Laplacian Growth

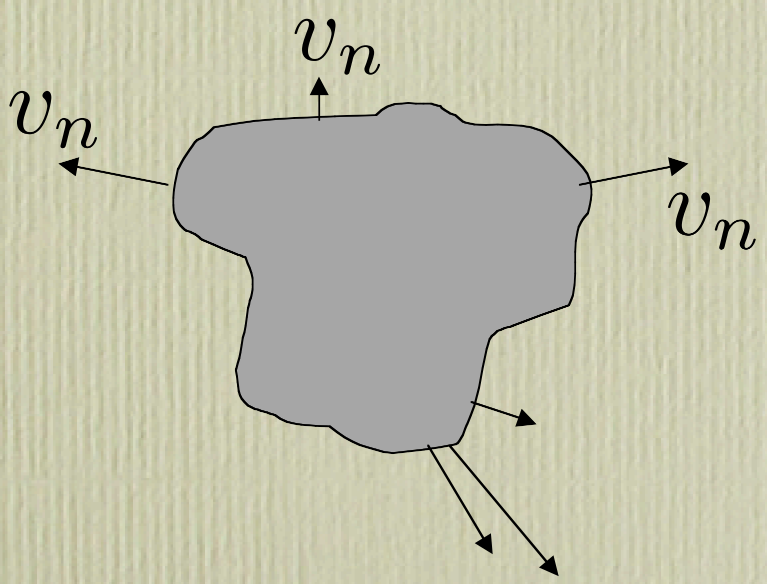
Singular limit of dispersive waves

and

related problems in quantum and classical physics

P. Wiegmann
University of Chicago

Laplacian growth is an evolution of a plane interface between two immiscible phases, driven by a gradient of a harmonic field



$$\vec{v}_n = \vec{\nabla} p$$

$$\Delta p = 0$$

$$p \to \log|z|, \quad |z| \to \infty$$

$$p = 0 \quad \text{on the boundary}$$

normal velocity = *gradient of harmonic field*

- Deterministic Growth
- Stochastic Growth

Zabrodin

Agam

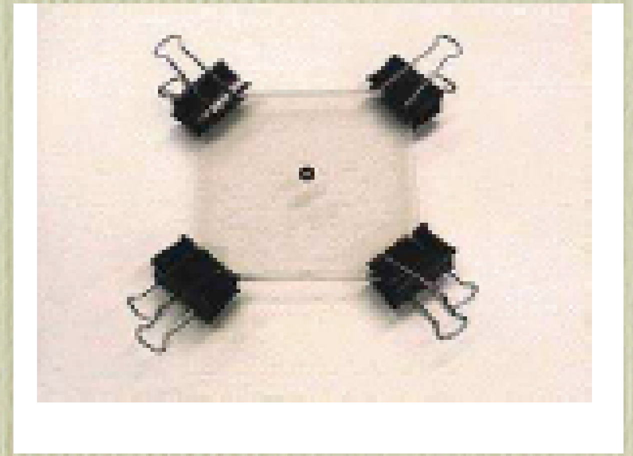
Bettelheim

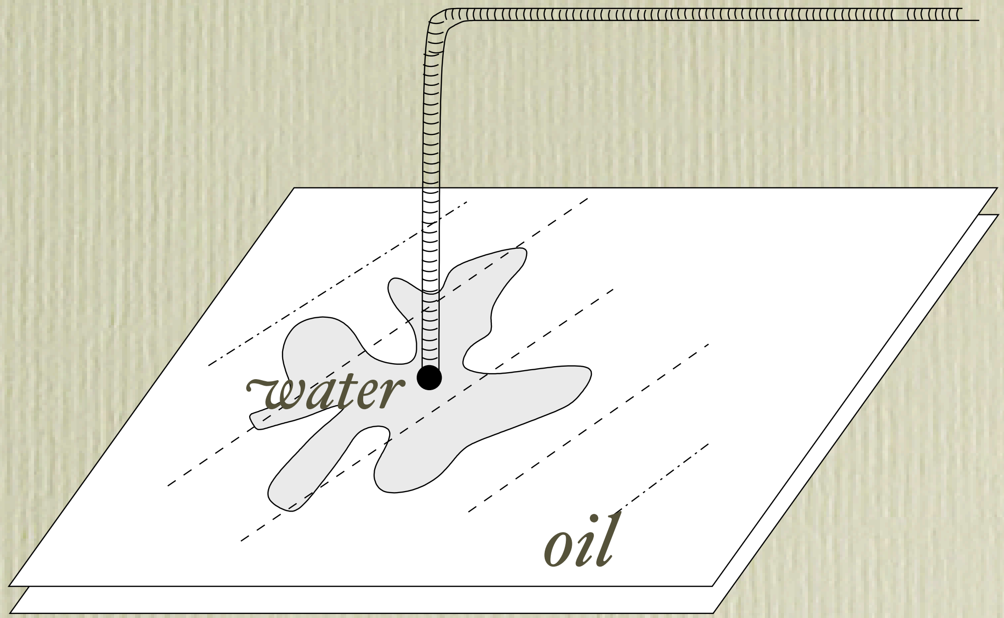
Krichever

Mineev-Weinstein

Teodorescu

Hele-Shaw cell





Oil (exterior)-incompressible liquid with high viscosity

Water (interior) - incompressible liquid with high viscosity

Laplacian growth;

diffusion driven patterns.

$$v_n = -\nabla_n P \quad \text{on the interface}$$

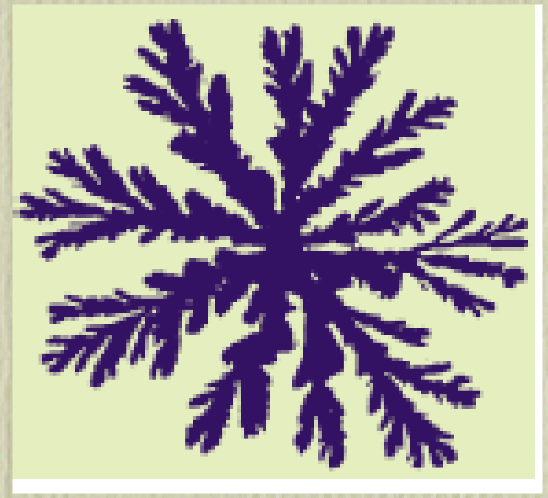
$$\Delta P = 0 \quad \text{in oil,}$$

$$P = \sigma \times \text{curvature} \quad \text{in water}$$

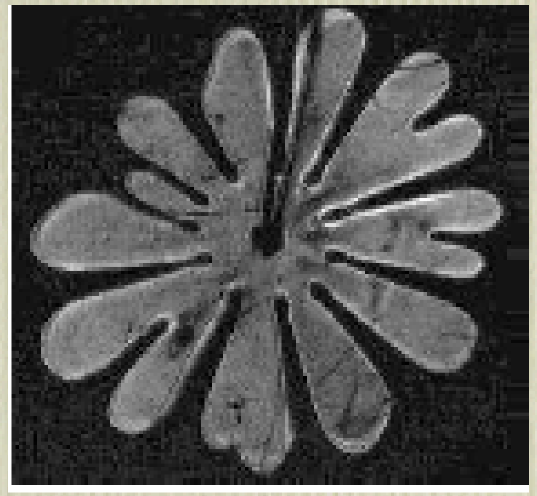
σ -surface tension

Velocity= gradient of a harmonic function

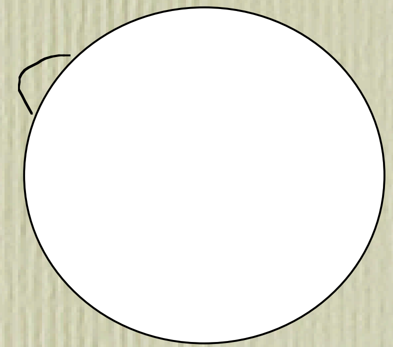
Fingering Instability

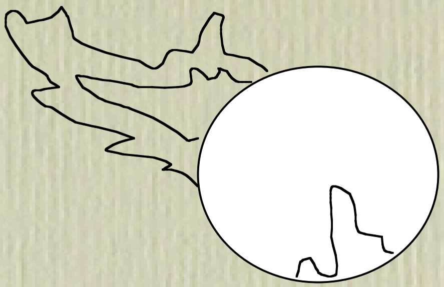


Large flux, small surface tension



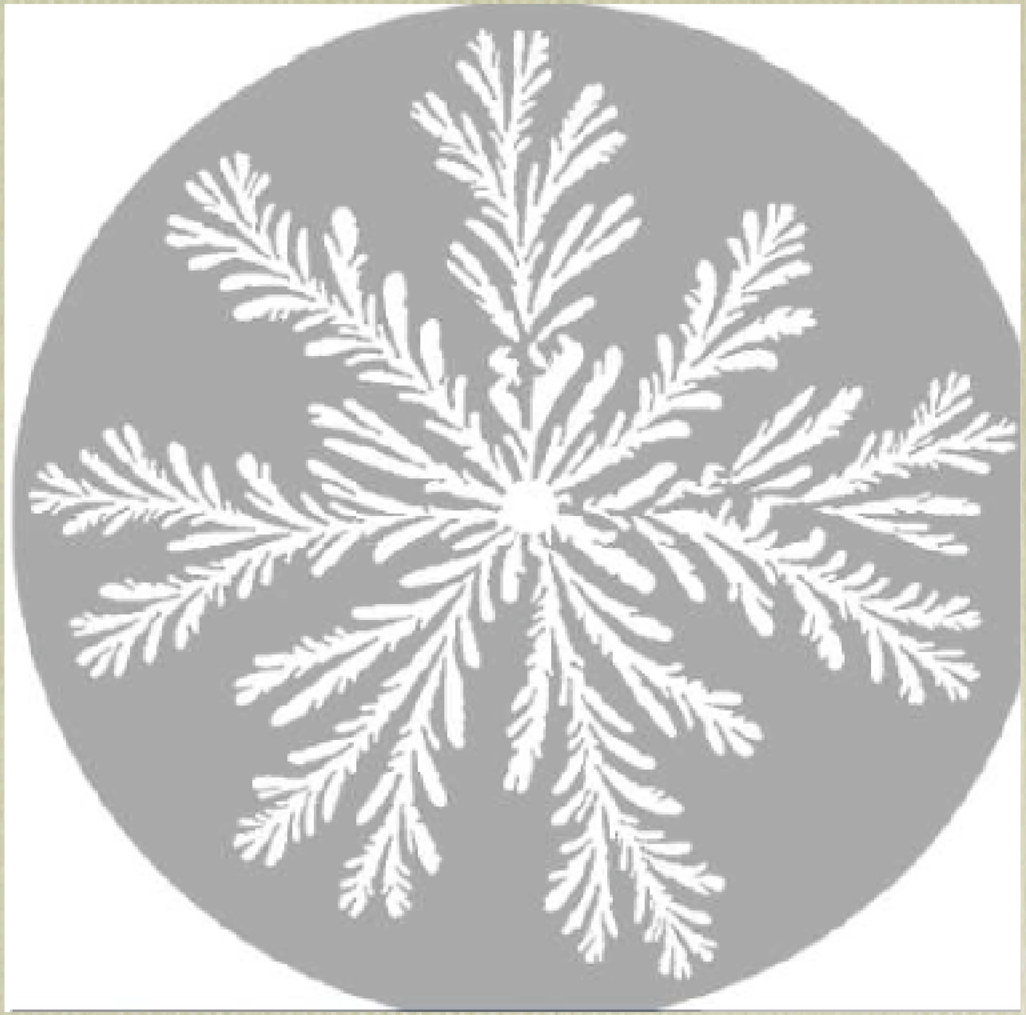
Small Flux, large surface tension





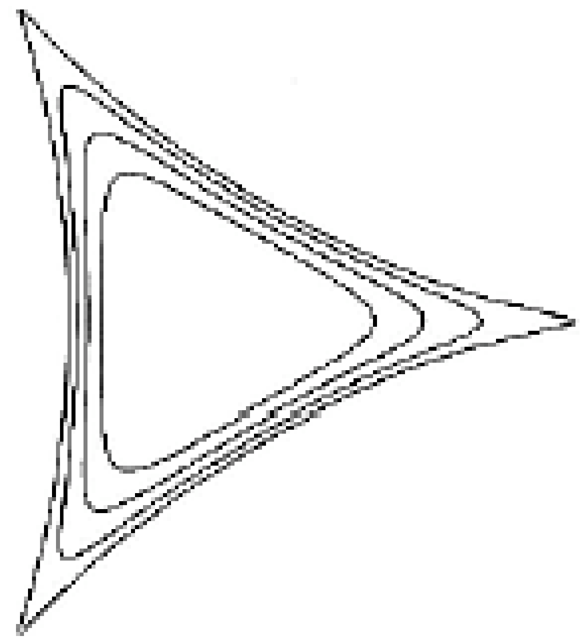
Fingering
instability

Flame with no convection



after H. Swinney et al

Finite time singularities



Hypotrochoid grows until it reaches a critical point.

Reformulation: Exterior harmonic moments

$$t_k = -\frac{1}{k}\int_{exterior}(x+iy)^{-k}dxdy$$

= Conserved

Find an evolution of a domain domain, which area grows, while the moments are conserved

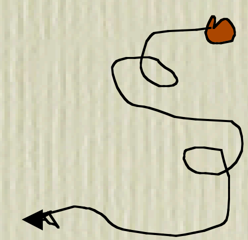
t_k - are coordinates of the moduli space of the droplet

- Fingering Instability
- Finite time cusp-like singularities
- Necessity of a regularization of short distances

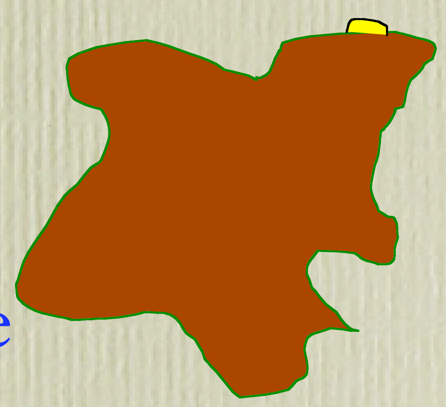
Interpretation:

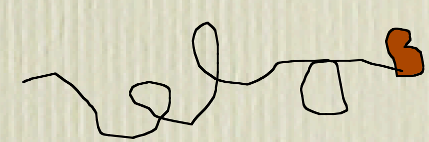
pressure - $$p = \log |w(z)| \quad \text{conformal map}$$

velocity - $$v = \nabla p = |w'(z)| \quad \text{conformal measure}$$

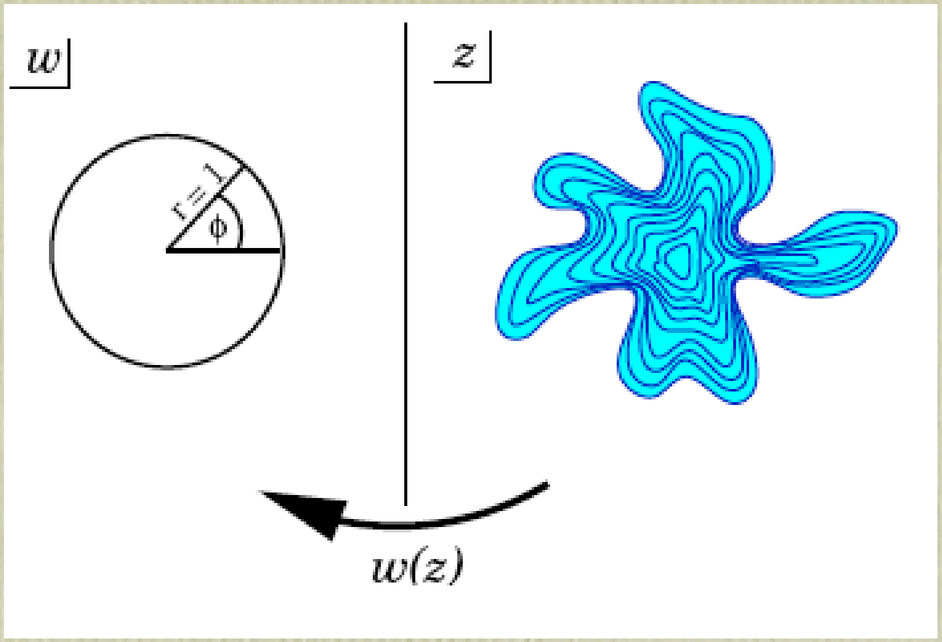


A probability of a Brownian mover to arrive and join the aggregate



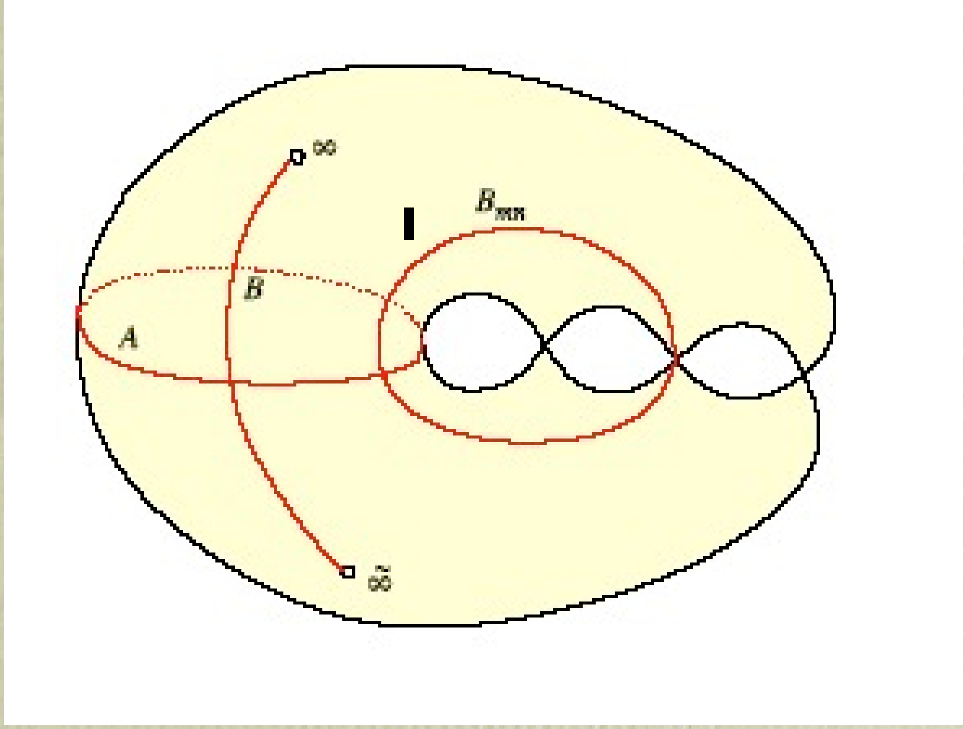


While joining the aggregate a mover becomes a *fermion*



Evolution of conformal map

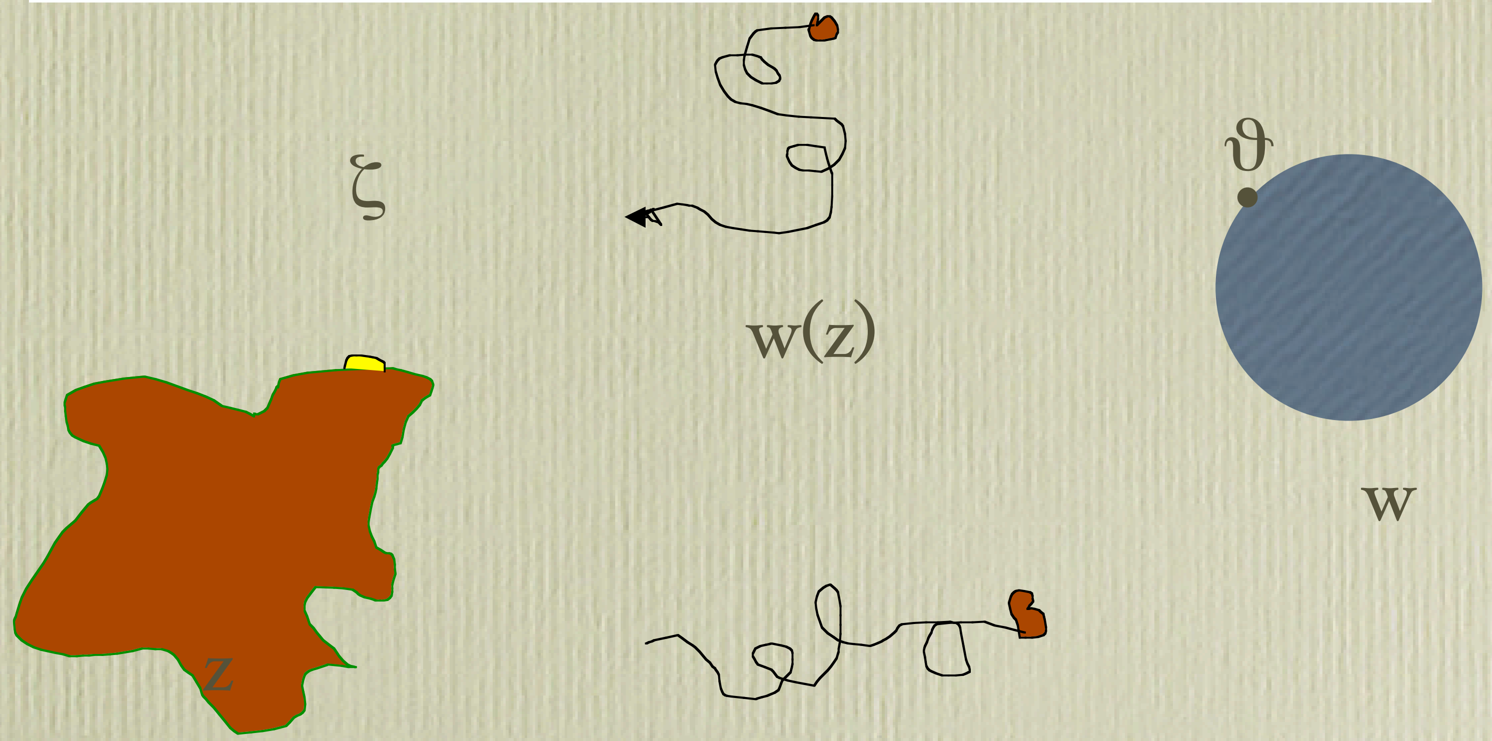
- Evolution of Riemann's surfaces:



Degenerate Riemann surface: all cycles are nulled, except one

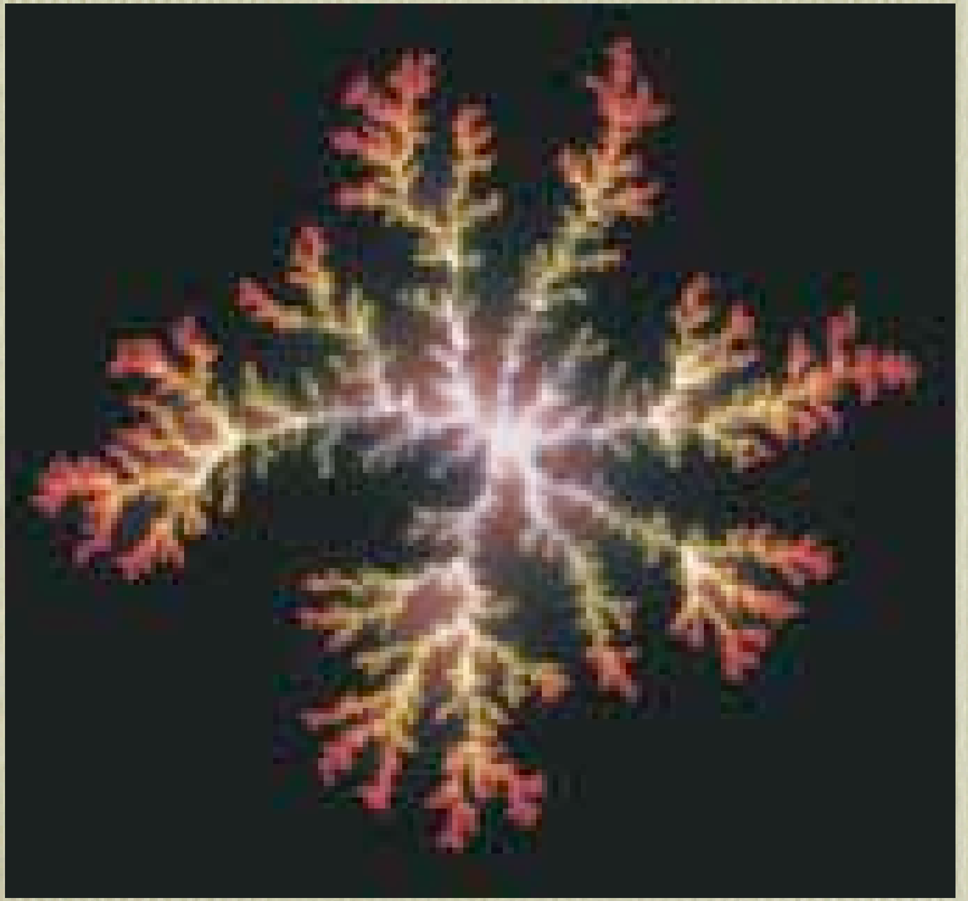
Hadamard formula

$$\frac{\delta w(z)}{w(z)} = \frac{w(z)+e^{i\theta}}{w(z)-e^{i\theta}} \cdot |w'(\zeta)|^2 \mathrm{d(Area)}$$

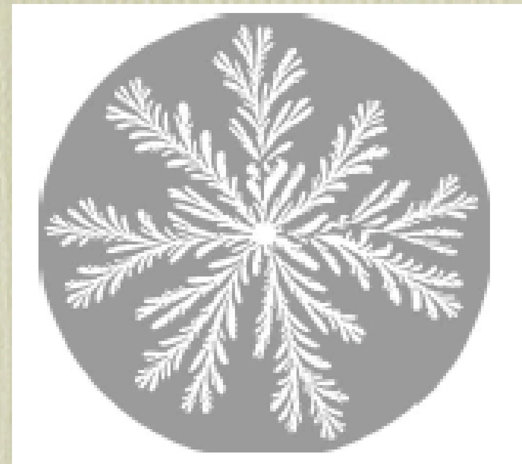


$$\frac{d}{dt}\log w(z) = \frac{w(z)+e^{i\theta}}{w(z)-e^{i\theta}} \cdot |w'(\zeta)|^2$$

Diffusion limited aggregation



turbulent regime - collection of singularities



Fluid dynamics

Turbulent Regime, DLA

Arriving particles have a finite size: $\hbar$

DLA - Diffusion limited aggregation

Noise - a mechanism of stabilization

- Saffman
- Taylor
- Kruskal
- Richardson
- Kadanoff
- Shraiman
- Hastings
- Levitov

Recent developments:

- Link to Random (non-Hermitian) Matrix Ensembles - particles are eigenvalues of a random matrix;

- Link to QHE and Correlation functions in quantum wires: water domain= Quantum droplet on the LLL, or a Fermi surface in the phase space;

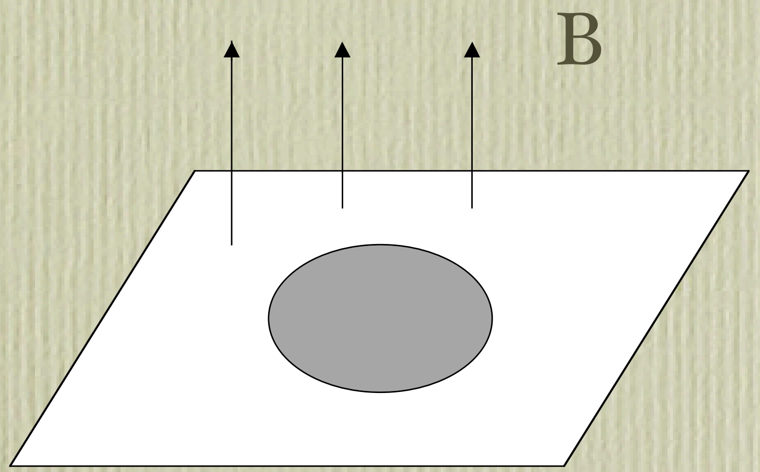
- Link to a Boundary Conformal Field Theory;

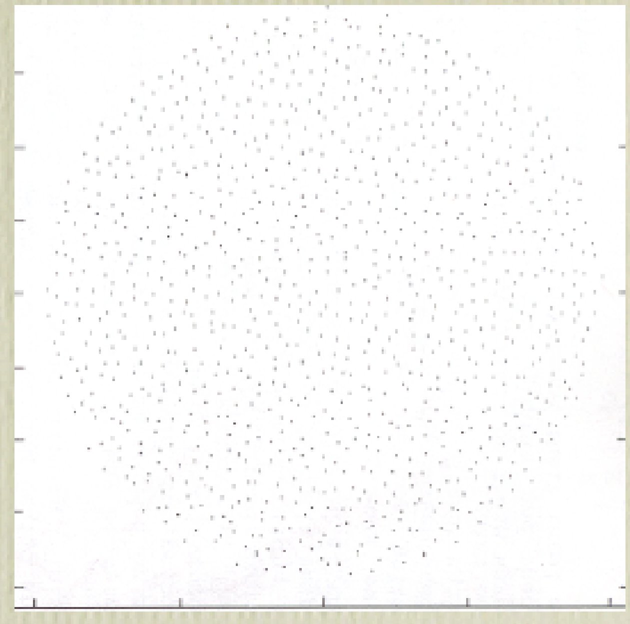
- Link to Integrable system - Toda lattice hierarchy (singular limit of dispersive waves)

- Link to problems of 2D - Quantum gravity

Aharonov-Bohm Effect on LLL

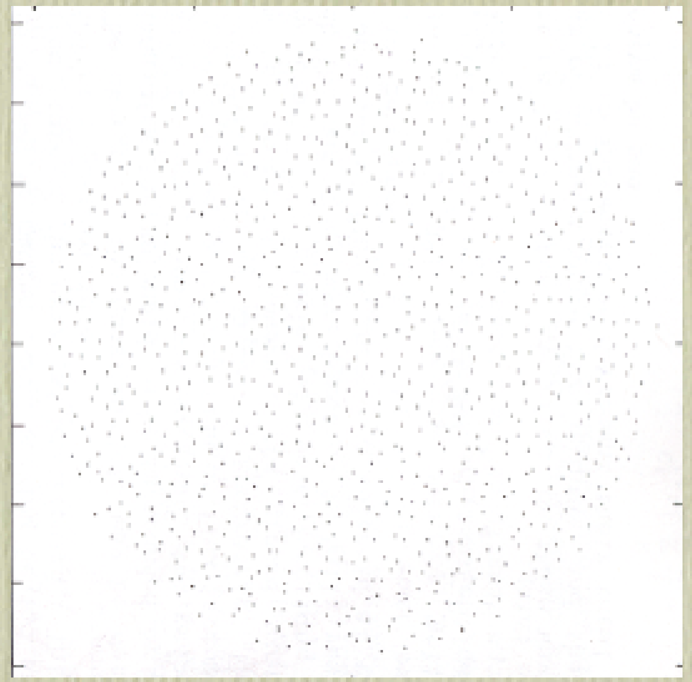
Particles on LLL in a uniform magnetic field form a circular droplet





$$\Psi(z_1, \ldots, z_N) = \prod_{m<n} (z_n - z_m) e^{-\sum_n \frac{|z_n|^2}{2\ell^2}}$$

Three interpretations:



- 2D-Electrons on the first Landau level (electronic droplet);

$$\Psi(z_1, ..., z_N) = \prod_{m<n} (z_n - z_m) e^{-\sum_n \frac{|z_n|^2}{2\ell^2}}$$

- Phase space (coherent state reprs) of 1D Electrons in parabolic potential;

$\Psi(z_1, ..., z_N)$ - Wigner (or coherent state repr.)

$$p^2 + E^2 = N\hbar$$

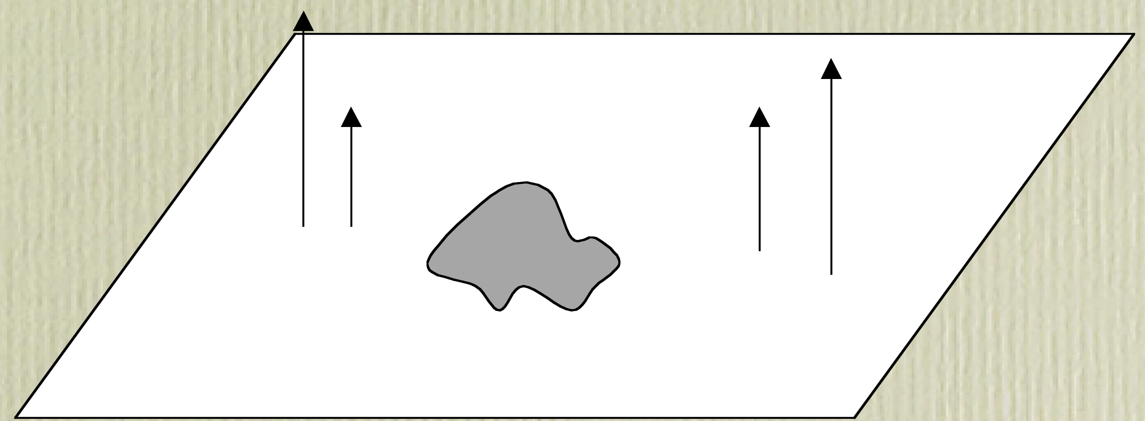
- Eigenvalues distribution of general Random Matrix ensemble

$$M = U^{-1}\mathrm{diag}(z_1, \ldots, z_N)U$$

$$e^{-\frac{1}{\hbar}\mathrm{Tr} MM^\dagger} dM \sim \prod_{m<n} |z_n - z_m|^2 e^{-\frac{1}{\hbar}\sum_n |z_n|^2}$$

The circular shape is unstable under gradients of magnetic field

$$V(z) = \sum_k t_k z^k$$



$$\prod_{m<n} |z_n - z_m|^2 e^{-\frac{1}{\hbar}\sum_n |z_n|^2 + V(z_n) + \overline{V(z_n)}}$$

Magnetic field $= B = -\partial_z \partial_{\bar{z}}(|z|^2 - V(z) - \overline{V(z)})$

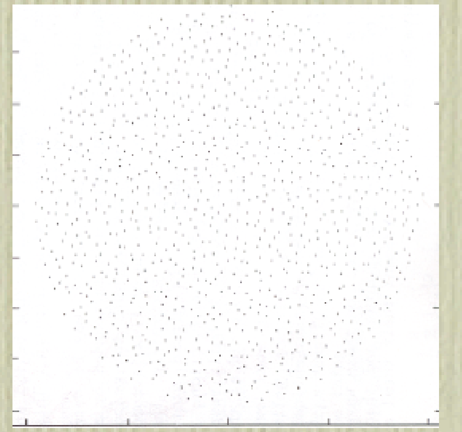
$$\Psi(z_1, \cdots, z_N) = \frac{1}{\sqrt{N! \tau_N}} \Delta(z) e^{-(\sum_n \frac{1}{2\ell^2}|z_n|^2 - V(z_n))}.$$

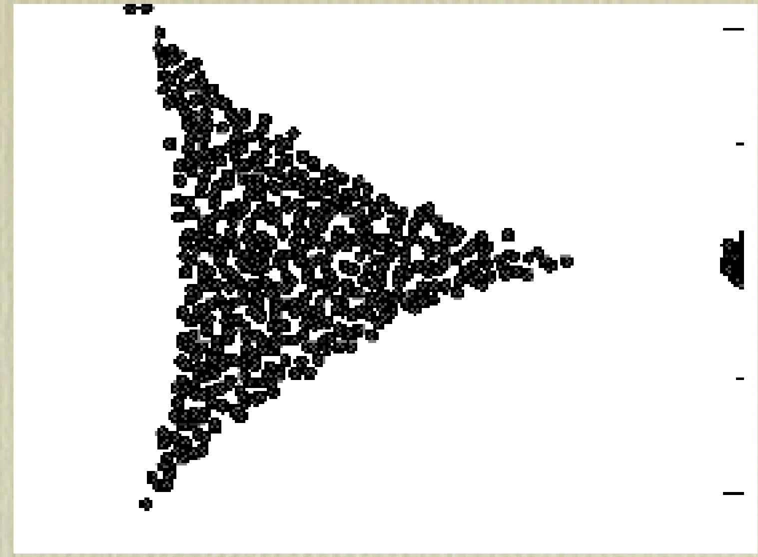
The saddle point equation (4) is transformed accordingly

$$\sum_{m \neq n}^{N} \frac{2\ell^2}{z_n - z_m} = \bar{z}_n - 2\ell^2 \frac{\partial}{\partial z} V(z).$$

Semiclassical approximation: $\hbar \to 0, \quad N \to \infty, \quad N\hbar = \text{fixed}$

The circular shape is unstable under gradients of magnetic field



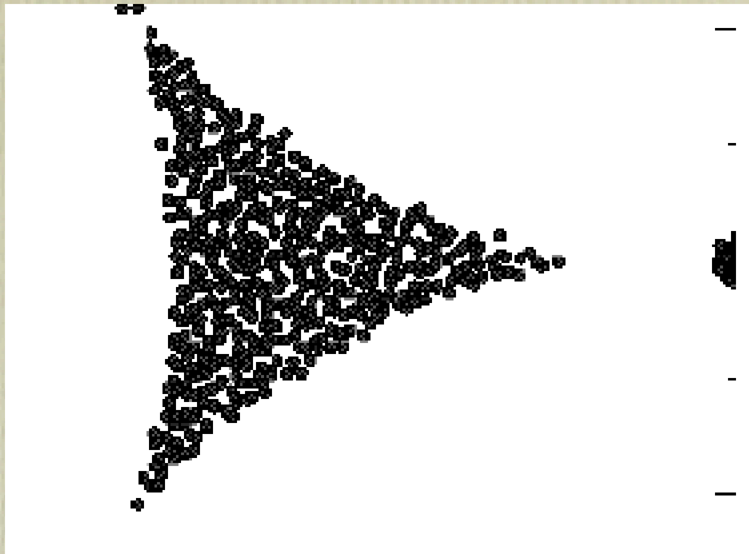


Semiclassical limit of the Growth: N→ N+1 provides the same hydrodynamics as Laplacian growth.

Quantum limit provides stochastic regularization

- Quantization - a way of regularization of classical systems at a singular regime:
- Quantization introduces a new scale $\hbar$

Cusp-like singularities



-
-
- Hydrodynamics of Dispersive waves;
- 2D-gravity
- Random matrices (edge of the spectrum)

Hermitian matrix ensemble:

eigenvalues are real, distributed in an interval of the real axis

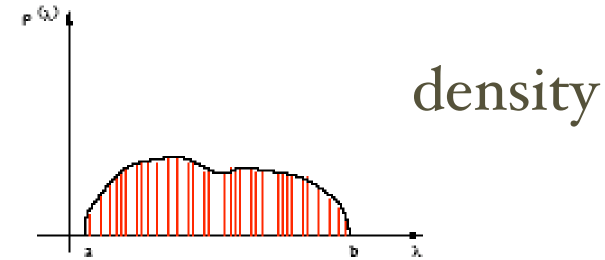
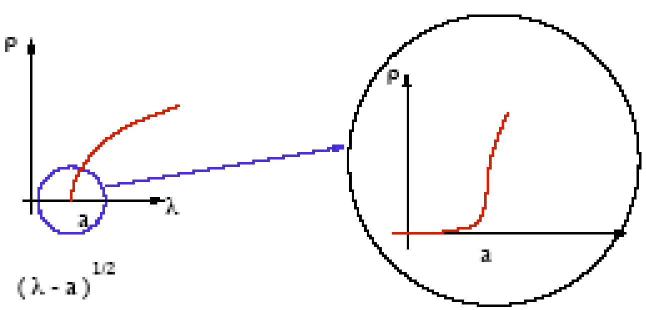


Figure 2.2: Typical spectrum of a matrix. In the large N limit, the spectrum is well approximated by a continuous density of levels $\rho(\lambda)$.

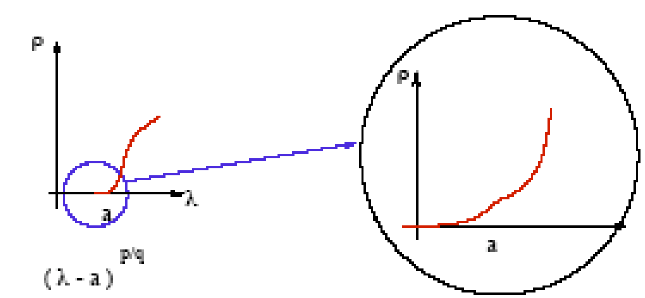
Bulk properties are universal (do not depend on potential V(M))

Edge properties are also universal but do depend on potential



Gaussian potential

e 2.6: The density of eigenvalues has square-root singularities at the edges. An gement of the edge's vicinity shows a universal tail related to the Airy function.



Non- Gaussian potential

Figure 2.7: At a critical point, the density of eigenvalues has power law singularities at the edges. An enlargement of the edge's vicinity shows a universal tail related to some integrable hierarchy of differential equations.

Edge singularities are characterized by p, q

Quantization can be seen as a noise

Noise is negligible in the semiclassical regime (a smooth droplet)

Important in the turbulent (singular) regime, where semiclassics fails.

Singular semiclassical expansion at the cusps

- Laplacian growth=Witham hierarchy of soliton equations
- Dispersive regularization of nonlinear waves

- Connection to integrable systems

How does the shape of the domain change with respect to a change of harmonic moments t_k ?

d (dispersionless) Toda hierarchy

How does the wave function change with respect to a change of harmonic moments t_k and $\hbar$?

Toda hierarchy

- Witham hierarchy = a semiclassical limit of soliton equations

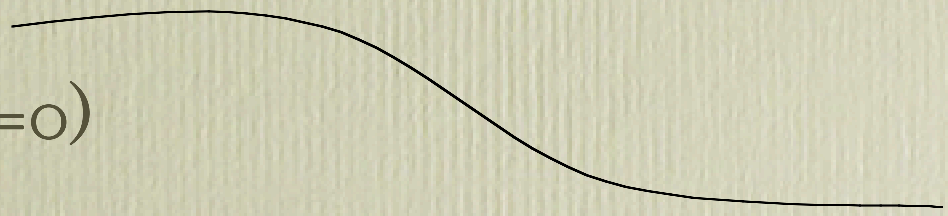
- Example KdV u(x,t):

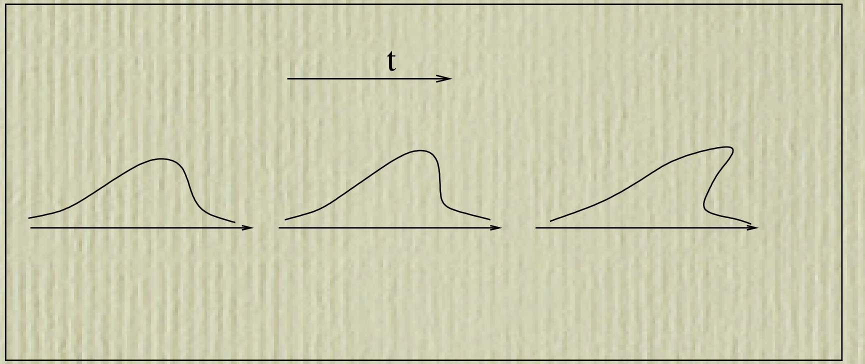
$$\boxed{\mathrm{u}_t - u \cdot u_x + \hbar \cdot u_{xxx} = 0}$$

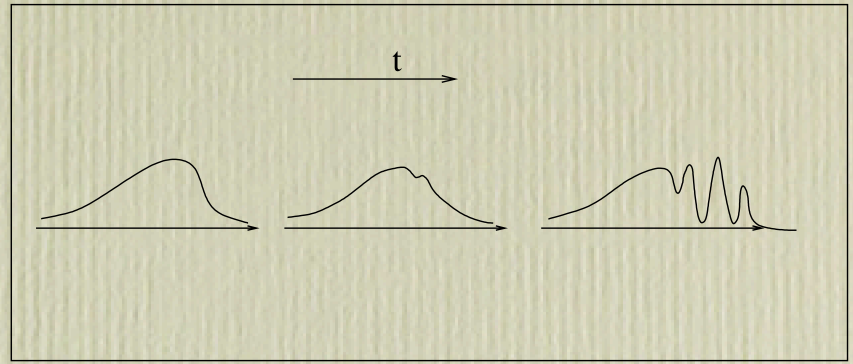
- Periodic (soliton-like) solution: waves in shallow water;

- Non-periodic solutions;

$u(x,t=0)$







Overhang - a result of
ill- approximation

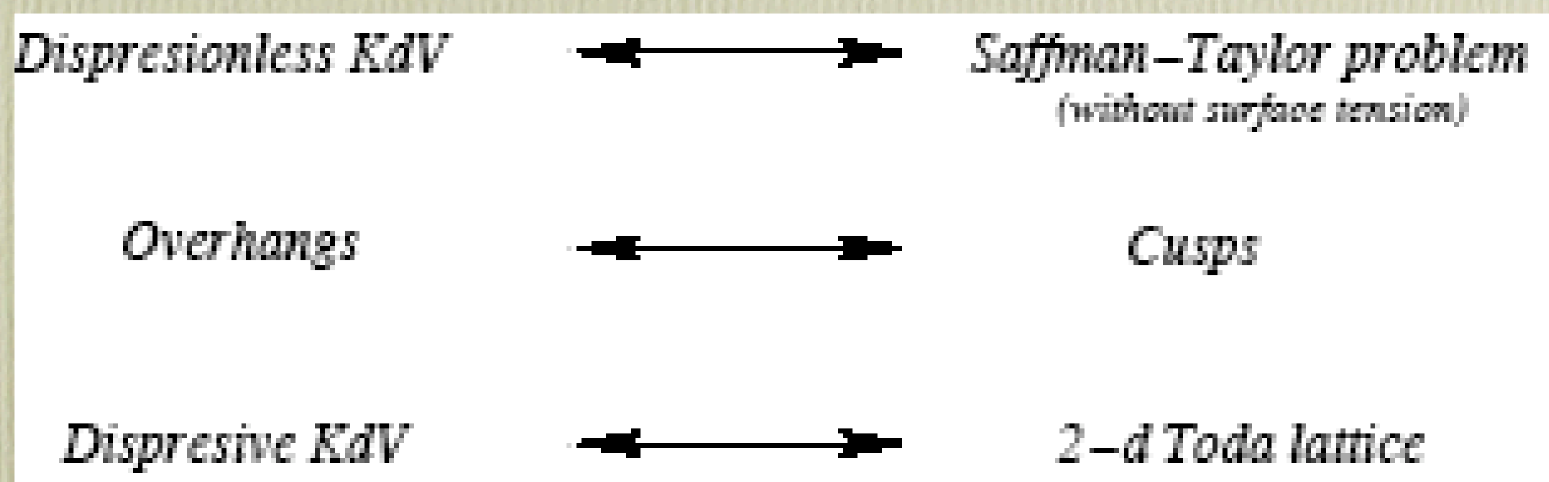
Oscillations is a
dispersive regularization

$$u_t - u \cdot u_x + \hbar \cdot u_{xxx} = 0 \quad \Rightarrow \quad u_t - u \cdot u_x = 0$$

Laplacian Growth is = dispersionless Toda hierarchy

Quantized Laplacian growth (QHE, or Random Matrix Models, etc.) =

Toda Integrable Hierarchy



Fine structure of a cusp

$$P(z) = |\Psi(z)|^2$$

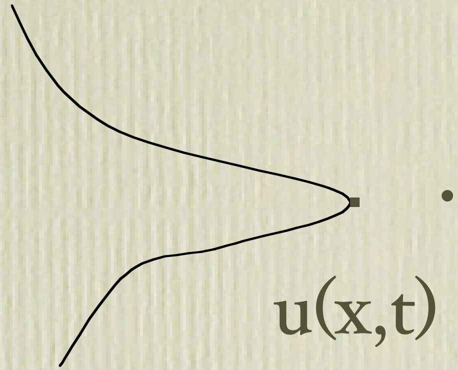
Cusp (2,3):

Painleve I equation

$$\left(-\frac{d^2}{dt^2} + u(t)\right)\Psi(z) = z\Psi(z)$$

u(t) - a coordinate of a tip

$$\boxed{\hbar \cdot u_{tt} - u^2 = t - t_c}$$



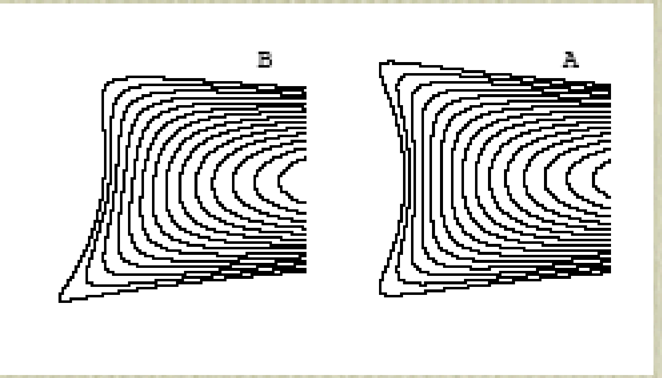
Universal character of the fine structure of singularities

- Singularities are classified by two integers (p,q) according to Kac table of CFT

- The simplest cusp (2,3) - Painleve I equation

- Higher order singularities - higher order Painleve like equations

Asymmetries and tip splitting



Set of operators perturbing singularities

Set of exponents - gravitational descendants

$$\Delta_{p,q}^{m} = \frac{q-p+m}{p+q-1}, \quad m = 1, \ldots, q < p$$

Are fractal exponents of DLA among this set?