^{(3)} := \frac{1}{2}(I^{(1)}I_1^{(2)} + Z^{(1)}Z_1^{(2)}), \quad I_3^{(3)} := \frac{1}{2}(I^{(1)}I_1^{(2)} - Z^{(1)}Z_1^{(2)}), \quad (24)$$

where $X^{(1)}X_0^{(2)} = X_a^{(1)} \otimes X_{0,bc}^{(2)}$, etc., a, b, c label the three qubits. In analogy to the two-qubit case, we note that all these operators anticommute and that if the orientations of the triples for each qubit are the same, the operators in Eq. (24) yield representations of the generalized Pauli group: $[X_x^{(3)}, Y_x^{(3)}]$

$= 2iZ_x^{(3)}$, for $x=0,1,2,3$. For convenience, we will indeed assume these orientations to be the same, unless noted otherwise. Choosing orientations differently would yield similar separability conditions, in the same vein as in the previous section. Under this choice we have, for all k ,

$$\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 + \langle Z_x^{(3)} \rangle^2 \leq \langle I_x^{(3)} \rangle^2, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}} \quad (25)$$

with equality only for pure states.

We now derive conditions for the different levels and classes of the partial separability classification. Most of the proofs are by straightforward generalization of the method of the previous section and these will be omitted.

Suppose first that the three-qubit state is pure and separable under split a -(bc). From the definitions (24) we obtain

$$\begin{aligned} \langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2 &= \frac{1}{4}(\langle X_a^{(1)} \rangle^2 + \langle Y_a^{(1)} \rangle^2)(\langle X_{0,bc}^{(2)} \rangle^2 + \langle Y_{0,bc}^{(2)} \rangle^2) \\ &= \langle X_1^{(3)} \rangle^2 + \langle Y_1^{(3)} \rangle^2 \\ &= \langle I_0^{(3)} \rangle^2 - \langle Z_0^{(3)} \rangle^2 \\ &= \frac{1}{4}(\langle I_a^{(1)} \rangle^2 - \langle Z_a^{(1)} \rangle^2)(\langle I_{0,bc}^{(2)} \rangle^2 - \langle Z_{0,bc}^{(2)} \rangle^2) \\ &= \langle I_1^{(3)} \rangle^2 - \langle Z_1^{(3)} \rangle^2, \end{aligned} \quad (26)$$

$$\begin{aligned} \langle X_2^{(3)} \rangle^2 + \langle Y_2^{(3)} \rangle^2 &= \frac{1}{4}(\langle X_a^{(1)} \rangle^2 + \langle Y_a^{(1)} \rangle^2)(\langle X_{1,bc}^{(2)} \rangle^2 + \langle Y_{1,bc}^{(2)} \rangle^2) \\ &= \langle X_3^{(3)} \rangle^2 + \langle Y_3^{(3)} \rangle^2 \\ &= \langle I_2^{(3)} \rangle^2 - \langle Z_2^{(3)} \rangle^2 \\ &= \frac{1}{4}(\langle I_a^{(1)} \rangle^2 - \langle Z_a^{(1)} \rangle^2)(\langle I_{1,bc}^{(2)} \rangle^2 - \langle Z_{1,bc}^{(2)} \rangle^2) \\ &= \langle I_3^{(3)} \rangle^2 - \langle Z_3^{(3)} \rangle^2. \end{aligned} \quad (27)$$

Similarly, for pure states that are separable under split b -(ac), we obtain analogous equalities by interchanging the labels $x=1$ and $x=3$ (denoted as $1 \leftrightarrow 3$); and for split c -(ab) by $1 \leftrightarrow 2$.

Of course, these equalities hold for pure states only, but by the convex analysis of Sec. III A we obtain from Eqs. (26) and (27) inequalities for all mixed states that are biseparable under the split a -(bc):

$$\begin{aligned} \max_{x \in \{0,1\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{0,1\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4}, \\ \max_{x \in \{2,3\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{2,3\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4}, \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{a-(bc)}. \quad (28)$$

For states that are biseparable under split b -(ac) the analogous inequalities with $1 \leftrightarrow 3$ hold, i.e.,

$$\begin{aligned} \max_{x \in \{0,3\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{0,3\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4}, \\ \max_{x \in \{1,2\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{1,2\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4}, \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{b-(ac)}, \quad (29)$$

and for the split c -(ab) we need to replace $1 \leftrightarrow 2$:

$$\begin{aligned} \max_{x \in \{0,2\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{0,2\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4} \\ \max_{x \in \{1,3\}} \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 &\leq \min_{x \in \{1,3\}} \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \leq \frac{1}{4} \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{c-(ab)}. \tag{30}$$

A general biseparable state $\rho \in \mathcal{D}_3^{2\text{-sep}}$ is a convex mixture of states that are separable under some bipartite split, i.e., $\rho = p_1 \rho_{a-(bc)} + p_2 \rho_{b-(ac)} + p_3 \rho_{c-(ab)}$ with $\sum_{j=1}^3 p_j = 1$. Since $\sqrt{\langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2}$ is convex in ρ we get from Eqs. (28)–(30) for such a state:

$$\begin{aligned} \sqrt{\langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2} &\leq p_1 \sqrt{\langle X_0^{(3)} \rangle_{\rho_{a-(bc)}}^2 + \langle Y_0^{(3)} \rangle_{\rho_{a-(bc)}}^2} + p_2 \sqrt{\langle X_0^{(3)} \rangle_{\rho_{b-(ac)}}^2 + \langle Y_0^{(3)} \rangle_{\rho_{b-(ac)}}^2} + p_3 \sqrt{\langle X_0^{(3)} \rangle_{\rho_{c-(ab)}}^2 + \langle Y_0^{(3)} \rangle_{\rho_{c-(ab)}}^2} \\ &\leq p_1 \sqrt{\langle I_1^{(3)} \rangle_{\rho_{a-(bc)}}^2 - \langle Z_1^{(3)} \rangle_{\rho_{a-(bc)}}^2} + p_2 \sqrt{\langle I_3^{(3)} \rangle_{\rho_{b-(ac)}}^2 - \langle Z_3^{(3)} \rangle_{\rho_{b-(ac)}}^2} + p_3 \sqrt{\langle I_2^{(3)} \rangle_{\rho_{c-(ab)}}^2 - \langle Z_2^{(3)} \rangle_{\rho_{c-(ab)}}^2}. \end{aligned} \tag{31}$$

Here $\langle \cdot \rangle_{\rho_{a-(bc)}}$ means taking the expectation value in the state $\rho_{a-(bc)}$, etc. Analogous bounds hold for the expressions $\sqrt{\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2}$ for $x=1, 2, 3$.

From the numerical upper bounds in the conditions (28)–(30) it is easy to obtain a first biseparability condition:

$$\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 \leq 1/4, \quad \forall \rho \in \mathcal{D}_3^{2\text{-sep}}, \quad x \in \{0, 1, 2, 3\}. \tag{32}$$

This is equivalent to the Laskowski-Żukowski condition (3) for $k=2$, as will be shown below. However, a stronger condition can be obtained by noting that $\sqrt{\langle I_y^{(3)} \rangle^2 - \langle Z_y^{(3)} \rangle^2}$ is concave in ρ so that

$$\begin{aligned} p_1 \sqrt{\langle I_y^{(3)} \rangle_{\rho_{a-(bc)}}^2 - \langle Z_y^{(3)} \rangle_{\rho_{a-(bc)}}^2} + p_2 \sqrt{\langle I_y^{(3)} \rangle_{\rho_{b-(ac)}}^2 - \langle Z_y^{(3)} \rangle_{\rho_{b-(ac)}}^2} \\ + p_3 \sqrt{\langle I_y^{(3)} \rangle_{\rho_{c-(ab)}}^2 - \langle Z_y^{(3)} \rangle_{\rho_{c-(ab)}}^2} \leq \sqrt{\langle I_y^{(3)} \rangle^2 - \langle Z_y^{(3)} \rangle^2}. \end{aligned} \tag{33}$$

After taking a sum over $y \neq x$ in Eq. (33), the left-hand side of Eq. (33) is larger than the right-hand side of Eq. (31). This yields a stronger condition for biseparability of three-qubit states

$$\begin{aligned} \sqrt{\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2} &\leq \sum_{y \neq x} \sqrt{\langle I_y^{(3)} \rangle^2 - \langle Z_y^{(3)} \rangle^2}, \\ \forall \rho \in \mathcal{D}_3^{2\text{-sep}}, \quad x, y &\in \{0, 1, 2, 3\}. \end{aligned} \tag{34}$$

That Eq. (34) is indeed a stronger than Eq. (32) will be shown below using the density matrix representation of this condition. If one would alter the orientation of the orthogonal triple of observables for a certain qubit, then the right-hand side of Eq. (34) changes by adding either 1, 2, or 3 (modulo 3) to x in the sum on the right-hand side, depending on for which qubit the orientation was changed.

Next, consider the case of a 3-separable state, $\rho \in \mathcal{D}_3^{3\text{-sep}}$. One might then use the fact that this split is contained in all three bipartite splits $a-(bc)$, $b-(ac)$, and $c-(ab)$ to conclude that the inequalities (28)–(30) must hold simultaneously. Thus 3-separable states must obey

$$\max_x \{\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2\} \leq \min_x \{\langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2\} \leq \frac{1}{4},$$

$$\forall \rho \in \mathcal{D}_3^{3\text{-sep}}. \tag{35}$$

However, a more stringent condition holds by virtue of the following equalities for pure 3-separable states:

$$\begin{aligned} \langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2 &= \frac{1}{16} (\langle X_a^{(1)} \rangle^2 + \langle Y_a^{(1)} \rangle^2) (\langle X_b^{(1)} \rangle^2 + \langle Y_b^{(1)} \rangle^2) \\ &\quad \times (\langle X_c^{(1)} \rangle^2 + \langle Y_c^{(1)} \rangle^2) = \langle X_1^{(3)} \rangle^2 + \langle Y_1^{(3)} \rangle^2 \\ &= \langle X_2^{(3)} \rangle^2 + \langle Y_2^{(3)} \rangle^2 = \langle X_3^{(3)} \rangle^2 + \langle Y_3^{(3)} \rangle^2, \end{aligned} \tag{36}$$

$$\begin{aligned} \langle I_0^{(3)} \rangle^2 - \langle Z_0^{(3)} \rangle^2 &= \frac{1}{16} (\langle I_a^{(1)} \rangle^2 - \langle Z_a^{(1)} \rangle^2) (\langle I_b^{(1)} \rangle^2 - \langle Z_b^{(1)} \rangle^2) (\langle I_c^{(1)} \rangle^2 \\ &\quad - \langle Z_c^{(1)} \rangle^2) = \langle I_1^{(3)} \rangle^2 - \langle Z_1^{(3)} \rangle^2 = \langle I_2^{(3)} \rangle^2 - \langle Z_2^{(3)} \rangle^2 \\ &= \langle I_3^{(3)} \rangle^2 - \langle Z_3^{(3)} \rangle^2. \end{aligned} \tag{37}$$

From these equalities for pure states it is easy to obtain, by a convexity argument similar to previous cases, an upper bound of 1/16 instead of 1/4 in Eq. (35):

$$\begin{aligned} \max_x \{\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2\} &\leq \min_x \{\langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2\} \leq \frac{1}{16}, \\ \forall \rho \in \mathcal{D}_3^{3\text{-sep}}. \end{aligned} \tag{38}$$

We have thus obtained different conditions for each of the ten classes in the full separability classification of three qubits, summarized in Table I.

Violations of these partial separability conditions give sufficient conditions for particular types of entanglement. For example, if inequality (38) is violated, then the state must be in one of the biseparable classes 2.1–2.8 or in class 1, which implies that the state is at least 2-partite entangled; if Eq. (34) violated it is in class 1 and thus fully inseparable (fully entangled), and so on.

In order to gain further familiarity with the above separability inequalities, we choose the ordinary Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ for the locally orthogonal observables $\{X^{(1)}, Y^{(1)}, Z^{(1)}\}$, and formulate them in terms of density matrix elements in the standard z basis. Inequalities (28)–(30) now read successively:

$$\begin{aligned} \max\{|\rho_{1,8}|^2, |\rho_{4,5}|^2\} &\leq \min\{\rho_{4,4}\rho_{5,5}, \rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{2,7}|^2, |\rho_{3,6}|^2\} &\leq \min\{\rho_{2,2}\rho_{7,7}, \rho_{3,3}\rho_{6,6}\} \leq 1/16, \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{a-(bc)}, \quad (39)$$

$$\begin{aligned} \max\{|\rho_{1,8}|^2, |\rho_{3,6}|^2\} &\leq \min\{\rho_{3,3}\rho_{6,6}, \rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{2,7}|^2, |\rho_{4,5}|^2\} &\leq \min\{\rho_{2,2}\rho_{7,7}, \rho_{4,4}\rho_{5,5}\} \leq 1/16, \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{b-(ac)}, \quad (40)$$

$$\begin{aligned} \max\{|\rho_{1,8}|^2, |\rho_{2,7}|^2\} &\leq \min\{\rho_{2,2}\rho_{7,7}, \rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{3,6}|^2, |\rho_{4,5}|^2\} &\leq \min\{\rho_{3,3}\rho_{6,6}, \rho_{4,4}\rho_{5,5}\} \leq 1/16, \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{c-(ab)}. \quad (41)$$

For a general biseparable state we can rewrite Eq. (32) as

$$\max\{|\rho_{1,8}|, |\rho_{2,7}|, |\rho_{3,6}|, |\rho_{4,5}|\} \leq 1/4, \quad \forall \rho \in \mathcal{D}_3^{2\text{-sep}}. \quad (42)$$

It can easily be seen that this is equivalent to Laskowski-Żukowski's condition (3) for $k=2$. The condition (34) for biseparability yields

$$\begin{aligned} |\rho_{1,8}| &\leq \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{3,3}\rho_{6,6}} + \sqrt{\rho_{4,4}\rho_{5,5}} \\ |\rho_{2,7}| &\leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{3,3}\rho_{6,6}} + \sqrt{\rho_{4,4}\rho_{5,5}}, \\ |\rho_{3,6}| &\leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{4,4}\rho_{5,5}} \\ |\rho_{4,5}| &\leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{3,3}\rho_{6,6}} \end{aligned} \quad \forall \rho \in \mathcal{D}_3^{2\text{-sep}}. \quad (43)$$

Finally, condition (35) for general 3-separable states becomes

$$\max\{|\rho_{1,8}|^2, |\rho_{2,7}|^2, |\rho_{3,6}|^2, |\rho_{4,5}|^2\} \leq \min\{\rho_{1,1}\rho_{8,8}, \rho_{2,2}\rho_{7,7}, \rho_{3,3}\rho_{6,6}, \rho_{4,4}\rho_{5,5}\} \leq \frac{1}{64}, \quad \forall \rho \in \mathcal{D}_3^{3\text{-sep}}. \quad (44)$$

Note that the separability inequalities (39)–(44) all give bounds on antidiagonal elements in terms of diagonal elements.

We will now show that these bounds improve upon the separability conditions discussed in Sec. II B. We focus on the antidiagonal element $\rho_{1,8}$ (i.e., we suppose that this is the largest antidiagonal matrix element) since this is easiest for comparison. However, the same argument holds for any other antidiagonal matrix element.

The Dür-Cirac conditions in terms of $|\rho_{1,8}|$ read as follows. For partial separability under the split $a-(bc)$: $2|\rho_{1,8}| \leq \rho_{4,4} + \rho_{5,5}$, under the split $b-(ac)$: $2|\rho_{1,8}| \leq \rho_{3,3} + \rho_{6,6}$, and lastly under the split $c-(ab)$: $2|\rho_{1,8}| \leq \rho_{2,2} + \rho_{7,7}$. Next, the

TABLE I. Separability conditions for the ten classes in the separability classification of three-qubit states.

Class	Separability conditions
1	(25)
2.1	(34)
2.2	(28)
2.3	(29)
2.4	(30)
2.5	(28) and (29) but not (30)
2.6	(28) and (30) but not (29)
2.7	(29) and (30) but not (28)
2.8	[(28)–(30)] \Leftrightarrow (35)
3	(38)

Laskowski-Żukowski condition (3) gives for $\rho \in \mathcal{D}_3^{2\text{-sep}}$ that $|\rho_{1,8}| \leq 1/4$ and for $\rho \in \mathcal{D}_3^{3\text{-sep}}$ that $|\rho_{1,8}| \leq 1/8$. The fidelity condition (9) gives that if $\rho \in \mathcal{D}_3^{2\text{-sep}}$ then $2|\rho_{1,8}| \leq \rho_{2,2} + \dots + \rho_{7,7}$.

In order to show that all these conditions are implied by our separability conditions, we employ some inequalities which hold for all states ρ : $|\rho_{1,8}|^2 \leq \rho_{1,1}\rho_{8,8}$ [this follows from Eq. (25)], and $(\sqrt{\rho_{4,4}} - \sqrt{\rho_{5,5}})^2 \geq 0 \Leftrightarrow 2\sqrt{\rho_{4,4}\rho_{5,5}} \leq \rho_{4,4} + \rho_{5,5}$, and similarly $2\sqrt{\rho_{3,3}\rho_{6,6}} \leq \rho_{2,2} + \rho_{6,6}$ and $2\sqrt{\rho_{2,2}\rho_{7,7}} \leq \rho_{2,2} + \rho_{7,7}$. Using these trivial inequalities one easily sees that the conditions (39)–(41) imply the Dür-Cirac conditions for separability under the three bipartite splits. It is also easy to see that the condition for 3-separability (44) strengthens the Laskowski-Żukowski condition (3) for $k=3$. However, it is not so easy to see that Eq. (43) strengthens both the fidelity and Laskowski-Żukowski condition for $k=2$. We will nevertheless show that this is indeed the case.

Let us use the symbols $\stackrel{A}{\leq}$ and $\stackrel{2\text{-sep}}{\leq}$ to denote inequalities that hold for all states or for biseparable states, respectively. Combining the above trivial inequalities with condition (43) yields the following sequence of inequalities:

$$\begin{aligned} 4|\rho_{1,8}| - (\rho_{1,1} + \rho_{8,8}) &\stackrel{A}{\leq} 2|\rho_{1,8}| \stackrel{2\text{-sep}}{\leq} 2\sqrt{\rho_{4,4}\rho_{5,5}} + 2\sqrt{\rho_{3,3}\rho_{6,6}} \\ &\quad + 2\sqrt{\rho_{2,2}\rho_{7,7}} \stackrel{A}{\leq} \rho_{2,2} + \dots + \rho_{7,7}. \end{aligned} \quad (45)$$

The inequality between the second and third expression is Eq. (43). It implies the other inequalities that follow from

Eq. (45). Comparing the first and fourth expression of Eq. (45) one obtains the Laskowski-Żukowski condition (3), while a comparison of the second and fourth yields the fidelity criterion (9). Comparing the first and third term gives a condition which was not previously mentioned. All these are implied by condition (43).

To end this section we show that the separability inequalities for $x=0$ give Mermin-type separability inequalities [28]. Consider the Mermin operator for three qubits:

$$M^{(3)} := X_a^{(1)} X_b^{(1)} Y_c^{(1)} + Y_a^{(1)} X_b^{(1)} X_c^{(1)} + X_a^{(1)} Y_b^{(1)} X_c^{(1)} - Y_a^{(1)} Y_b^{(1)} Y_c^{(1)}, \tag{46}$$

and define $M'^{(3)}$ in the same way, but with all X and Y interchanged. We can now use the identity $16(\langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2) = \langle M^{(3)} \rangle^2 + \langle M'^{(3)} \rangle^2$ to obtain from the separability conditions (32) and (38) the following quadratic inequality for k -separability:

$$16(\langle X_0^{(3)} \rangle^2 + \langle Y_0^{(3)} \rangle^2) = \langle M^{(3)} \rangle^2 + \langle M'^{(3)} \rangle^2 \leq 64 \left(\frac{1}{4} \right)^k, \tag{47}$$

$$\forall \rho \in \mathcal{D}_3^{k\text{-sep}}.$$

Of course, a similar bound holds when $\langle X_0 \rangle^2 + \langle Y_0 \rangle^2$ in the left-hand side is replaced by $\langle X_x \rangle^2 + \langle Y_x \rangle^2$ for $x=1, 2, 3$. This reproduces, for $N=3$, the result (7) of Ref. [1]. From the density matrix representation, we see that these Mermin-type

separability conditions are in fact equivalent to the Laskowski-Żukowski condition (3). Note that these conditions do not distinguish the different classes within level $k=2$, as was the case in Eqs. (39)–(41).

C. N -qubit case

In this section we generalize the analysis of the previous section to N qubits to obtain conditions for k -separability and α_k -separability. The proofs are analogous to the previous cases, and will be omitted. Explicit conditions for k -separability are presented for all levels $k=1, \dots, N$. Further, we give a recursive procedure to derive α_k -separability conditions for each k -partite split α_k at all level k . From these, one can easily construct the conditions that distinguish all the classes in N -partite separability classification by enumerating all possible logical combinations of separability or inseparability under each of these splits at a given level. We will, however, not attempt to write down these latter conditions explicitly since the number of classes grows exponentially with the number of qubits. We start by considering bipartite splits, and biseparable states (level $k=2$), and then move upwards to obtain separability conditions for splits on higher levels.

We define $2^{(N-1)}$ sets of four observables $\{X_x^{(N)}, Y_x^{(N)}, Z_x^{(N)}, I_x^{(N)}\}$, with $x \in \{0, 1, \dots, 2^{(N-1)} - 1\}$ recursively from the $(N-1)$ -qubit observables:

$$\begin{aligned} X_y^{(N)} &:= \frac{1}{2}(X^{(1)} \otimes X_{y/2}^{(N-1)} - Y^{(1)} \otimes Y_{y/2}^{(N-1)}), & X_{y+1}^{(N)} &:= \frac{1}{2}(X^{(1)} \otimes X_{y/2}^{(N-1)} + Y^{(1)} \otimes Y_{y/2}^{(N-1)}), \\ Y_y^{(N)} &:= \frac{1}{2}(Y^{(1)} \otimes X_{y/2}^{(N-1)} + X^{(1)} \otimes Y_{y/2}^{(N-1)}), & Y_{y+1}^{(N)} &:= \frac{1}{2}(Y^{(1)} \otimes X_{y/2}^{(N-1)} - X^{(1)} \otimes Y_{y/2}^{(N-1)}), \\ Z_y^{(N)} &:= \frac{1}{2}(Z^{(1)} \otimes I_{y/2}^{(N-1)} + I^{(1)} \otimes Z_{y/2}^{(N-1)}), & Z_{y+1}^{(N)} &:= \frac{1}{2}(Z^{(1)} \otimes I_{y/2}^{(N-1)} - I^{(1)} \otimes Z_{y/2}^{(N-1)}), \\ I_y^{(N)} &:= \frac{1}{2}(I^{(1)} \otimes I_{y/2}^{(N-1)} + Z^{(1)} \otimes Z_{y/2}^{(N-1)}), & I_{y+1}^{(N)} &:= \frac{1}{2}(I^{(1)} \otimes I_{y/2}^{(N-1)} - Z^{(1)} \otimes Z_{y/2}^{(N-1)}), \end{aligned} \tag{48}$$

with y even, i.e., $y \in \{0, 2, 4, \dots\}$. Analogous relations between these observables hold as those between the observables (17) and (24). In particular, if the orientations of each triple of local orthogonal observables is the same, these sets form representations of the generalized Pauli group, and every N -qubit state obeys $\langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2$, with equality only for pure states.

1. Biseparability

Consider a state that is separable under some bipartite split α_2 of the N qubits. For each such split we get $2^{(N-1)}$

separability inequalities in terms of the sets $\{X_x^{(N)}, Y_x^{(N)}, Z_x^{(N)}, I_x^{(N)}\}$ labeled by $x \in \{0, 1, \dots, 2^{(N-1)} - 1\}$. These separability inequalities provide necessary conditions for the N -qubit state to be separable under the split under consideration. In order to find these inequalities, we first determine the N -qubit analogs of the three-qubit pure state equalities (26) and (27) corresponding to this bipartite split. We have not found a generic expression that lists them all for each possible split and all x . However, for the split where the first qubit is separated from the $(N-1)$ other qubits, i.e., $\alpha_2 = a-(bc \cdots n)$ a generic form can be given:

TABLE II. Solution sets for the seven different bipartite splits of four qubits.

Split α_2	$a-(bcd)$	$b-(acd)$	$c-(abd)$	$d-(abc)$	$(ab)-(cd)$	$(ac)-(bd)$	$(ad)-(bc)$
$z_1^{\alpha_2}$	{0,1}	{0,3}	{0,6}	{0,4}	{0,2}	{0,7}	{0,5}
$z_2^{\alpha_2}$	{2,3}	{1,2}	{1,7}	{1,5}	{1,3}	{1,6}	{1,4}
$z_3^{\alpha_2}$	{4,5}	{5,6}	{2,4}	{2,6}	{4,6}	{2,5}	{2,7}
$z_4^{\alpha_2}$	{6,7}	{4,7}	{3,5}	{3,7}	{5,7}	{3,4}	{3,6}

$$\begin{aligned}
 \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 &= \frac{1}{4} (\langle X_a^{(1)} \rangle^2 + \langle Y_a^{(1)} \rangle^2) (\langle X_{x/2}^{(N-1)} \rangle^2 + \langle Y_{x/2}^{(N-1)} \rangle^2) \\
 &= \langle X_{x+1}^{(N)} \rangle^2 + \langle Y_{x+1}^{(N)} \rangle^2 = \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \\
 &= \frac{1}{4} (\langle I_a^{(1)} \rangle^2 - \langle Z_a^{(1)} \rangle^2) (\langle I_{x/2}^{(N-1)} \rangle^2 - \langle Z_{x/2}^{(N-1)} \rangle^2) \\
 &= \langle I_{x+1}^{(N)} \rangle^2 - \langle Z_{x+1}^{(N)} \rangle^2, \tag{49}
 \end{aligned}$$

where, without loss of generality, x is chosen to be even, i.e., $x \in \{0, 2, 4, \dots\}$. For other bipartite splits the sets of observables labeled by x are permuted, in a way depending on the particular split.

For example, for $N=4$ where $x \in \{0, 1, \dots, 7\}$ the equalities (49) give the result for the split $a-(bcd)$. The corresponding equalities for other bipartite splits are obtained by the following permutations of x : for split $b-(acd)$: $1 \leftrightarrow 3$ and $5 \leftrightarrow 7$; for split $c-(abd)$: $1 \leftrightarrow 6$ and $3 \leftrightarrow 4$; and for split $d-(abc)$: $1 \leftrightarrow 4$ and $3 \leftrightarrow 6$. For the split $(ab)-(cd)$: $1 \leftrightarrow 2$ and $5 \leftrightarrow 6$; for $(ac)-(bd)$: $1 \leftrightarrow 7$ and $3 \leftrightarrow 5$; and lastly, for $(ad)-(bc)$: $1 \leftrightarrow 5$ and $3 \leftrightarrow 7$.

For mixed states that are separable under a given bipartite split the equalities (49) (and their analogs obtained via suitable permutations) become inequalities. We again state them for the split $a-(bc \cdots n)$:

$$\max \left\{ \begin{array}{l} \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \\ \langle X_{x+1}^{(N)} \rangle^2 + \langle Y_{x+1}^{(N)} \rangle^2 \end{array} \right\} \leq \min \left\{ \begin{array}{l} \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \\ \langle I_{x+1}^{(N)} \rangle^2 - \langle Z_{x+1}^{(N)} \rangle^2 \end{array} \right\} \leq \frac{1}{4}, \quad \forall \rho \in \mathcal{D}_N^{\alpha-(bc \cdots n)} \quad \text{with } x \in \{0, 2, 4, \dots\}. \tag{50}$$

The proof of Eq. (50) is a straightforward generalization of the convex analysis in Sec. III A. Again, for the other bipartite splits, the labels x are permuted in a way depending on the particular split.

For a general biseparable state $\rho \in \mathcal{D}_N^{2\text{-sep}}$, we thus obtain the following biseparability conditions:

$$\langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq 1/4, \quad \forall x, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}, \tag{51}$$

which is equivalent to the Laskowski-Żukowski condition for $k=2$ (as will be shown below); and just as in the three-qubit case, we also obtain a stronger condition

$$\sqrt{\langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2} \leq \sum_{y \neq x} \sqrt{\langle I_y^{(N)} \rangle^2 - \langle Z_y^{(N)} \rangle^2}, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}, \quad \text{with } x, y = 0, 1, \dots, 2^{(N-1)} - 1. \tag{52}$$

Violation of this inequality is a sufficient condition for full inseparability, i.e., for full N -partite entanglement.

The inequalities (52) are stronger than the fidelity criterion (9) and the Laskowski-Żukowski criterion (3) for $k=2$, and inequalities (50) are stronger than the Dür-Cirac condition (14) for separability under bipartite splits. This will be shown below in Sec. III C 3.

2. Partial separability criteria for levels $2 < k \leq N$

For levels $k > 2$ we sketch a procedure to find α_{k+1} -separability inequalities recursively from inequalities at the preceding level. Suppose that at level k the inequalities are given for separability under each k -partite split α_k of the N qubits, and that these α_k -separability inequalities take the form

$$\max_{x \in z_i^{\alpha_k}} \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \min_{x \in z_i^{\alpha_k}} (I_x)^2 - (Z_x)^2 \leq \frac{1}{4^{(k-1)}}, \quad \forall \rho \in \mathcal{D}_N^{\alpha_k}, \quad i \in \{1, 2, \dots, 2^{(N-k)}\}, \tag{53}$$

where $z_i^{\alpha_k}$ denote ‘‘solution sets’’ for the specific k -partite split α_k . For example, in the case of three qubits, the solution sets for the bipartite split $a-(bc)$ are $z_1^{\alpha-(bc)} = \{0, 1\}$ and $z_2^{\alpha-(bc)} = \{2, 3\}$, as can be seen from Eq. (28). The solution sets for other bipartite splits can be read off Eqs. (29) and (30) so as to give $z_1^{\alpha-(ac)} = \{0, 3\}$, $z_2^{\alpha-(ac)} = \{1, 2\}$, and $z_1^{\alpha-(ab)} = \{0, 2\}$, $z_2^{\alpha-(ab)} = \{1, 3\}$; and for future purposes we list them for the case of four qubits in Table II above. These were obtained by determining Eq. (50) for $N=4$ and for all bipartite splits α_2 .

Now move one level higher and consider a given $(k+1)$ -partite split $\alpha_{(k+1)}$. This split is contained in a total number of $\binom{k+1}{2} = k(k+1)/2$ k -partite splits α_k . Call the collection of these k -partite splits $\mathcal{S}_{\alpha_{(k+1)}}$. We then obtain preliminary separability

TABLE III. Solution sets for the six different 3-partite splits of four qubits.

Split α_3	$a-b-(cd)$	$(ab)-c-d$	$a-b-(cd)$	$(ac)-b-d$	$(ad)-b-c$	$(bd)-a-c$
$z_1^{\alpha_3}$	{0,1,2,3}	{0,2,4,6}	{0,1,4,5}	{0,3,4,7}	{0,3,5,6}	{0,1,6,7}
$z_2^{\alpha_3}$	{4,5,6,7}	{1,3,5,7}	{2,3,6,7}	{1,2,5,6}	{1,2,4,7}	{2,3,4,5}

inequalities for the split α_{k+1} from the conjunction of all separability inequalities for the splits α_k in the set $\mathcal{S}_{\alpha_{k+1}}$. To be specific, this yields

$$\max_{\alpha_k \in \mathcal{S}_{\alpha_{k+1}}} \max_{x \in z_i^{\alpha_k}} \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \min_{\alpha_k \in \mathcal{S}_{\alpha_{k+1}}} \min_{x \in z_i^{\alpha_k}} \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \leq \frac{1}{4^{k-1}}, \quad \forall \rho \in \mathcal{D}_N^{\alpha_{k+1}}. \quad (54)$$

This may be written more compactly as

$$\max_{x \in z_i^{\alpha_{k+1}}} \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \min_{x \in z_i^{\alpha_{k+1}}} \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \leq \frac{1}{4^{k-1}}, \quad \forall \rho \in \mathcal{D}_N^{\alpha_{k+1}}, \quad i \in \{1, 2, \dots, 2^{(N-k-1)}\}. \quad (55)$$

(In fact, this can be regarded as an implicit definition of the solution sets $z_i^{\alpha_{k+1}}$.) More importantly, by an argument similar to that leading from Eq. (35) to Eq. (38) one finds a stronger numerical bound in the utmost right-hand side of these inequalities, namely 4^{-k} instead of $4^{-(k-1)}$. Thus the final result is

$$\max_{x \in z_i^{\alpha_{k+1}}} \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \min_{x \in z_i^{\alpha_{k+1}}} \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \leq \frac{1}{4^k}, \quad \forall \rho \in \mathcal{D}_N^{\alpha_{k+1}}, \quad i \in \{1, 2, \dots, 2^{(N-k-1)}\}. \quad (56)$$

This shows that the α_k -separability inequalities indeed take the same form as Eq. (53) at all levels.

As an example of this recursive procedure, take $N=4$, set $k=3$, and choose the split $a-b-(cd)$. This split is contained in three 2-partite splits $a-(bcd)$, $b-(acd)$ and $(ab)-(cd)$. Using Eq. (54) and the first, second, and fifth column of Table II one obtains the following two solutions sets for the split $a-b-(cd)$: $z_1^{a-b-(cd)} = \{0, 1, 2, 3\}$ and $z_2^{a-b-(cd)} = \{4, 5, 6, 7\}$. This leads to the separability inequalities

$$\begin{aligned} \max_{x \in \{0,1,2,3\}} \langle X_x^{(4)} \rangle^2 + \langle Y_x^{(4)} \rangle^2 &\leq \min_{x \in \{0,1,2,3\}} \langle I_x^{(4)} \rangle^2 - \langle Z_x^{(4)} \rangle^2 \leq \frac{1}{16}, \\ \max_{x \in \{4,5,6,7\}} \langle X_x^{(4)} \rangle^2 + \langle Y_x^{(4)} \rangle^2 &\leq \min_{x \in \{4,5,6,7\}} \langle I_x^{(4)} \rangle^2 - \langle Z_x^{(4)} \rangle^2 \leq \frac{1}{16}, \end{aligned} \quad \forall \rho \in \mathcal{D}_4^{a-b-(cd)}. \quad (57)$$

For other 3-partite splits the inequalities can be obtained in a similar way so as to give Table III above.

As a special case, we mention the result for full separability, i.e., for $k=N$. There is only one N -partite split, namely where all qubits end up in a different set. Further, there is only one solution set $z_i^{\alpha_N}$ and it contains all $x \in \{0, 1, \dots, 2^{(N-1)} - 1\}$. States ρ that are separable under this split thus obey

$$\max_x \langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \min_x \langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 \leq \frac{1}{4^{(N-1)}}, \quad \forall \rho \in \mathcal{D}_N^{N\text{-sep}}. \quad (58)$$

Violation of this inequality is a sufficient condition for some entanglement to be present in the N -qubit state. The condition (58) strengthens the Laskowski-Żukowski condition (3) for $k=N$ (to be shown below).

For an N -qubit k -separable state $\rho \in \mathcal{D}_N^{k\text{-sep}}$, i.e., a state that is a convex mixture of states that are separable under some k -partite split, we obtain from Eq. (56) the following k -separability conditions:

$$\langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2 \leq \frac{1}{4^{(k-1)}}, \quad \forall x, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}, \quad (59)$$

which is equivalent to the Laskowski-Żukowski condition (3) for all N and k (this will be shown below using the density matrix formulation of these conditions). However, in analogy to Eq. (34) we also obtain the stronger condition:

$$\sqrt{\langle X_x^{(N)} \rangle^2 + \langle Y_x^{(N)} \rangle^2} \leq \min_l \sum_{y \in \mathcal{T}_{k,l}^{N,x}} \sqrt{\langle I_y^{(N)} \rangle^2 - \langle Z_y^{(N)} \rangle^2}, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}, \quad (60)$$

where, for given N, k , and x , $\mathcal{T}_{k,l}^{N,x}$ denotes a tuple of values of $y \neq x$, each one being picked from each of the solutions sets $z_i^{\alpha_k}$ that contain x , where α_k ranges over all the k -partite splits of the N qubits. In general, there will be many ways of picking such values, and we use l as an index to label such tuples.

For example, in the case $N=3$, there are a total of six solution sets (two for each of the three bipartite splits): $\{0, 1\}$, $\{2, 3\}$, $\{0, 2\}$, $\{1, 3\}$, $\{0, 3\}$, $\{1, 2\}$. If we set $x=0$ and pick a member different from 0 from each of those sets that contain 0, we find $\mathcal{T}_{2,1}^3 = \{1, 2, 3\}$. This is in fact the only such choice and thus $l=1$. Thus in this example condition (60) reproduces the result (34).

As a more complicated example, take $N=4$, $k=3$, and choose again $x=0$. In this case there are six 3-partite splits each of which has two solution sets, as given in Table III. The solution sets that contain 0 are all on the top row of this table. There are now many ways of constructing a tuple by picking elements that differ from 0 from each of these sets, for example, $\mathcal{T}_{3,1}^4 = \{1, 2, 1, 3, 3, 1\}$, $\mathcal{T}_{3,2}^4 = \{1, 2, 1, 3, 3, 6\}$, etc. In this case one has to take a minimum in Eq. (60) over all these $l=1, \dots, 3^6$ tuples.

For $k=2$, condition (60) reduces to Eq. (52) and for $k=N$ to Eq. (58). For these values of k , the condition is stronger than Eq. (59) (see the next section). For $k \neq 2, N$, this is still an open question.

To conclude this section, let us recapitulate. We have found separability conditions in terms of local orthogonal observables for each of the N parties that are necessary for k -separability and for separability under splits α_k at each level on the hierarchic separability classification. Violations

of these separability conditions give sufficient criteria for k -separable entanglement and m -partite entanglement with $[N/k] \leq m \leq N-k+1$. The separability conditions are stronger than the Dür-Cirac condition for separability under specific splits, and stronger than the fidelity condition and the Laskowski-Żukowski condition for biseparability. The latter condition is also strengthened for $k=N$. These implications are shown in the next section.

3. The conditions in terms of matrix elements

Choosing the Pauli matrices $\{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}\}$ as local orthogonal observables, with the same orientation at each qubit, allows one to formulate the separability conditions in terms of the density matrix elements $\rho_{i,j}$ on the standard z basis [31]. For these choices we obtain:

$$X_0^{(N)} = |0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N}, \quad \langle X_0^{(N)} \rangle = 2 \operatorname{Re} \rho_{1,d},$$

$$Y_0^{(N)} = -i|0\rangle\langle 1|^{\otimes N} + i|1\rangle\langle 0|^{\otimes N}, \quad \langle Y_0^{(N)} \rangle = -2 \operatorname{Im} \rho_{1,d},$$

$$I_0^{(N)} = |0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N}, \quad \langle I_0^{(N)} \rangle = \rho_{1,1} + \rho_{d,d},$$

$$Z_0^{(N)} = |0\rangle\langle 0|^{\otimes N} - |1\rangle\langle 1|^{\otimes N}, \quad \langle Z_0^{(N)} \rangle = \rho_{1,1} - \rho_{d,d}, \quad (61)$$

where $d=2^N$. Analogous relations hold for $X_x^{(N)}, Y_x^{(N)}, Z_x^{(N)}, I_x^{(N)}$ for $x \neq 0$.

Let us treat the case $N=4$ in detail. First, consider the level $k=2$. Biseparability under the split a - (bcd) gives the following inequalities for the antidiagonal matrix elements:

$$\begin{aligned} \max\{|\rho_{1,16}|^2, |\rho_{8,9}|^2\} &\leq \min\{\rho_{1,1}\rho_{16,16}, \rho_{8,8}\rho_{9,9}\} \leq 1/16 \\ \max\{|\rho_{2,15}|^2, |\rho_{7,10}|^2\} &\leq \min\{\rho_{2,2}\rho_{15,15}, \rho_{7,7}\rho_{10,10}\} \leq 1/16 \\ \max\{|\rho_{3,14}|^2, |\rho_{6,11}|^2\} &\leq \min\{\rho_{3,3}\rho_{14,14}, \rho_{6,6}\rho_{11,11}\} \leq 1/16 \\ \max\{|\rho_{5,12}|^2, |\rho_{4,13}|^2\} &\leq \min\{\rho_{5,5}\rho_{12,12}, \rho_{4,4}\rho_{13,13}\} \leq 1/16 \end{aligned} \quad \forall \rho \in \mathcal{D}_4^{a-(bcd)}. \quad (62)$$

The analogous inequalities for separability under other bipartite splits are obtained by suitable permutations on the labels. Indeed, for split b - (acd) labels 8 and 5, 9 and 12, 2 and 3, and 5 and 14 are permuted, which we denote as $(8, 9, 2, 15) \leftrightarrow (5, 12, 3, 14)$; for split c - (abd) : $(8, 9, 2, 15) \leftrightarrow (3, 14, 5, 12)$; for split d - (abc) : $(8, 9, 3, 14) \leftrightarrow (2, 15, 5, 12)$; for the split (ab) - (cd) : $(8, 9, 3, 14) \leftrightarrow (4, 13, 7, 10)$; for (ac) - (bd) : $(8, 9, 5, 12) \leftrightarrow (6, 11, 7, 10)$; and lastly, for the split (ad) - (bc) : $(8, 9, 5, 12) \leftrightarrow (7, 10, 6, 11)$. For a general biseparable state we obtain

$$|\rho_{1,16}| \leq \sqrt{\rho_{2,2}\rho_{15,15}} + \sqrt{\rho_{3,3}\rho_{14,14}} + \dots + \sqrt{\rho_{8,8}\rho_{9,9}}, \quad \forall \rho \in \mathcal{D}_4^{2\text{-sep}}, \quad (63)$$

and analogous for the other antidiagonal elements.

Next, consider one level higher, i.e., $k=3$. There are six different 3-partite splits for a system consisting of four qubits. For separability under each such split a different set of inequalities can be obtained from Eq. (54). To be more precise, such a set consists of the conjunction of all the separability inequalities for the bipartite splits at level $k=2$ this particular 3-partite split is contained in. For $N=4$ each 3-partite split is contained in three bipartite splits. For example, for separability under split a - b - (cd) we obtain

$$\begin{aligned} \max\{|\rho_{1,16}|^2, |\rho_{8,9}|^2, |\rho_{4,13}|^2, |\rho_{5,12}|^2\} &\leq \min\{\rho_{1,1}\rho_{16,16}, \rho_{8,8}\rho_{9,9}, \rho_{4,4}\rho_{13,13}, \rho_{5,5}\rho_{12,12}\} \leq 1/64. \\ \max\{|\rho_{2,15}|^2, |\rho_{3,14}|^2, |\rho_{6,11}|^2, |\rho_{7,10}|^2\} &\leq \min\{\rho_{2,2}\rho_{15,15}, \rho_{3,3}\rho_{14,14}, \rho_{6,6}\rho_{11,11}, \rho_{7,7}\rho_{10,10}\} \leq 1/64, \end{aligned} \quad \forall \rho \in \mathcal{D}_4^{a-b-(cd)}. \quad (64)$$

This is the density matrix formulation of Eq. (57).

A general 3-separable state $\rho \in \mathcal{D}_4^{3\text{-sep}}$ is a convex mixture of states that each are separable under some such 3-partite split. The separability condition follows from Eq. (60):

$$|\rho_{1,16}| \leq \min_l \left(\sum_{j \in \tilde{T}_{3,l}^{4,0}} \sqrt{\rho_{j,j} \rho_{17-j,17-j}} \right), \quad \forall \rho \in \mathcal{D}_4^{3\text{-sep}}, \quad (65)$$

where $\tilde{T}_{3,l}^{4,0}$ is the tuple of indices j in $\{1,16\}$ that label the antidiagonal density matrix elements $\rho_{j,17-j}$ corresponding to the density matrix formulation of the set of operators $\langle X_y^{(4)} \rangle^2 + \langle Y_y^{(4)} \rangle^2$ with y determined by $\mathcal{T}_{3,l}^{4,0}$. Here we have used that the antidiagonal element $\rho_{1,16}$ corresponds to $\langle X_0^{(4)} \rangle^2 + \langle Y_0^{(4)} \rangle^2$. For $N=4$, $k=3$ there are six possible splits, so for each l , j is picked from a total of six sets. For the case under consideration the sets are $\{1, 4, 5, 8\}$, $\{1, 2, 3, 4\}$, $\{1, 3, 5, 7\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 7, 8\}$, and $\{1, 3, 6, 8\}$. For each l one chooses a tuple of values of j where one value is picked from each of these six sets, except for the value 1 which is excluded. Analogous inequalities are obtained for the other antidiagonal matrix elements.

Finally for full separability ($k=4$) we get

$$\max\{|\rho_{1,16}|^2, |\rho_{2,15}|^2, \dots, |\rho_{8,9}|^2\} \leq \min\{\rho_{1,1}\rho_{16,16}, \rho_{2,2}\rho_{15,15}, \dots, \rho_{8,8}\rho_{9,9}\} \leq 1/256, \quad \forall \rho \in \mathcal{D}_4^{4\text{-sep}}. \quad (66)$$

For general N , it is easy to see that Eq. (51) yields the Laskowski-Żukowski condition (3). It is instructive to look at the extremes of biseparability and full separability, since for them explicit forms can be given. For $k=2$ condition (52) reads

$$|\rho_{l,\bar{l}}| \leq \sum_{n \neq l, \bar{l}} \sqrt{\rho_{n,n} \rho_{\bar{n},\bar{n}}} / 2, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}, \quad \text{where } \bar{l} = d+1 - l, \quad \bar{n} = d+1 - n, \quad l, n \in \{1, \dots, d\}. \quad (67)$$

For $k=N$, we can reformulate condition (58) as

$$\max\{|\rho_{1,d}|^2, |\rho_{2,d-1}|^2, \dots\} \leq \min\{\rho_{1,1}\rho_{d,d}, \rho_{2,2}\rho_{d-1,d-1}, \dots\} \leq 1/4^N, \quad \forall \rho \in \mathcal{D}_N^{N\text{-sep}}. \quad (68)$$

It is easily seen that the condition (68) is stronger than the Laskowski-Żukowski condition (3) for this case.

Again, these inequalities give bounds on antidiagonal matrix elements in terms of diagonal ones on the z basis. These density matrix representations depend on the choice of the Pauli matrices as the local observables. However, every other triple of locally orthogonal observables with the same orientation can be obtained from the Pauli matrices by suitable local basis transformations, and therefore this matrix representation does not lose generality. Choosing different orientations of the triples one obtains the corresponding inequalities by suitable permutations of antidiagonal matrix elements.

We will now show that Eq. (67) is indeed stronger than the fidelity condition (9) and the Laskowski-Żukowski condition (3) for $k=2$ by following the same analysis as in the three-qubit case. We again assume, for convenience, that the antidiagonal element $\rho_{1,d}$ is the largest of all antidiagonal elements. Using some inequalities that hold for all states together with the condition (67) for biseparability we get the following sequence of inequalities for $\rho_{1,d}$:

$$\begin{aligned} 4|\rho_{1,d}| - (\rho_{1,1} + \rho_{d,d}) &\stackrel{A}{\leq} 2|\rho_{1,d}| \stackrel{2\text{sep}}{\leq} 2\sqrt{\rho_{2,2}\rho_{d-1,d-1}} + \dots \\ &+ 2\sqrt{\rho_{d/2,d/2}\rho_{d/2+1,d/2+1}} \leq \rho_{22} + \dots \\ &+ \rho_{d-1,d-1}. \end{aligned} \quad (69)$$

The inequality in the middle is Eq. (67). It implies all other

inequalities in the sequence (69). The inequality between the first and fourth term yields the Laskowski-Żukowski condition for $k=2$, and between the second and fourth gives the fidelity criterion in the formulation (11). One also sees that the fidelity criterion is stronger than the Laskowski-Żukowski condition for $k=2$.

We finally discuss two examples showing that the biseparability condition (67) is stronger in detecting full entanglement than other methods. First, consider the family of N -qubit states

$$\begin{aligned} \rho'_N = \lambda_0^+ |\psi_0^+\rangle\langle\psi_0^+| + \lambda_0^- |\psi_0^-\rangle\langle\psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\psi_k^+\rangle + |\psi_j^-\rangle) & \\ & + \langle\psi_j^-|. \end{aligned} \quad (70)$$

The states (70) violate Eq. (67) for all $|\lambda_0^+ - \lambda_0^-| \neq 0$ and are thus detected as fully entangled by that condition. In that case they are also inseparable under any split. The fidelity criterion (11), however, detects these states as fully entangled only for $|\lambda_0^+ - \lambda_0^-| \geq \sum_j \lambda_j$. Violation of Eq. (67) thus allows for detecting more states of the form ρ'_N as fully entangled than violation of the fidelity criterion. Further, the Dür-Cirac criteria detects these states as inseparable under any split for $|\lambda_0^+ - \lambda_0^-| > 2\lambda_j$, $\forall j$, which includes less states than a violation of Eq. (67). This generalizes the observation of Ref. [32] from two qubits to the N -qubit case.

Second, consider the N -qubit GHZ-like states $|\theta\rangle = \cos\theta|0\rangle^{\otimes N} + \sin\theta|1\rangle^{\otimes N}$. We can easily read off from the density matrix $|\theta\rangle\langle\theta|$ that the far off-antidiagonal matrix elements $\rho_{1,d} = \rho_{d,1}$ is equal to $\cos\theta \sin\theta$ and that the diagonal matrix elements $\rho_{2,2}, \dots, \rho_{d-1,d-1}$ are all equal to zero. Using Eq. (67) we see that these states are fully N -partite entangled for $\rho_{1,d} = \cos\theta \sin\theta \neq 0$, i.e., for all $\theta \neq 0, \pi/2 \pmod{\pi}$. Thus all fully entangled states of this form are detected by condition (67), including those not detectable by any standard multipartite Bell inequality [33].

4. Relationship to Mermin-type inequalities for partial separability and LHV models

We will now show that the separability inequalities of the previous section imply already known Mermin-type inequalities [28] for partial separability. Using the identity

$2^{(N+1)}(\langle X_0^{(N)} \rangle^2 + \langle Y_0^{(N)} \rangle^2) = \langle M^{(N)} \rangle^2 + \langle M'^{(N)} \rangle^2$ for the Mermin operators (6) together with the upper bound for the separability inequality of Eq. (59) for $x=0$ gives the following sharp quadratic inequality:

$$\langle M^{(N)} \rangle^2 + \langle M'^{(N)} \rangle^2 \leq 2^{(N+3)} \left(\frac{1}{4}\right)^k, \quad \forall \rho \in \mathcal{D}_N^{k\text{-sep}}, \quad (71)$$

which reproduces the result (7) found by [1]. Since Eq. (51) is equivalent to Eq. (3) we see that the Mermin-type separability condition is in fact one of the Laskowski-Żukowski conditions written in terms of local observables X and Y .

As a special case we consider a split of the form $\{1\}, \dots, \{\kappa\}, \{\kappa+1, \dots, n\}$. Any state that is separable under this split is $(\kappa+1)$ -separable so we get the condition $\langle M^{(N)} \rangle^2 + \langle M'^{(N)} \rangle^2 \leq 2^{(N-2\kappa+1)}$, and hence $|\langle M^{(N)} \rangle| \leq 2^{(N-2\kappa+1)/2}$. This strengthens the result of Gisin and Bechmann-Pasquinucci [9] by a factor $2^{\kappa/2}$ for these specific Mermin operators (6).

As another special case of the inequalities (71), consider $k=N$. In this case, the inequalities express a condition for full separability of ρ . These inequalities are maximally violated by fully entangled states by an exponentially increasing factor of 2^{N-1} , since the maximal value of $|\langle M^{(N)} \rangle|$ for any quantum state ρ is $2^{(N+1)/2}$ [34]. Furthermore, LHV models violate them also by an exponentially increasing factor of $2^{(N-1)/2}$, since for all N , LHV models allow a maximal value for $|\langle M^{(N)} \rangle|$ of 2 [9,13], which is a factor $2^{(N-1)/2}$ smaller than the quantum maximum using entangled states. This bound for LHV models is sharp since the maximum is attained by choosing the LHV expectation values $\langle \sigma_x^i \rangle = \langle \sigma_y^i \rangle = 1$ for all $i \in \{1, \dots, N\}$. This shows that there are exponentially increasing gaps between the values of $|\langle M^{(N)} \rangle|$ attainable by fully separable states, fully entangled states, and LHV models. This is shown in Fig. 2.

That the maximum violation of multipartite Bell inequalities allowed by quantum mechanics grows exponentially with N with respect to the value obtainable by LHV models has been known for quite some years [28,34]. However, it is equally remarkable that the maximum value obtainable by separable quantum states *exponentially decreases* in comparison to the maximum value obtainable by LHV models, cf. Fig. 2. We thus see exponential divergence between separable quantum states and LHV theories: as N grows, the latter are able to give correlations that need more and more entanglement in order to be reproducible in quantum mechanics.

But why does quantum mechanics have correlations larger than those obtainable by a LHV model? Here we give an argument showing that it is not the degree of entanglement but the degree of inseparability that is responsible. The degree of entanglement of a state may be quantified by the value m that indicates the m -partite entanglement of the state, and the degree of inseparability by the value of k that indicates the k -separability of the state. Now suppose we have 100 qubits. For partial separability of $k \geq 51$ no state of these 100 qubits can violate the Mermin inequality (8) above the LHV bound, although the state could be up to 50-partite entangled ($m \leq 50$). However, for $k=2$, a state is possible

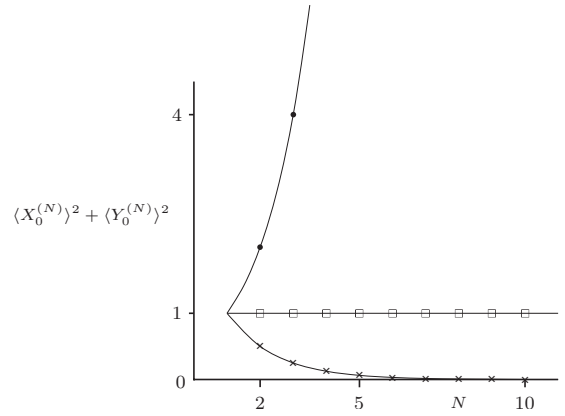


FIG. 2. The maximum value for $\langle X_0 \rangle^2 + \langle Y_0 \rangle^2$ obtainable by entangled quantum states (dots), by separable quantum states (crosses), and by LHV models (squares), plotted as a function of the number of qubits N . Note the exponential divergence between both the maxima obtained for entangled states as well as for separable states compared to the LHV value, where the former maximum is exponentially increasing and the latter maximum is exponentially decreasing.

that is also 50-partite entangled, but which violates the Mermin inequality by an exponentially large factor of $2^{97/2}$. For $k < N$, a k -separable state is always entangled in some way, so we see that it is the degree of partial separability, not the amount of entanglement in a multiqubit state that determines the possibility of a violation of the Mermin inequality. Of course, some entanglement must be present, but the inseparability aspect of the state determines the possibility of a violation. This is also reflected in the fact that for a given N it is the value of k , and not that of m , which determines the sharp upper bounds of the Mermin inequalities.

IV. EXPERIMENTAL STRENGTH OF THE CONDITIONS FOR k -SEPARABLE ENTANGLEMENT DETECTION

Violations of the above conditions for partial separability provide sufficient criteria for detecting k -separable entanglement (and m -partite entanglement with $\lfloor N/k \rfloor \leq m \leq N-k+1$). It has already been shown that these criteria are stronger than the Laskowski-Żukowski criterion for k -inseparability for $k=2$, N (i.e., detecting some and full entanglement), the fidelity criterion for full inseparability (i.e., full entanglement), and the Dür-Cirac criterion for inseparability under splits. In this section we will elaborate further on the experimental usefulness and strength of these entanglement criteria when focusing on specific N -qubit states. The strength of an entanglement criterion to detect a given entangled state may be assessed by determining how well it copes with two desiderata [11]: the noise robustness of the criterion for this given state should be high, and the number of local measurements settings needed for its implementation should be small.

In this section we will first take a closer look at the issue of noise robustness and at the number of required settings for implementation of the separability criteria, both in the general state-independent case and in the case of detecting target states. We then show the strength of the criteria for a variety of specific N -qubit states.

A. Noise robustness and the number of measurement settings

White noise robustness of an entanglement criterion for a given entangled state is the maximal fraction p_0 of white noise which may be admixed to this state so that the state can no longer be detected as entangled by the criterion. Thus for a given entangled state ρ , the noise robustness of a criterion is the threshold value p_0 for which the state $\rho = p\mathbb{1}/2^N + (1-p)\rho$, with $p \geq p_0$, can no longer be detected by that criterion.

So, for the criterion for detecting full entanglement (67), the white noise robustness is found by solving the threshold equation for p_0 :

$$|(1-p_0)\rho_{l,\bar{l}}| = \sum_{j \neq l} \sqrt{\left[\frac{p_0}{2^N} + (1-p_0)\rho_{j,j} \right] \left[\frac{p_0}{2^N} + (1-p_0)\rho_{\bar{j},\bar{j}} \right]}. \quad (72)$$

The state is fully entangled for $p < p_0$.

For the criterion (68), for detecting some entanglement, one finds a similar threshold equation:

$$\begin{aligned} & \max_l \{ |(1-p_0)\rho_{l,\bar{l}}|^2 \} \\ & = \min_j \left\{ \left[\frac{p_0}{2^N} + (1-p_0)\rho_{j,j} \right] \left[\frac{p_0}{2^N} + (1-p_0)\rho_{\bar{j},\bar{j}} \right] \right\}. \end{aligned} \quad (73)$$

This equation is quadratic and easily solved. Again, the state is entangled for $p < p_0$.

A local measurement setting [35–37] is an observable such as $\mathcal{M} = \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_N$, where σ_l denote single qubit observables for each of the N qubits. Measuring such a setting (determining all coincidence probabilities of the 2^N outcomes) also enables one to determine the probabilities for observables like $\mathbb{1} \otimes \sigma_2 \cdots \otimes \sigma_N$, etc. [15]. Now consider the observables $X_x^{(N)}$ and $Y_x^{(N)}$ that appear in the separability criteria of Eqs. (49)–(60). As it is easily seen from their definitions in Eq. (48), one can measure such an observable using 2^N local settings. However, these same 2^N settings then suffice to measure the observables $X_x^{(N)}$ and $Y_x^{(N)}$ for all other x since these are linear combinations of the same settings. Thus 2^N measurement settings are sufficient to determine $\langle X_x^{(N)} \rangle$ and $\langle Y_x^{(N)} \rangle$ for all x . It remains to determine the number of settings needed for the terms $\langle I_x^{(N)} \rangle$ and $\langle Z_x^{(N)} \rangle$. For all x these terms contain only two single-qubit observables: $Z^{(1)}$ and $I^{(1)} = \mathbb{1}$. They can thus be measured by a single setting, i.e., $(Z^{(1)})^{\otimes N}$.

Thus in total $2^N + 1$ settings are needed in order to test the separability conditions. This number grows exponentially with the number of qubits. However, this is the price we pay for being so general, i.e., for having criteria that work for all states. If we apply the criteria to detecting forms of inseparability and entanglement of specific entangled N -qubit states, this number can be greatly reduced. Knowledge of the target state enables one to select a single separability inequality for an optimal value of x in Eqs. (49)–(54) and (56)–(60). Violation of this single inequality is then sufficient for detecting the entanglement in this state, and, as we will now show, the required number of settings then grows only

linear in N , with $N+1$ being the optimum for many states of interest.

For simplicity, assume that the local observables featuring in the criteria are the Pauli spin observables with the same orientation for each qubit. We can then readily use the density matrix representations of the separability criteria given at the end of each section in Sec. III. Choosing the local observables differently amounts to performing suitable bases changes to the density matrix representations and would not affect the argument.

The matrix representations of the conditions show that only some antidiagonal matrix elements and the values of some diagonal matrix elements have to be determined in order to test whether these inequalities are violated. Indeed, observe that for all x $\langle I_x^{(N)} \rangle^2 - \langle Z_x^{(N)} \rangle^2 = 4\rho_{j,j}\rho_{\bar{j},\bar{j}}$ with $\bar{j} = d+1-j$ for some $j \in \{1, 2, \dots, d\}$ and $\langle X_x^{(N)} \rangle^2 - \langle Y_x^{(N)} \rangle^2 = 4|\rho_{j,\bar{j}}|^2$ denotes some antidiagonal matrix element. It suffices to consider $x=0$ since conditions for other values of x are obtained by some local unitary basis changes that will be explicitly given later on. We now want to rewrite the density matrix representation for this single separability inequality with $x=0$ in terms of less than $2^N + 1$ settings.

Determining the diagonal matrix elements requires only a single setting, namely $\sigma_z^{\otimes N}$. Next, we should determine the modulus of the far-off antidiagonal element $\rho_{1,d}$ ($d=2^N$) by measuring $X_0^{(N)}$ and $Y_0^{(N)}$, since $\langle X_0^{(N)} \rangle = 2\text{Re } \rho_{1,d}$ and $\langle Y_0^{(N)} \rangle = 2\text{Im } \rho_{1,d}$ [cf. Eq. (61)]. Following the method of [15], these matrix elements can be obtained from two settings \mathcal{M}_l and $\tilde{\mathcal{M}}_l$, given by

$$\mathcal{M}_l = \left[\cos\left(\frac{l\pi}{N}\right)\sigma_x + \sin\left(\frac{l\pi}{N}\right)\sigma_y \right]^{\otimes N}, \quad l = 1, 2, \dots, N, \quad (74)$$

$$\begin{aligned} \tilde{\mathcal{M}}_l &= \left[\cos\left(\frac{l\pi + \pi/2}{N}\right)\sigma_x + \sin\left(\frac{l\pi + \pi/2}{N}\right)\sigma_y \right]^{\otimes N}, \\ & \quad l = 1, 2, \dots, N. \end{aligned} \quad (75)$$

These operators obey

$$\sum_{l=1}^N (-1)^l \mathcal{M}_l = NX_0^{(N)}, \quad (76)$$

$$\sum_{l=1}^N (-1)^l \tilde{\mathcal{M}}_l = NY_0^{(N)}. \quad (77)$$

The proof of Eq. (76) is given in [15] and Eq. (77) can be proven in the same way.

These relations show that the imaginary and the real part of an antidiagonal element can be determined by the N settings \mathcal{M}_l and $\tilde{\mathcal{M}}_l$, respectively. This implies that the biseparability condition (67) needs only $2N+1$ measurement settings. However, if each antidiagonal term is real valued (which is often the case for states of interest) it can be determined by the N settings \mathcal{M}_l , so that in total $N+1$ settings suffice.

Implementation of the criteria for other x involves determining the modulus of some other antidiagonal matrix element instead of the far-off antidiagonal element $\rho_{1,d}$. The settings that allow for this determination can be obtained from a local unitary rotation on the settings \mathcal{M}_l and $\tilde{\mathcal{M}}_l$ needed to measure $|\rho_{1,d}|$. This can be done as follows.

Suppose we want to determine the modulus of the matrix element $\rho_{j,\bar{j}}$. The unitary rotation to be applied is given by $U_j = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \dots \otimes \sigma_{j_N}$ with $j = j_1 j_2 \dots j_N$ in binary notation, with $\sigma_0 = 1$ and $\sigma_1 = \sigma_x$. The settings that suffice are then given by $\mathcal{M}_{j,l} = U_j \mathcal{M}_l U_j^\dagger$ and $\tilde{\mathcal{M}}_{j,l} = U_j \tilde{\mathcal{M}}_l U_j^\dagger$ ($l = 1, 2, \dots, N$). For example, take $N = 4$ and suppose we want to determine $\rho_{5,4}$. We obtain the required settings by applying the local unitary $U_5 = 1 \otimes \sigma_x \otimes 1 \otimes \sigma_x$ (since the binary notation of 5 on four bits is 0101) to the two settings \mathcal{M}_l and $\tilde{\mathcal{M}}_l$ given in Eqs. (74) and (77), respectively, that for $N = 4$ allow for determining $|\rho_{1,16}|$. In conclusion, using the above procedure the modulus of each antidiagonal element can be determined using $2N$ settings, and in case they are real (or imaginary) N settings suffice.

Since the strongest separability inequality for the specific target state under consideration is chosen, this reduction in the number of settings does not reduce the noise robustness for detecting forms of entanglement as compared to that obtained using the entanglement criteria in terms of the usual settings $X_x^{(N)}$, etc.

In conclusion, if the state to be detected is known, the $2N$ settings of Eqs. (74) and (75) together with the single setting $\sigma_z^{\otimes N}$ suffice, and in case this state has solely real or imaginary antidiagonal matrix elements only $N + 1$ settings are needed. The white noise robustness using these settings is just as great as using the general condition that use the observables $X_x^{(N)}$ and $Y_x^{(N)}$, and is found by solving Eqs. (72) or (73) for detecting full and some entanglement, respectively.

As a final note, we observe that in order to determine the modulus of not just one but of all antidiagonal matrix elements it is more efficient to use the observables $X_x^{(N)}$, $Y_x^{(N)}$ than the observables of Eqs. (74) and (75). The first method needs 2^N settings to do this and the second needs $2^N N / 2$ settings (since there are $2^N / 2$ independent antidiagonal elements), i.e., the latter needs more settings than the former for all N .

Let us apply the above procedure to an example, taken from Ref. [15], the so-called four-qubit singlet state, which is given by

$$|\Phi_4\rangle = \frac{1}{\sqrt{3}} \left(|0011\rangle + |1100\rangle - \frac{1}{2} (|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle) \right). \tag{78}$$

For detecting it as fully entangled Eq. (72) gives a noise robustness $p_0 = 12/29 \approx 0.41$, and for detecting it as entangled Eq. (73) gives a noise robustness of $16/19 \approx 0.84$. The implementation needs $16 + 1 = 17$ settings.

This number of settings can be reduced by using the fact that this state has only real antidiagonal matrix elements and that we need only look at the largest antidiagonal element. As shown above, this matrix element can be mea-

sured in four settings. Thus the total number of settings required is reduced to only five. The off-diagonal matrix element to be determined is $|0011\rangle\langle 1100|$. The four settings that allow for this determination are obtained from the four settings given in Eq. (74) by applying the unitary operator $U_3 = 1 \otimes 1 \otimes \sigma_x \otimes \sigma_x$ to these settings.

For comparison, note that in Ref. [15] it was shown that the so-called projector-based witness for the state (78) detects full entanglement with a white noise robustness $p_0 = 0.267$ and uses 15 settings, whereas the optimal witness from [15] uses only three settings and has $p_0 = 0.317$. Here we obtain $p_0 \approx 0.41$ using five settings, implying a significant increase in white noise robustness using only two settings more.

This example gives the largest noise robustness when the conditions are measured in the standard z basis. However, sometimes one obtains larger noise robustness when the state is first rotated so as to be expressed in a different basis before it is analyzed. For example, consider the four qubit Dicke state $|2,4\rangle$, where $|l,N\rangle = \binom{N}{l}^{-1/2} \sum_k \pi_k (|1_1, \dots, 1_l, 0_{l+1}, \dots, 0_N\rangle)$ are the symmetric Dicke states [38] [with $\{\pi_k(\cdot)\}$ the set of all distinct permutations of the N qubits]. In the standard basis this state does not violate any of the separability conditions we have discussed above. However, if each qubit is rotated around the x axis by 90° all of the separability conditions can be violated with quite high noise robustness. Indeed, it is detected as inseparable under all splits through violation of conditions (50) for $p < p_0 = 16/19 \approx 0.84$ and as fully entangled through violation of condition (52) for $p < p_0 = 4/11 \approx 0.36$ using five settings. For comparison, Chen *et al.* [16] used specially constructed entanglement witnesses for detection of full entanglement in these states, and they obtained as noise robustness $p_0 = 2/9 \approx 0.22$ using only two settings. We have not performed an optimization procedure, so it is unclear whether or not the values obtained for p_0 can be improved.

B. Noise and decoherence robustness for the N -qubit GHZ state

In this section we determine the robustness of our separability criteria for detecting the N -qubit GHZ state in five kinds of noise processes (admixing white and colored noise, and three types of decoherence: depolarization, dephasing, and dissipation of single qubits). We give the noise robustness as a function of N for detecting some entanglement, inseparability with respect to all splits and full entanglement. We compare the results for white noise robustness of the criteria for full entanglement to that of the fidelity criterion (10) and to that of the so-called stabilizer criteria of Refs. [11,39].

The N -qubit GHZ state $|\Psi_{\text{GHZ},0}^N\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ can be transformed into a mixed state ρ_N by admixing noise to this state or by decoherence. Let us consider the following five such processes.

(i) Mixing in a fraction p of white noise (also called ‘‘generalized Werner states’’ [40]) gives

$$\rho_N^{(i)} = (1 - p) |\Psi_{\text{GHZ},0}^N\rangle\langle \Psi_{\text{GHZ},0}^N| + p \frac{1}{2^N}. \tag{79}$$

(ii) Mixing in a fraction p of colored noise [17] gives

$$\rho_N^{(ii)} = (1-p)|\Psi_{\text{GHZ},0}^N\rangle\langle\Psi_{\text{GHZ},0}^N| + \frac{p}{2}(|0\dots 0\rangle\langle 0\dots 0| + |1\dots 1\rangle\langle 1\dots 1|). \quad (80)$$

(iii) A depolarization process [18] with a depolarization degree p of a single qubit gives

$$\rho_N^{(iii)} = \frac{1}{2} \left\{ \left[\left(1 - \frac{p}{2}\right) |0\rangle\langle 0| + \frac{p}{2} |1\rangle\langle 1| \right]^{\otimes N} + \left[\frac{p}{2} |0\rangle\langle 0| + \left(1 - \frac{p}{2}\right) |1\rangle\langle 1| \right]^{\otimes N} + (1-p)^N (|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N}) \right\}. \quad (81)$$

(iv) A dephasing process [18] with a dephasing degree p of a single qubit gives

$$\rho_N^{(iv)} = \frac{1}{2} [|0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N} + (1-p)^N (|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N})]. \quad (82)$$

(v) A dissipation process [18] with a dissipation degree p of a single qubit (where the ground state is taken to be $|0\rangle$) gives

$$\rho_N^{(v)} = \frac{1}{2} \{ |0\rangle\langle 0|^{\otimes N} + [p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|]^{\otimes N} + (1-p)^{N/2} (|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N}) \}. \quad (83)$$

We now consider the question for what values of p these states $\rho_N^{(i)}$ to $\rho_N^{(v)}$ are detected as (i) containing some entanglement by the condition (58), and (ii) inseparable under any split by the conditions of the form (50) for all bipartite splits. In other words, we determine the noise (or decoherence) robustness of violations of all these conditions for $\rho_N^{(i)}$ to $\rho_N^{(v)}$. We find the following threshold values p_0 :

$$\begin{aligned} \text{(i)} \quad p_0 &= \frac{1}{1+2^{(1-N)}}, \\ \text{(ii)} \quad p_0 &= 1, \quad \forall N, \\ \text{(iii)} \quad (1-p_0)^N &= \left(1 - \frac{p_0}{2}\right)^\alpha \left(\frac{p_0}{2}\right)^{(N-\alpha)} \\ &\quad + \left(1 - \frac{p_0}{2}\right)^{(N-\alpha)} \left(\frac{p_0}{2}\right)^\alpha, \\ \text{(iv)} \quad p_0 &= 1, \quad \forall N, \\ \text{(v)} \quad p_0 &= 1, \quad \forall N, \end{aligned} \quad (84)$$

For cases (i), (ii), (iv), and (v) the threshold values p_0 for detecting some entanglement and inseparability with respect to all splits are the same because for these cases the product of the diagonal matrix elements $\rho_{j,j}\rho_{j,j}$ is the same for all $j \neq 1, d$. Only in case (iii) is this product different for differ-

ent j . We then have to take the minimum and maximum value, respectively, from which it follows that α is to be set to $\lfloor N/2 \rfloor$ for detecting some entanglement and to 1 for detecting inseparability with respect to all splits. Here $\lfloor N/2 \rfloor$ is the largest integer smaller or equal to $N/2$.

The result in case (i) is in accordance with the results of Refs. [2,4], where it is furthermore shown that the opposite holds as well, i.e., if and only if $p < 1/(1+2^{(1-N)})$ then $\rho_N^{(i)}$ is inseparable under any split and otherwise it is fully separable. Thus all states of the form (79) that are inseparable under any split are detected by violations of the conditions of the form (50) for all bipartite splits. The same holds for cases (ii), (iv), and (v), since all states $\rho_N^{(ii)}$, $\rho_N^{(iv)}$, and $\rho_N^{(v)}$ are inseparable under any split for all $p < 1$. In other words, as soon as a fraction of the GHZ state is present, these states are inseparable under any split. In case (i) p_0 increases monotonically from $p_0=2/3$ for $N=2$ to $p_0=1$ for large N . For process (iii) these limiting values are not so straightforward: $p_0=(3-\sqrt{3})/3 \approx 0.42$ for $N=2$, and $p_0=(5-\sqrt{5})/5 \approx 0.55$ for large N . In conclusion, the noise and decoherence robustness is high for all N , except maybe for case (iii).

Next, consider the noise robustness for detecting full entanglement by means of the biseparability condition (52). The result is the following:

$$\begin{aligned} \text{(i)} \quad p_0 &= 1/[2(1-2^{-N})], \\ \text{(ii)} \quad p_0 &= 1, \quad \forall N, \\ \text{(iii)} \quad p_0 &\approx 0.42, 0.28, 0.22, 0.18, \quad N=2, 3, 4, 5, \\ \text{(iv)} \quad p_0 &= 1, \quad \forall N, \\ \text{(v)} \quad p_0 &\approx 1, 0.48, 0.39, 0.35, \quad N=2, 3, 4, 5. \end{aligned} \quad (85)$$

For case (i) the noise robustness is equivalent to the fidelity criterion (10). For large N p_0 decreases to the limit value $p_0=1/2$. Cases (ii) and (iv) have $p_0=1$, thus as soon as the states $\rho_N^{(ii)}$ and $\rho_N^{(iv)}$ are entangled they are fully entangled. For cases (iii) and (v) we listed the noise robustness found numerically for $N=2$ to 5. These values decrease for increasing N .

Let us compare the results for white noise robustness [case (i)] to the results obtained from the so-called stabilizer formalism. This formalism [41] is used by Tóth and Gühne to derive entanglement witnesses [11,39] that are especially useful for minimizing the number of settings required to detect either full or some entanglement. Here we will only consider the criteria formulated for detecting entanglement of the N -qubit GHZ states. The stabilizer witness by Tóth and Gühne that detects some entanglement has $p_0=2/3$, independent of N , and requires only three settings [cf. Eq. (13) in [11]]. The strongest witness for full entanglement of Tóth and Gühne has a robustness $p_0=1/(3-2^{(2-N)})$ and requires only two settings [cf. Eq. (23) in [11]].

Figure 3 shows these threshold noise ratios for detecting full entanglement for these three criteria. Note that the criterion of Tóth and Gühne [11] needs only two measurement settings, whereas our criteria need $N+1$ settings. So although the former are less robust against white noise admixture,

they compare favorably with respect to minimizing the number of measurement settings.

Although we give a criterion for full entanglement that is generally stronger than the fidelity criterion, for the N -partite GHZ state this does not lead to better noise robustness. It appears that for large N the noise threshold $p_0=1/2$ is the best one can do. However, in the limit of large N the GHZ state is inseparable under all splits for all $p_0 < 1$, as was shown in (i) in Eq. (84), see also Fig. 3. Furthermore, we have seen that if the state $\rho_N^{(i)}$ (i.e., the GHZ state with a fraction p of white noise) is entangled it is also inseparable under any split. Because of the high symmetry of both the GHZ state and white noise, one might conjecture that if the state $\rho_N^{(i)}$ is entangled it is also fully entangled. At present, however, it is unknown whether this is indeed true. Detecting the states $\rho_N^{(i)}$ as fully entangled appears to be a much more demanding task than detecting them as inseparable under all splits. In the first case, for large N , only a fraction of 50% noise is permitted, in the second case one can permit any noise fraction (less than 100%). Note that we have given explicit examples of states that are diagonal in GHZ basis [cf. Eq. (14) of Sec. II B], and that are inseparable under any split, but not fully entangled; but these are not of the form $\rho_N^{(i)}$.

Lastly, we mention that our criteria detect the various forms of entanglement and inseparability also if the state $|\Psi_{\text{GHZ},0}^N\rangle$ is replaced by any other maximally entangled state [i.e., any state of the GHZ basis, cf. Eq. (13)], a feature which is not possible using linear entanglement witnesses. There is no single linear witness that detects entanglement of all maximally entangled states.

C. Detecting bound entanglement for $N \geq 3$

Violation of the separability inequality (58) allows for detecting all bound entangled states of Ref. [42]. These states have the form

$$\rho_B = \frac{1}{N+1} \left(|\Psi_{\text{GHZ},\alpha}^N\rangle\langle\Psi_{\text{GHZ},\alpha}^N| + \frac{1}{2} \sum_{l=1}^N P_l + \bar{P}_l \right), \quad (86)$$

with P_l the projector on the state $|0\rangle_1 \cdots |1\rangle_l \cdots |0\rangle_N$, and where \bar{P}_l is obtained from P_l by replacing all zeros by ones and vice versa. For $N \geq 4$ these states are entangled and have positive partial transposition (PPT) with respect to transposition of any qubit. This means they are bound entangled [43]. Note that they are detected as entangled by the N -partite Mermin inequality $|M_N| \leq 2$ of Sec. III C only for $N \geq 8$ [42]. However, the condition (58) detects them as entangled for $N \geq 4$. Thus all bound entangled states of this form are detected as entangled by this latter condition. The white noise robustness for this purpose is $p_0 = 2^N / (2 + 2N + 2^N)$, which for $N=4$ gives $p_0 = 8/13 \approx 0.615$ and goes to 1 for large N . Note that for $N=4$, this state violates the condition for 4-separability, and the condition for 3-separability (60), but not the condition for 2-separability. It is thus at least 2-separable entangled. It is not detected as fully entangled by these criteria. (Of course, it could still be fully entangled since these criteria are only sufficient and not

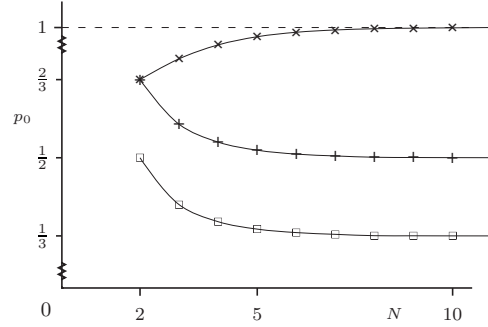


FIG. 3. The threshold noise ratios p_0 for detection of full N -qubit entanglement when admixing white noise to the N -qubit GHZ state for the criterion (52) derived here (plus signs) and for the stabilizer witness of Ref. [11] (squares). The noise robustness for detecting inseparability under all splits as given in (i) in Eq. (84) is also plotted (crosses).

necessary for entanglement.) For general N we have not investigated the k -separable entanglement of the states (86), although this can be readily performed using the criteria of Eq. (60).

Another interesting bound entangled state is the so-called four-qubit Smolin state [44]

$$\rho_S = \frac{1}{4} \sum_{j=1}^4 |\Psi_{ab}^j\rangle\langle\Psi_{ab}^j| \otimes |\Psi_{cd}^j\rangle\langle\Psi_{cd}^j|, \quad (87)$$

where $\{|\Psi^j\rangle\}$ is the set of four Bell states $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$, and a, b, c, d label the four qubits. This state is also detected as entangled by the criterion (58), and with white noise robustness $p_0=2/3$. The Smolin state violates the separability conditions (50) for biseparability under the splits a -(bcd), b -(acd), c -(abd), d -(abc). However, it is separable under the splits (ab) -(cd), (ac) -(bd), (ad) -(bc) (cf. [44]). This state is thus inseparable under splits that partition the system into two subsets with one and three qubits, but it is separable when each subset contains two qubits.

So far we have detected bound entanglement for $N \geq 4$. What about $N=3$? Consider the three-qubit bound entangled state of [3]:

$$\rho = \frac{1}{3} |\Psi_{\text{GHZ},0}^3\rangle\langle\Psi_{\text{GHZ},0}^3| + \frac{1}{6} (|001\rangle\langle 001| + |010\rangle\langle 010| + |101\rangle\langle 101| + |110\rangle\langle 110|). \quad (88)$$

This state is detected as entangled by the criterion (35), with white noise robustness $p_0=4/7 \approx 0.57$. It violates the biseparability condition (28) for the split a -(bc) so it is at least biseparable entangled, but does not violate the condition (34) for biseparability, i.e., it is not detected as fully entangled. In fact, it can be shown using the results of Ref. [4] that this state is separable under the splits b -(ac) and c -(ab).

V. DISCUSSION

We have discussed partial separability of quantum states by distinguishing k -separability and α_k -separability and used these distinctions to extend the classification proposed by

Dür and Cirac. We discussed the relationship of partial separability to multipartite entanglement and distinguished the notions of a k -separable entangled state and a m -partite entangled state and indicated the interrelations of these kinds of entanglement.

Next, we have presented necessary conditions for partial separability in the hierarchic separability classification. These are formulated in terms of experimentally accessible correlation inequalities for operators defined by products of local orthogonal observables. Violations of these inequalities provide, for all N -qubit states, criteria for the entire hierarchy of k -separable entanglement, ranging from the levels $k=1$ (full or genuine N -particle entanglement) to $k=N$ (full separability, no entanglement), as well as for specific classes within each level. Choosing the Pauli matrices as the locally orthogonal observables provided matrix representations of the criteria that bound antidiagonal matrix elements in terms of diagonal ones.

Further, the N -qubit Mermin-type separability inequalities for partial separability were shown to follow from the partial separability conditions derived in this paper. The biseparability conditions are stronger than the fidelity criterion and the Laskowski-Żukowski criterion, and the latter criterion is also shown to be strengthened for full separability and biseparability. For separability under splits the conditions are stronger than the Dür-Cirac conditions. Violation of these conditions thus give entanglement criteria that detect more entangled states than violations of these three other separability conditions.

We have furthermore shown that the required number of measurement settings for implementation of these criteria, which is 2^N+1 in general, can be drastically reduced if entanglement of a given target state is to be detected. In that case, it may be reduced to $2N+1$, and for multiqubit states with either real or imaginary antidiagonal matrix elements, only $N+1$ settings are needed.

When comparing the entanglement criteria to other state-specific multiqubit entanglement criteria it was found that the white noise robustness was high for a great variety of interesting multiqubit states, whereas the number of required settings was only $N+1$. However, these other state-specific entanglement criteria need less settings although for the states analyzed here they give lower noise robustness. Analyzing some specific target states shows that the entanglement criteria detect bound entanglement for $N \geq 3$.

Furthermore, we applied the entanglement criteria for some and full entanglement to the N -qubit GHZ state subjected to two different kinds of noise and three different kinds of decoherence. The robustness against colored noise and against dephasing turns out to be maximal (i.e., $p_0=1$) both for detecting some and full entanglement. It is remarkable that for large N the GHZ state allows for maximal white noise robustness for the state to remain inseparable under all possible splits, whereas for detecting full entanglement the best known result—to our best knowledge—only allows for a white noise robustness of $p_0=1/2$. It would be very interesting to search for full entanglement criteria that can close this gap, or if this is shown to be impossible to understand why this is the case.

Orthogonality of the local observables is crucial in the above derivation of separability conditions. It is due to this assumption that the multiqubit operators form representations of the generalized Pauli group. It would be interesting to analyze the role of orthogonality in deriving the inequalities. For two qubits it has been shown [45] that when orthogonality is relaxed the separability conditions become less strong, and we conjecture the same holds for their multiqubit analogs. Relaxing the requirement of orthogonality has the advantage that some uncertainty in the angles may be accommodated, which is desirable since in real experiments it may be hard to measure perfectly orthogonal observables.

It is also interesting that the separability inequalities are equivalent to bounds on antidiagonal matrix elements in terms of products of diagonal ones. We thus gain a novel perspective on why they allow for entanglement detection: they probe the values of antidiagonal matrix elements, which encode entanglement information about the state; and if these elements are large enough, this entanglement is detected. Note, furthermore, that compared to the Mermin-type separability inequalities we need not do much more to obtain our stronger inequalities. We must solely determine some diagonal matrix elements, and this can be easily performed using the single extra setting $\sigma_z^{\otimes N}$. It is also noteworthy that the comparison to the Mermin-type separability inequalities shows that the strength of the correlations allowed for by separable states is exponentially decreasing when compared to the strength of the correlations allowed for by LHV models.

Our recursive definition of the multipartite correlation operators [see Eq. (48)] is by no means unique. One can generate many new inequalities by choosing the locally orthogonal observables differently, e.g., by permuting their order in each triple of local observables. It could well be that combining such inequalities with those presented here yield even stronger separability conditions, as is indeed the case for pure two-qubit states, cf. [30]. Unfortunately, we have no conclusive answers for this open question.

We end by suggesting three further lines of future research. First, it would be interesting to apply the entanglement criteria to an even larger variety of N -qubit states than analyzed here, including, for example, all N -qubit graph and Dicke states. Second, the generalization from qubits to qudits (i.e., d -dimensional quantum systems) would, if indeed possible, prove very useful since strong partial separability criteria for N qudits have—to our knowledge—not yet been obtained. Finally, it would be beneficial to have optimization procedures for choosing the set of local orthogonal observables featuring in the entanglement criteria that gives the highest noise robustness for a given set of states. We believe we have chosen such optimal sets for the variety of states analyzed here, but since no rigorous optimization was performed, our choices could perhaps be improved.

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- [1] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. Lett. **89**, 260401 (2002).
- [2] W. Dür and J. I. Cirac, Phys. Rev. A **61**, 042314 (2000).
- [3] W. Dür and J. I. Cirac, J. Phys. A **34**, 6837 (2001).
- [4] W. Dür, J. I. Cirac, and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999).
- [5] W. Laskowski and M. Żukowski, Phys. Rev. A **72**, 062112 (2005).
- [6] J. Uffink, Phys. Rev. Lett. **88**, 230406 (2002).
- [7] M. Seevinck and G. Svetlichny, Phys. Rev. Lett. **89**, 060401 (2002).
- [8] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).
- [9] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).
- [10] S. M. Roy, Phys. Rev. Lett. **94**, 010402 (2005).
- [11] G. Tóth and O. Gühne, Phys. Rev. A **72**, 022340 (2005).
- [12] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, Nature (London) **404**, 256 (2000).
- [13] M. Seevinck and J. Uffink, Phys. Rev. A **65**, 012107 (2001).
- [14] B. Zeng, D. L. Zhou, P. Zhang, Z. Xu, and L. You, Phys. Rev. A **68**, 042316 (2003).
- [15] O. Gühne, C.-Y. Lu, W.-B. Gao, and J.-W. Pan, Phys. Rev. A **76**, 030305(R) (2007).
- [16] L. Chen and Y.-X. Chen, Phys. Rev. A **76**, 022330 (2007).
- [17] A. Cabello, A. Feito, and A. Lamas-Linares, Phys. Rev. A **72**, 052112 (2005).
- [18] S. S. Jang, Y. W. Cheong, J. Kim, and H. W. Lee, Phys. Rev. A **74**, 062112 (2006).
- [19] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
- [20] O. Gühne, G. Tóth, and H. J. Briegel, New J. Phys. **7**, 229 (2005).
- [21] G. Tóth and O. Gühne, Appl. Phys. B: Lasers Opt. **82**, 237 (2006).
- [22] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
- [23] Häffner *et al.*, Nature (London) **438**, 643 (2005).
- [24] For example, consider the following construction (inspired by Ref. [21]) where we use the N -qubit states $|\psi_1\rangle = |e\rangle_{12}|0\rangle_3|0\rangle_4\cdots|0\rangle_N$; $|\psi_2\rangle = |0\rangle_1|e\rangle_{23}|0\rangle_4\cdots|0\rangle_N$; \dots , $|\psi_N\rangle = |0\rangle_1|0\rangle_2|0\rangle_3\cdots|e\rangle_{N-1,N}$, where $|e\rangle_{ij}$ is any entangled pure state of the two parties i and j . Then the state $\rho = \sum_{i=1}^N |\psi_i\rangle\langle\psi_i|/N$ is inseparable under all splits, yet by construction $(N-1)$ -separable.
- [25] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999); T. Eggeling and R. F. Werner, Phys. Rev. A **63**, 042111 (2001); A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
- [26] S. Walck and D. Lyons, Phys. Rev. Lett. **100**, 050501 (2008).
- [27] W. Dür, Phys. Rev. A **63**, 020303(R) (2001).
- [28] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); S. M. Roy and V. Singh, *ibid.* **67**, 2761 (1991); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A. V. Belinskii and D. N. Klyshko, Usp. Fiz. Nauk **163-165**, 1 (1993) [Phys. Usp. **36**, 653 (1993)].
- [29] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. A **65**, 042314 (2002).
- [30] J. Uffink and M. Seevinck, Phys. Lett. A **372**, 1205 (2008).
- [31] In the standard z basis, $\rho_{i,j} = \langle i'|\rho|j'\rangle$ with $i' = i-1$, $j' = j-1$ and where $i' = i_1i_2\cdots i_N$ and $j' = j_1j_2\cdots j_N$ are in binary notation. For example, for $N=4$: $\rho_{1,16} = \langle 0000|\rho|1111\rangle$ and $\rho_{9,12} = \langle 1000|\rho|1011\rangle$.
- [32] Y. Ota, M. Yoshida, and I. Ohba, e-print arXiv:0704.1375.
- [33] M. Żukowski, Č. Brukner, W. Laskowski, and M. Wiesniak, Phys. Rev. Lett. **88**, 210402 (2002).
- [34] R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000).
- [35] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).
- [36] B. M. Terhal, Theor. Comput. Sci. **287**, 313 (2002).
- [37] O. Gühne and P. Hyllus, Int. J. Theor. Phys. **42**, 1001 (2003).
- [38] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
- [39] G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005).
- [40] A. O. Pittenger and M. H. Rubin, Opt. Commun. **179**, 447 (2000).
- [41] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
- [42] W. Dür, Phys. Rev. Lett. **87**, 230402 (2001).
- [43] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [44] J. A. Smolin, Phys. Rev. A **63**, 032306 (2001).
- [45] M. Seevinck and J. Uffink, Phys. Rev. A **76**, 042105 (2007).