Fractal Geometry of Critical Phenomena

John Cardy

University of Oxford

PITP Showcase Conference – May 2005

- self-similar fractal objects are ubiquitous in nature
  - clouds
  - coastlines
  - broccoli

- self-similar fractal objects are ubiquitous in nature
  - clouds
  - coastlines
  - broccoli
- deterministic mathematical fractals are well-studied
- but most natural fractals are random and therefore more difficult to characterize

- critical phenomena are an important area in which random fractals arise
  - they are well-studied from other points of view (renormalization group, field theory, exact solutions,...)

- critical phenomena are an important area in which random fractals arise
  - they are well-studied from other points of view (renormalization group, field theory, exact solutions,...)
  - in 2d they obey the powerful symmetry of conformal invariance

Random fractals in general

- they are only statistically self-similar
- all physical fractals are cut-off at short scales a and long scales R
- fractal dimension d_f:

Random fractals in general

- they are only statistically self-similar
- all physical fractals are cut-off at short scales a and long scales R
- fractal dimension d_f:
  - mean mass $\sim (R/a)^{d_f}$

Random fractals in general

- they are only statistically self-similar
- all physical fractals are cut-off at short scales a and long scales R
- fractal dimension d_f:
  - mean mass $\sim (R/a)^{d_f}$
  - box-counting: minimum # of discs of radius ϵ required to cover object $\sim (R/\epsilon)^{d_f}$

Random fractals in general

- they are only statistically self-similar
- all physical fractals are cut-off at short scales a and long scales R
- fractal dimension d_f:
  - mean mass $\sim (R/a)^{d_f}$
  - box-counting: minimum # of discs of radius ϵ required to cover object $\sim (R/\epsilon)^{d_f}$
  - if fractal embedded in D-dimensional euclidean space, probability that a point r lies within ϵ of the object $\sim \epsilon^{D-d_f}$

2d critical lattice models

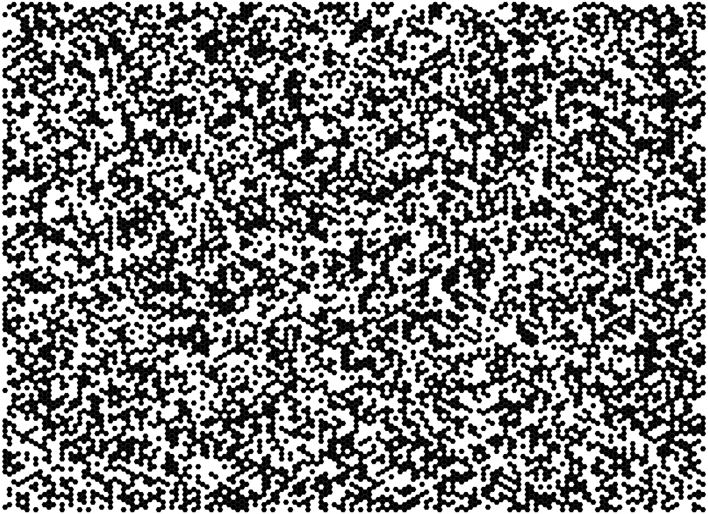
- prototype is Ising model:

$$Z = \prod_r \sum_{s(r)=\pm 1} e^{-\beta E(\{s(r)\})}$$

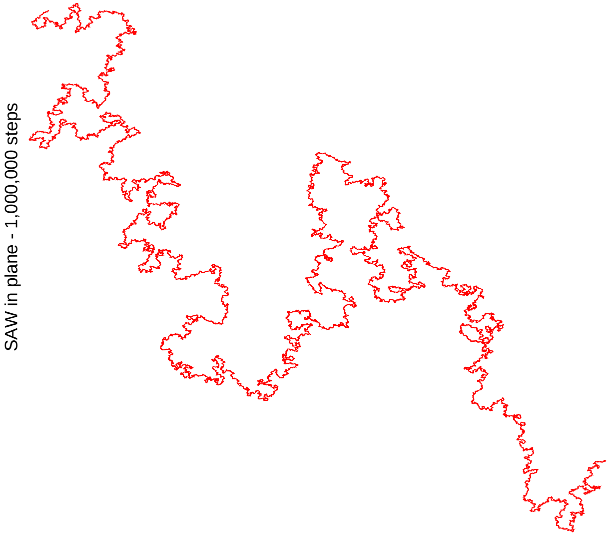
where $s(r) = \pm 1$ and e.g. $E = -J \sum_{\mathrm{nn}} s(r)s(r')$

- critical point at $\beta = \beta_c$
- but, from the geometrical point of view, even $\beta = 0$ is interesting:

Percolation



Self-avoiding Walk



- conventional questions involve local correlation functions:
    - Ising spin-spin correlation $\langle s(r_1)s(r_2)\rangle$
    - probability r_1 and r_2 are in same percolation cluster
    - number of SAWs (weighted by $\mu^{-\text{length}}$) from r_1 to r_2
- these all $\sim |a/(r_1 - r_2)|^{2x}$ at criticality
    - x is a critical exponent which depends on what kind of quantity is being measured but is otherwise universal

- conventional questions involve local correlation functions:
  - Ising spin-spin correlation $\langle s(r_1)s(r_2)\rangle$
  - probability r_1 and r_2 are in same percolation cluster
  - number of SAWs (weighted by $\mu^{-\text{length}}$) from r_1 to r_2
- these all $\sim |a/(r_1 - r_2)|^{2x}$ at criticality
  - x is a critical exponent which depends on what kind of quantity is being measured but is otherwise universal
- but there are other more geometrical universal quantities, eg:
  - probability that r_1 and r_2 lie near the same Ising cluster boundary
  - probability two opposite sides of a rectangle connected by a percolation cluster
  - probability that a self-avoiding loop starting and ending at r_1 encloses a given point r_2
- not obviously related to local correlation functions

Methods

- exact solution of the lattice model (1950s)
  - limited to a few models
  - not clear what is universal

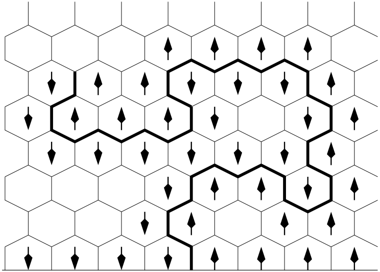
Methods

- exact solution of the lattice model (1950s)
  - limited to a few models
  - not clear what is universal
- conformal field theory/Coulomb gas methods (1980s)
  - very powerful, but
  - lack of systematic mathematical basis
  - often some guesswork involved

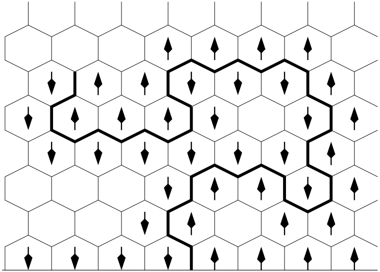
Methods

- exact solution of the lattice model (1950s)
  - limited to a few models
  - not clear what is universal
- conformal field theory/Coulomb gas methods (1980s)
  - very powerful, but
  - lack of systematic mathematical basis
  - often some guesswork involved
- SLE (stochastic Loewner evolution) `[O Schramm; G Lawler and W Werner]` (2000s)
  - clear mathematical basis
  - in some cases (eg percolation) a rigorous connection to lattice model
  - adapted to answering geometrical questions

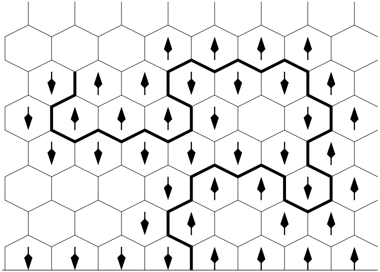
- describing the fractal soup of all these curves is too hard



- describing the fractal soup of all these curves is too hard
- focus on just one of them, picked out by the boundary conditions
- this curve can be 'grown' dynamically on the lattice:



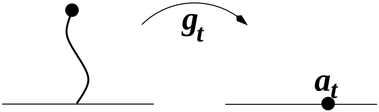
- describing the fractal soup of all these curves is too hard
- focus on just one of them, picked out by the boundary conditions
- this curve can be ‘grown’ dynamically on the lattice:



- SLE describes the continuous version of this process

Loewner evolution

- instead of the curve itself, consider the (unique) conformal mapping $g_t(z)$ which sends the region outside the curve to the half-plane

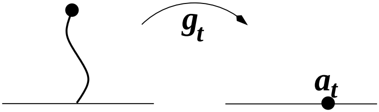


- g_t sends the growing tip to a_t
- it satisfies a simple equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

Loewner evolution

- instead of the curve itself, consider the (unique) conformal mapping $g_t(z)$ which sends the region outside the curve to the half-plane



- g_t sends the growing tip to a_t
- it satisfies a simple equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

- so instead of thinking about random curves we can think about random (continuous) functions a_t

Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:

Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:
  - $\kappa = \frac{8}{3}$ self-avoiding walks

Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:
  - $\kappa = \frac{8}{3}$ self-avoiding walks
  - $\kappa = 3$ Ising domain walls

Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:
  - $\kappa = \frac{8}{3}$ self-avoiding walks
  - $\kappa = 3$ Ising domain walls
  - $\kappa = 4$ level lines at roughening transition

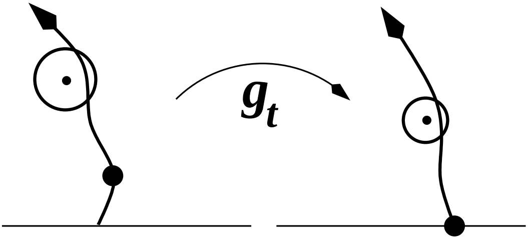
Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:
  - $\kappa = \frac{8}{3}$ self-avoiding walks
  - $\kappa = 3$ Ising domain walls
  - $\kappa = 4$ level lines at roughening transition
  - $\kappa = 6$ percolation cluster boundaries

Stochastic Loewner evolution

- if the measure on the curves is conformally invariant, a_t must be 1d Brownian motion `[Schramm]`
- diffusion constant κ: $\langle a_t^2 \rangle = \kappa t$
- different values of κ gives different universality classes of 2d critical behavior, eg:
  - $\kappa = \frac{8}{3}$ self-avoiding walks
  - $\kappa = 3$ Ising domain walls
  - $\kappa = 4$ level lines at roughening transition
  - $\kappa = 6$ percolation cluster boundaries
- problems in 2d critical behavior reduced to calculations in 1d Brownian motion

Fractal dimension



$$d_f = 1 + \kappa/8 \qquad (\kappa \leq 8)$$

- many more analytic results which characterize the fractal geometry of 2d critical phenomena, eg:
  - crossing probabilities
  - multifractal exponents
  - as well as all the more conventional exponents ν, η, etc

- many more analytic results which characterize the fractal geometry of 2d critical phenomena, eg:
  - crossing probabilities
  - multifractal exponents
  - as well as all the more conventional exponents ν, η, etc
- outlook: extensions to other 2d random fractals
  - diffusion limited aggregation
  - turbulence

- many more analytic results which characterize the fractal geometry of 2d critical phenomena, eg:
  - crossing probabilities
  - multifractal exponents
  - as well as all the more conventional exponents ν, η, etc
- outlook: extensions to other 2d random fractals
  - diffusion limited aggregation
  - turbulence
  - broccoli?

Some References for Physicists

- W. Kager and B. Nienhuis, *A guide to stochastic Loewner evolution and its applications,* `math-ph/0312251`
- J. Cardy, *SLE for theoretical physicists,* `cond-mat/0503313`