Decoherence limits quantum computation

Or does it?

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Outline

Fighting decoherence:

- Magic state distillation

Befriending decoherence:

- Measurement-based quantum computation
- Quantum computation by dissipation

Part I:

Quantum computation with magic states

S. Bravyi and A. Kitaev, Phys. Rev. A, 2005

Undoing decoherence

Capstone result:

Threshold theorem. Given [fill in a suitable error model], if the error per elementary gate in a quantum computer is below a critical threshold, arbitrarily long and arbitrarily accurate quantum computation is possible.

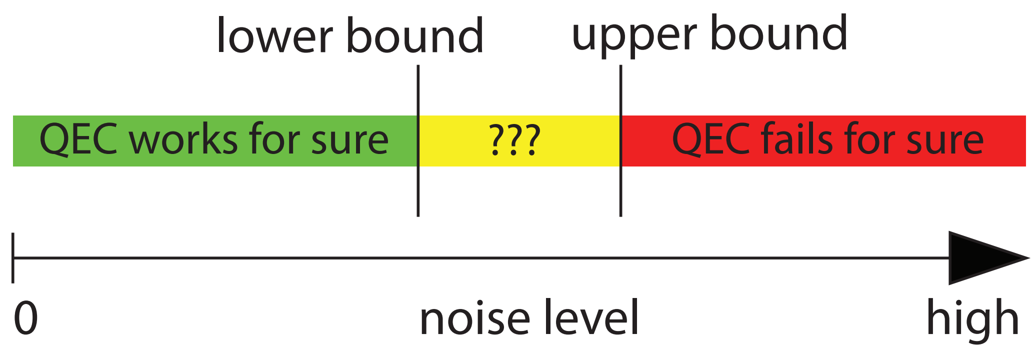
Q: *What is the noise threshold?*

Noise threshold for fault-tolerance

... exact value may be hard to calculate. Instead derive

- **Upper bound:** For a given set of computational primitives, if the noise level exceeds the upper bound, then *no method*, however clever, can achieve fault-tolerance.

- **Lower bound:** For a given set of computational primitives, if the noise level is less than the lower bound, then at least *one method* makes the computation fault-tolerant.



Quantum computation using magic states

We consider the computational primitives

$$\{\text{CNOT-gate}, \text{Hadamard-gate}, |T\rangle\}$$

Therein,

$$|T\rangle = \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}. \tag{1}$$

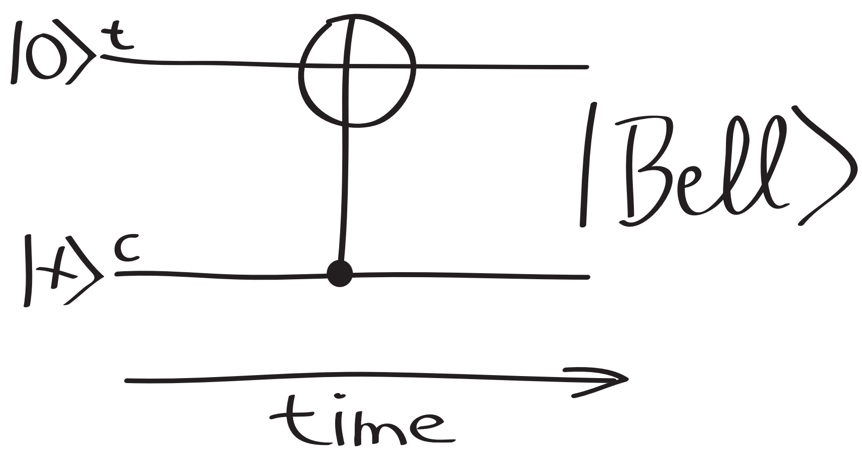
is the "magic" state.

Computational primitives - CNOT

- The CNOT gate is a two-qubit gate. It acts as

$$\text{CNOT}_{c,t} = |0\rangle_c\langle 0| \otimes I^{(t)} + |1\rangle_c\langle 1| \otimes \sigma_x^{(t)}. \qquad (2)$$

- The CNOT is the only computational primitive in the set which has the power to *entangle*.



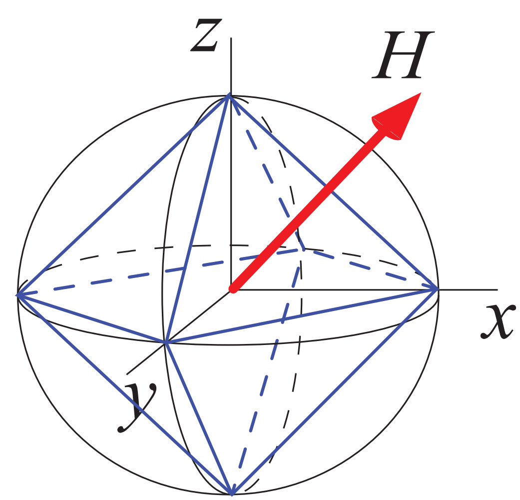
Computational primitives - Hadamard

- The Hadamard gate H acts as

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|, \tag{3}$$

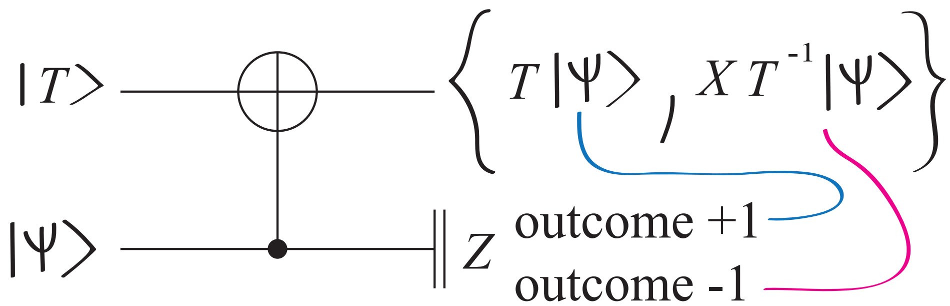
where $|\pm\rangle := 1/\sqrt{2}(|0\rangle \pm |1\rangle)$.

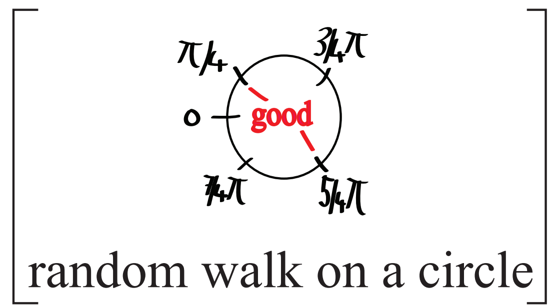
- The Hadamard gate rotates the Bloch sphere of a qubit by an angle of π about the axis in the middle between x and z.



Computational primitives - magic state $|T\rangle$

- State $|T\rangle = \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}$ implements gate $T = \exp(i\pi/8Z)$.





- Gate construction is probabilistic. Repeat until success.

- The used primitives are universal for quantum computation!
  Any unitary in $U \in SU(2^n)$, for any n, can be built as a sequence of the above gates.

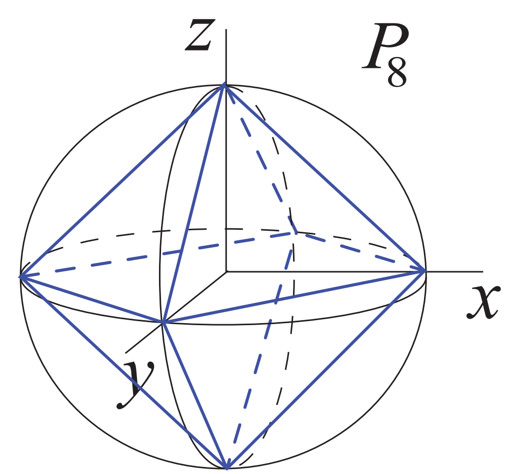
Decoherence model

Computational primitive	Quality
CNOT-gate	perfect
Hadamard-gate	perfect
magic state $\|T\rangle$	noisy

- Instead of pure states $|T\rangle$ have states $\rho_T \approx |T\rangle\langle T|$. Use fidelity $F(\rho_T) = \sqrt{\langle T|\rho_T|T\rangle}$ as measure for quality.
- Motivation for this noise model: certain versions of topological quantum computation which are non-universal.

Bravyi & Kitaev's results

Result 1. If the noisy magic states ρ_T are inside the octahedron P_8 inscribed in the Bloch sphere, then quantum computation using the primitives $\{\mathrm{CNOT}, H, \rho_T\}$ can be efficiently classically simulated.

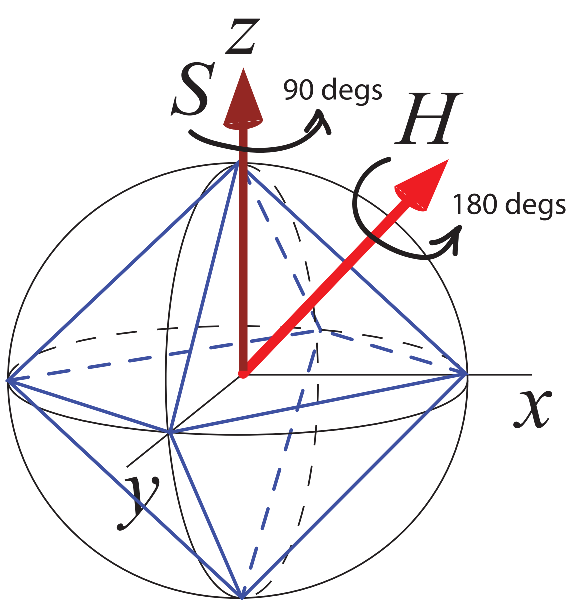


Result 2 [*Magic state distillation*]. If the noisy magic states ρ_T are such that $F(\rho_T) \geq 0.927$, then arbitrarily long and accurate universal quantum computation is possible with the primitives $\{\mathrm{CNOT}, H, \rho_T\}$.

Derivation of Result 1

- *How powerful is the gate set* $\{CNOT, H, S = \exp(2\times i\pi/8Z)\}$*?*
- *Not powerful at all. It is efficiently classically simulatable.*

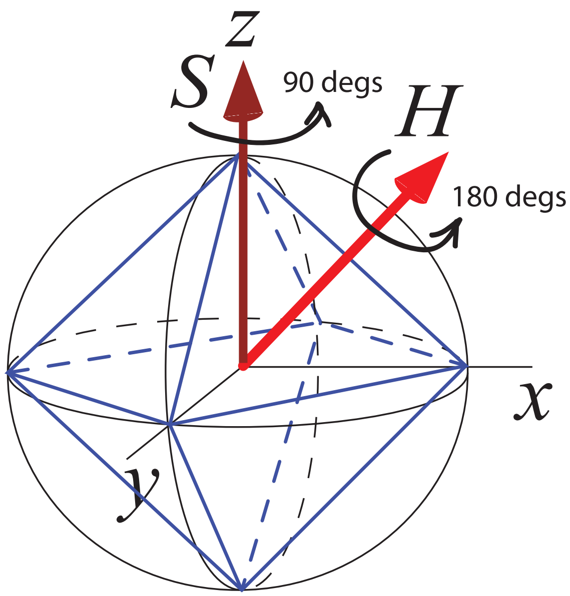
First, consider the one-qubit gates H, $S = \exp(i\pi/4\,Z)$:



- H, S leave the octahedron invariant.
- H, S generate the octahedral group. Not dense in $SU(2)$, hence no 1-qubit universality.

Heisenberg picture

Consider the action of H, $S = \exp(i\pi/4\,Z)$, $CNOT$ on Pauli operators:



- $HXH^\dagger = Z$, $HZH^\dagger = X$.
- $SXS^\dagger = Y$, $SZS^\dagger = Z$.
- CNOT $X^{(c)}\,CNOT^\dagger = X^{(c)} \otimes X^{(t)}$ ...

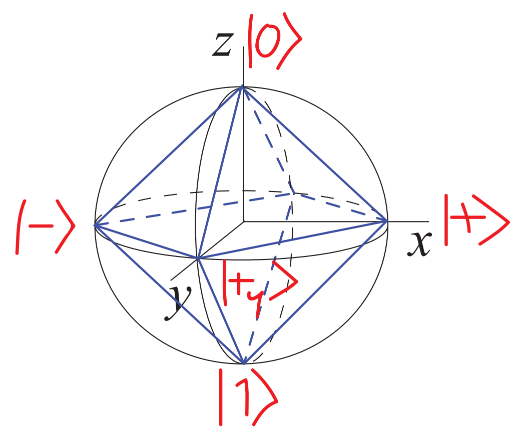
- Above gates map Pauli operators onto Pauli operators.

$\Rightarrow$ Evolution easily trackable in Heisenberg picture.

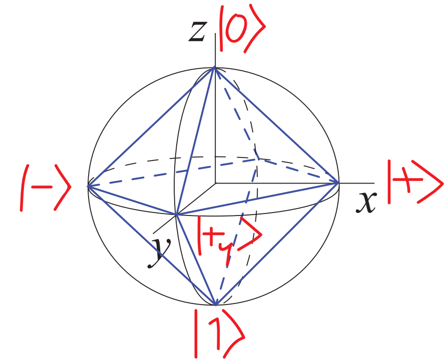
Gottesman-Knill theorem

$\Rightarrow$ Evolution easily trackable in Heisenberg picture. Leads to Gottesman-Knill Theorem:

Theorem 1. Quantum computation with $\{\text{CNOT}, H, S\}$, on initial qubit states $|0/1\rangle$, $|\pm\rangle$, $|\pm_y\rangle$, and with readout measurements in the X, Y or Z-basis can be efficiently classically simulated.



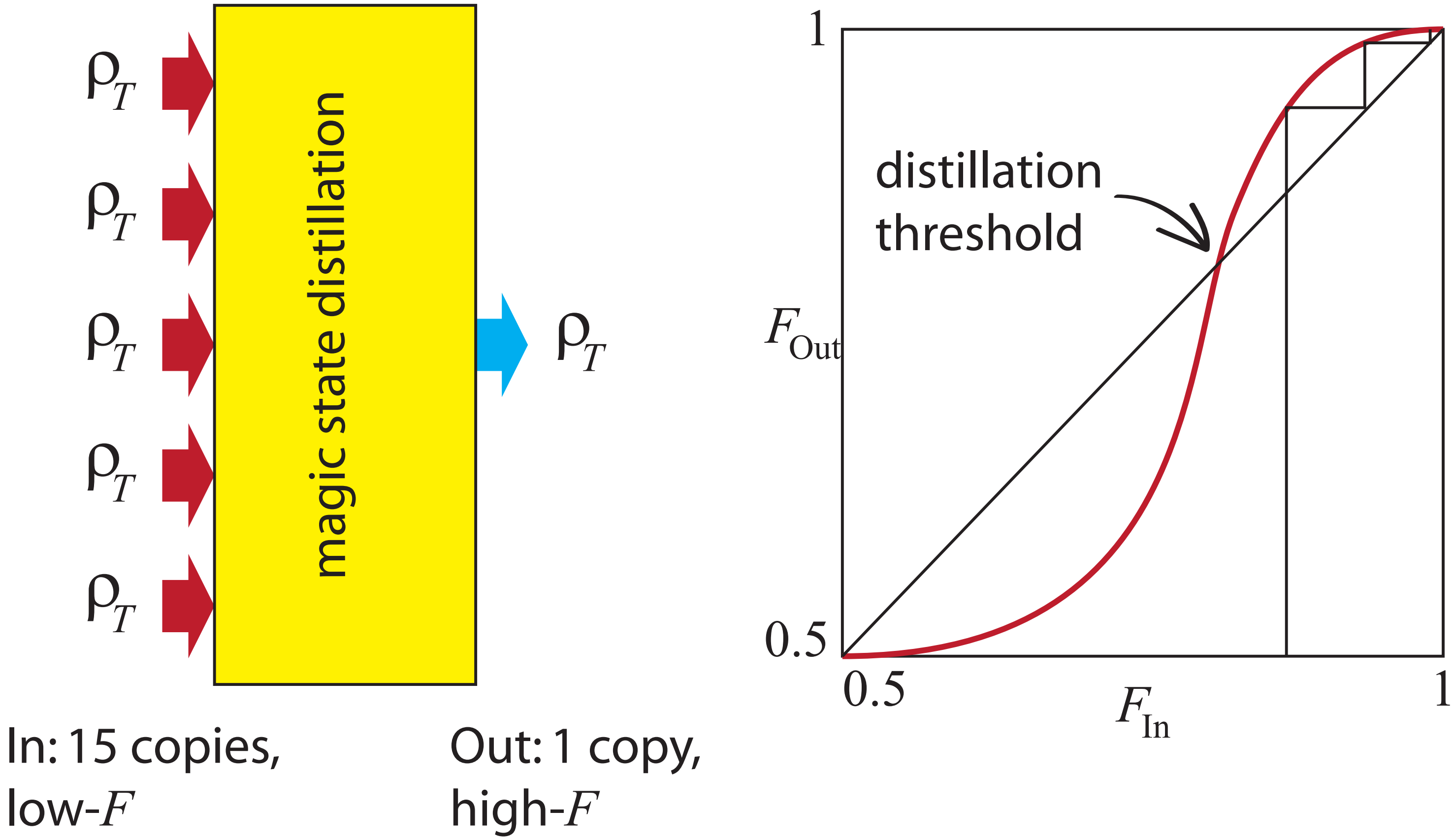
Result 1 of Bravyi & Kitaev



- The only computational primitive that evades the Gottesman-Knill theorem is the magic state $|T\rangle$.
- Can the noisy state ρ_T be described as a *probabilistic mixture* of $\{|0, 1\rangle, |\pm\rangle, |\pm_y\rangle\}$?
- If yes, then the computation can be efficiently simulated using the Gottesman-Knill theorem + Monte Carlo sampling.

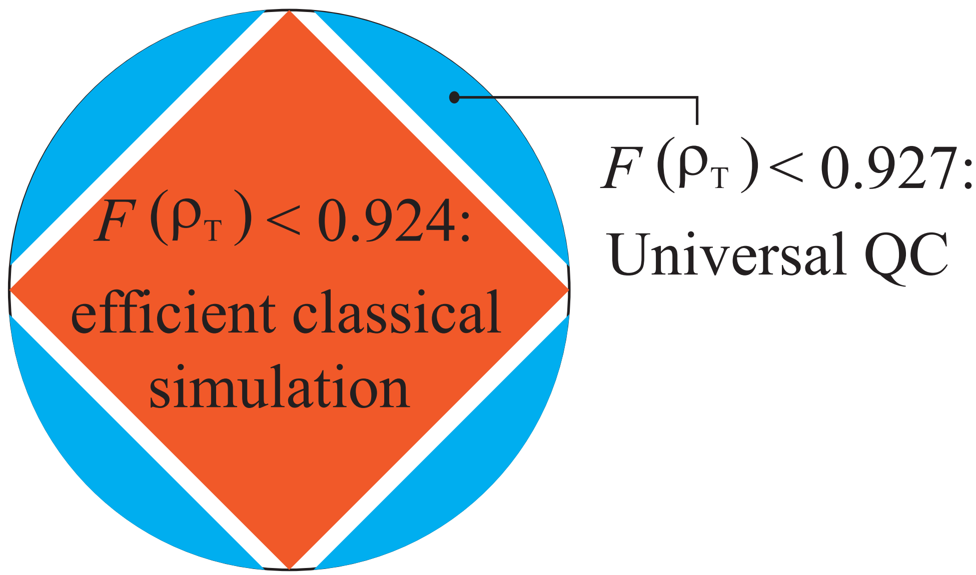
Result 1. If the noisy magic states ρ_T are inside the octahedron P_8 inscribed in the Bloch sphere, then quantum computation using the primitives $\{\text{CNOT}, H, \rho_T\}$ can be efficiently classically simulated.

Result 2: Magic state distillation



Result 2 [*Magic state distillation*]. If the noisy magic states ρ_T are such that $F(\rho_T) \geq 0.927$, then arbitrarily long and accurate universal quantum computation is possible using the primitives $\{\mathrm{CNOT}, H, \rho_T\}$.

Results 1 & 2



X/*Y* - equator of the Bloch sphere

But where's the magic?

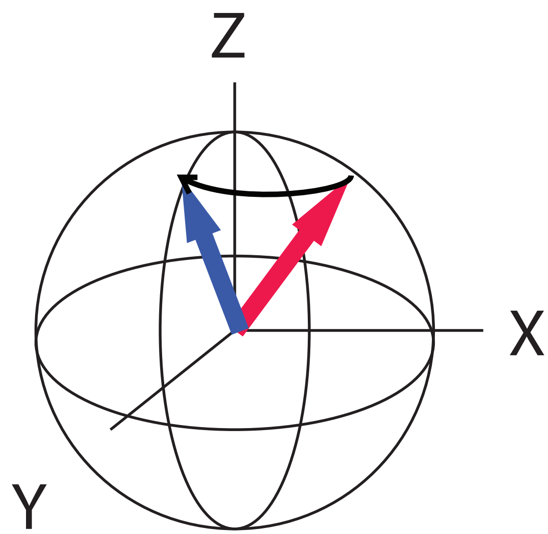
Part II:

Measurement-based quantum computation

R. Raussendorf and H.J. Briegel, PRL 86, 5188 (2001).

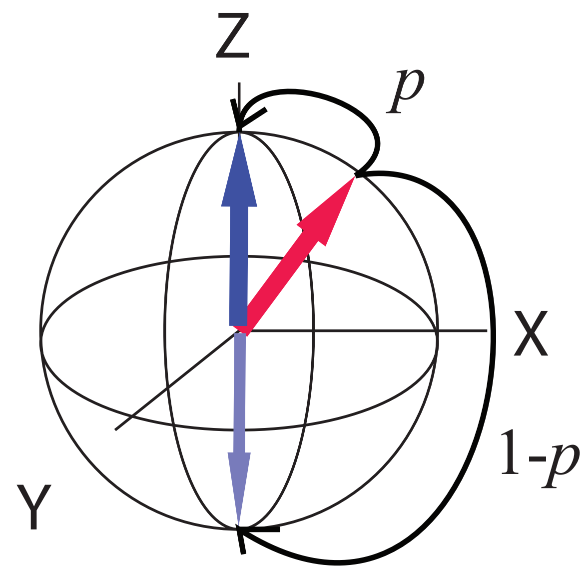
Measurement-based Quantum Computation

Unitary transformation



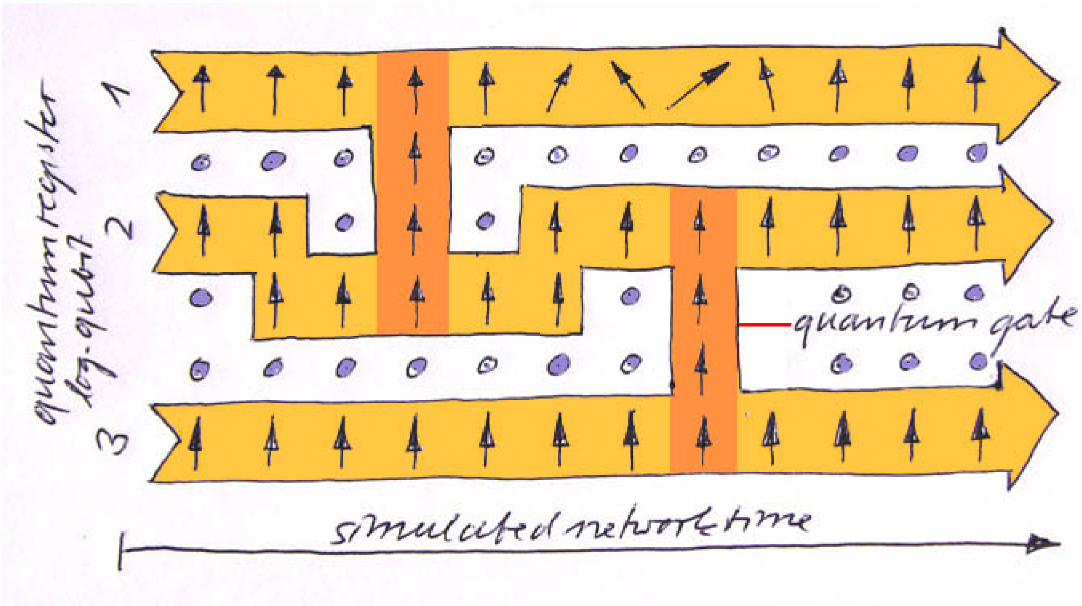
deterministic,
reversible

Projective measurement



probabilistic,
irreversible

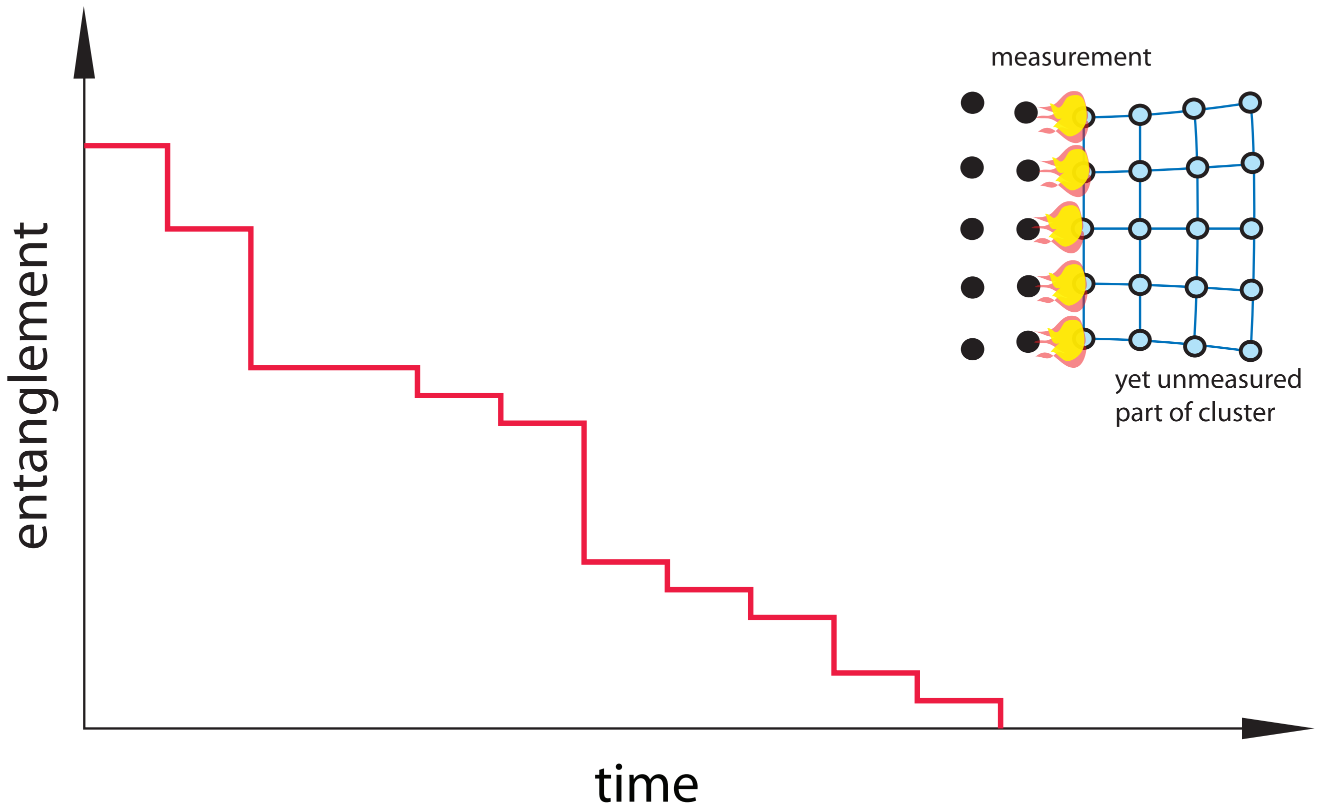
The one-way quantum computer



measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

Trading entanglement for output



- Intuition: **Entanglement = Resource**

Part III:

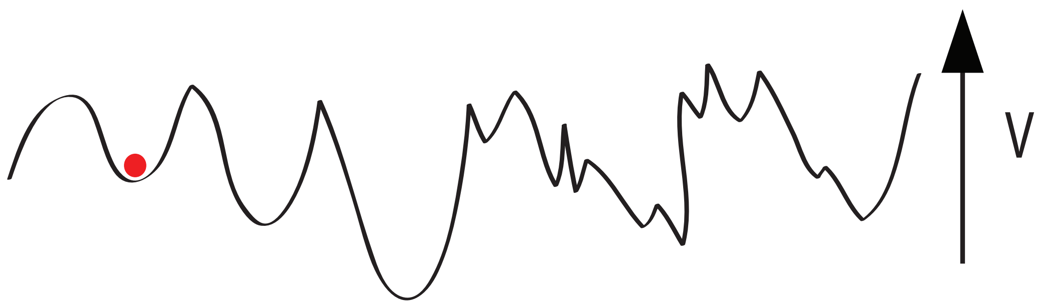
Quantum computation by dissipation

F. Vertraete, M. Wolf and J.I. Cirac, Nature Phys. 5 (2009).

Computing fridges

- Cooling into the ground state of a simple (3-body, say) Hamiltonian were an incredibly powerful computational tool ...

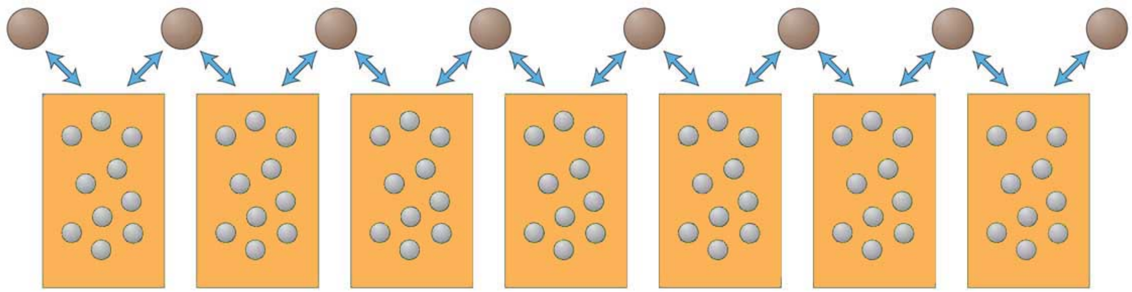
  If one could avoid local minima.



- **Bold task:** Solve NP-complete problems by cooling.
- *Task:* Universal quantum computation by cooling.

Computing fridges

Result 3. Consider a quantum circuit of n qubits and T gates, $|\Psi_{\text{out}}\rangle = U_T U_{T-1} ..U_2 U_1 |0\rangle$. The output of this quantum computation can be efficiently simulated by local *dissipative* evolution on $n + T$ qubits.



*

Why $n + T$ qubits?

Total Hilbert space $\mathcal{H} = \underbrace{\mathcal{H}_{\text{Q-register}}}_{n \text{ qubits}} \otimes \underbrace{\mathcal{H}_{\text{clock}}}_{T \text{ qubits}}.$

*: Image adapted from Nature Physics.

Computing fridges

Consider dissipative evolution described by Lindblad equation with a Liouville operator

$$\mathcal{L}(\rho) = \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho\right\}_+, \tag{4}$$

where

$$\begin{aligned} L_i &= |0\rangle_i\langle 1| \otimes |0\rangle\langle 0|, & \forall i = 1..n, && \text{(initialize QR)} \\ L_t &= U_t \otimes |t+1\rangle\langle t| + \text{h.c.}, & \forall t = 1..T, && \text{(advance clock)} \end{aligned} \tag{5}$$

Computing fridges

The above dissipative evolution has the following properties

1. Unique fixpoint is a *history state*

$$\rho_0 = \frac{1}{T+1} \sum_t |\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t|, \tag{6}$$

   where $|\psi\rangle_t =$ state of QR at time t.

2. Liouville operator has a spectral gap $\Delta \sim 1/T^2$.
   Good approximation to ρ_0 is reached in poly time $\tau \sim T^2$.

Final step of the computation: After evolution for time τ, measure the clock register. If obtain $t = T$ then read out $|\Psi_{\text{out}}\rangle$. Otherwise start over.

Why decoherence did not hurt in Ex. **II**, **III**?

•

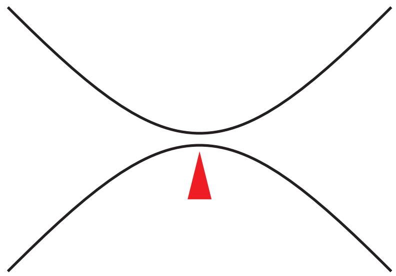
Closing remark on entanglement

Why did decoherence not hurt in Ex. II, III ?

Because we only depleted
coherences
that we didn't care about.

Example III - Universal AQC vs. DQC

Adiabatic QC (unitary)

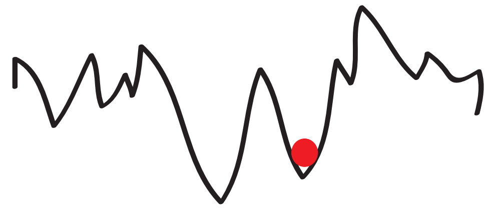


$H(t) = H_I\,(1-t) + H_F\,t$

initial Hamiltonian:
GS easy to prepare

final Hamiltonian:
encodes comp. result
in its ground state

Dissipative QC



H_I, H_F exist such that[1]

1. GS is history state:
   $|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_t |\psi_t\rangle |t\rangle$
2. Min gap $\sim 1/T^2$.

$\mathcal{L}$ exists such that

1. FP is history state:
   $\rho_0 = \frac{1}{T+1} \sum_t |\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t|$
2. Gap $\sim 1/T^2$.

1: D. Aharonov et al., arXiv:quat-ph/0405098 (2004).

Example III - Universal AQC vs. DQC

Adiabatic QC:

H_I, H_F exist such that

1. GS is history state:
   $|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_t |\psi_t\rangle |t\rangle$
2. Min gap $\sim 1/T^2$.

Dissipative QC:

$\mathcal{L}$ exists such that

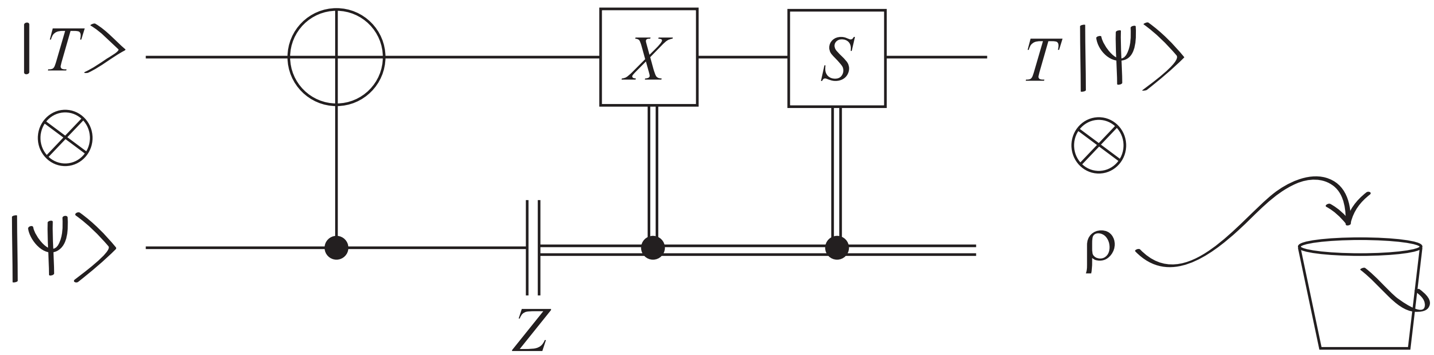
1. FP is history state:
   $\rho_0 = \frac{1}{T+1} \sum_t |\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t|$
2. Gap $\sim 1/T^2$.

- Adiabatic QC: history state is a coherent superposition,
  Dissipative QC: history state is a mixture.
- Recall: Measure clock at end of computation.

$\Rightarrow$ Coherence between clock states is not important.

Example II - Measurement-based QC

Instead of the one-way QC, look at simpler example:



Recall: $S = \exp(i\pi/4\,Z)$, $T = \exp(i\pi/8\,Z)$

- Decohered is only the post-measurement state or the lower qubit, which we discard.
- Again, decoherence does not affect the computational degrees of freedom.

Remark on entanglement

Do the dissipative evolutions discussed in Examples II, III drive the respective system to a "classical" state?

- One-way QC: Yes. Final state is a product state.
- Dissipative QC: No. Final state is highly entangled*.

*: The entanglement of the history state ρ_0 equals the time-averaged entanglement of the circuit quantum register.