Bäcklund Transformations and D-Branes in Minimal String Theory

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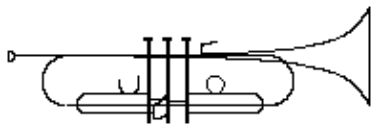
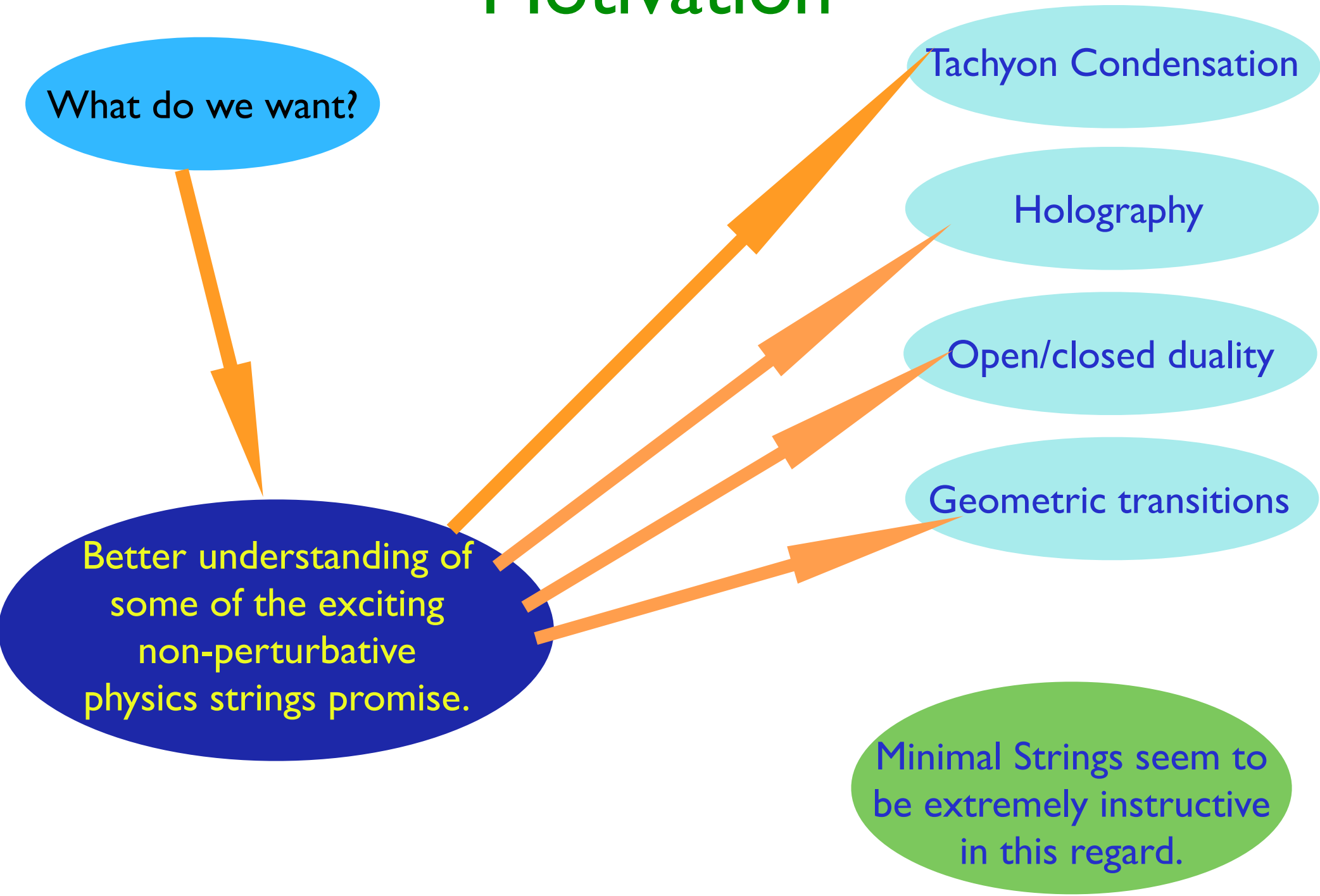


Motivation


What do we want?
Better understanding of some of the exciting non-perturbative physics strings promise.
Tachyon Condensation
Holography
Open/closed duality
Geometric transitions
Minimal Strings seem to be extremely instructive in this regard.

Based on lots of work, including:
Dalley, CVJ, Morris, 1991-1992: (several papers)
Dalley, CVJ, Morris, and Watterstam, 1993: hep-th/9206060.

Also work by CVJ: hep-th/0408049.

New work by:
J.E. Carlisle (Durham), CVJ, and J.S.Pennington (USC): *hep-th/0501006*

See also related work which recently appeared, by:
N. Seiberg and D. Shih (IAS): *hep-th/0412315*

String Equations

Consider the following equation:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0 \ .$$

$u(z)$ real function

$$' \equiv \nu\frac{\partial}{\partial z}$$

$$\mathcal{R} = \sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)t_k R_k[u]$$

$$R'_{k+1} = \frac{1}{4}R'''_k - uR'_k - \frac{1}{2}u'R_k$$

$$R_0 = \frac{1}{2}\,; \quad R_1 = -\frac{1}{4}u\,; \quad R_2 = \frac{1}{16}(3u^2 - u'')\,; \ldots$$

String Equations

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0 \ .$$

Choose *k*th model by:

$t_k = 0 \quad$ except $\quad t_0 = z \quad t_k = 1 \ .$ (up to some number)

$$u(z) = \nu^2 \frac{\partial^2 F}{\partial \mu^2}\bigg|_{\mu=z} \ , \qquad Z = \exp(-F) \ .$$

String Equations

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0 \ .$$

An obvious family of solutions is: $\mathcal{R} = 0$.

$$-\frac{1}{3}u'' + u^2 = z \ .$$

$$u(z) = z^{\frac{1}{2}} - \frac{\nu^2}{24z^2} + \frac{49\nu^4}{1152}z^{-\frac{9}{2}} + \cdots$$

$$z \to +\infty \ .$$

$$F = \frac{4}{15}g_s^{-2} + \frac{1}{24}\log\mu - \frac{7}{1440}g_s^2 + \cdots$$

$$g_s = \nu\mu^{-5/4}$$

(dimensionless string coupling)

sphere

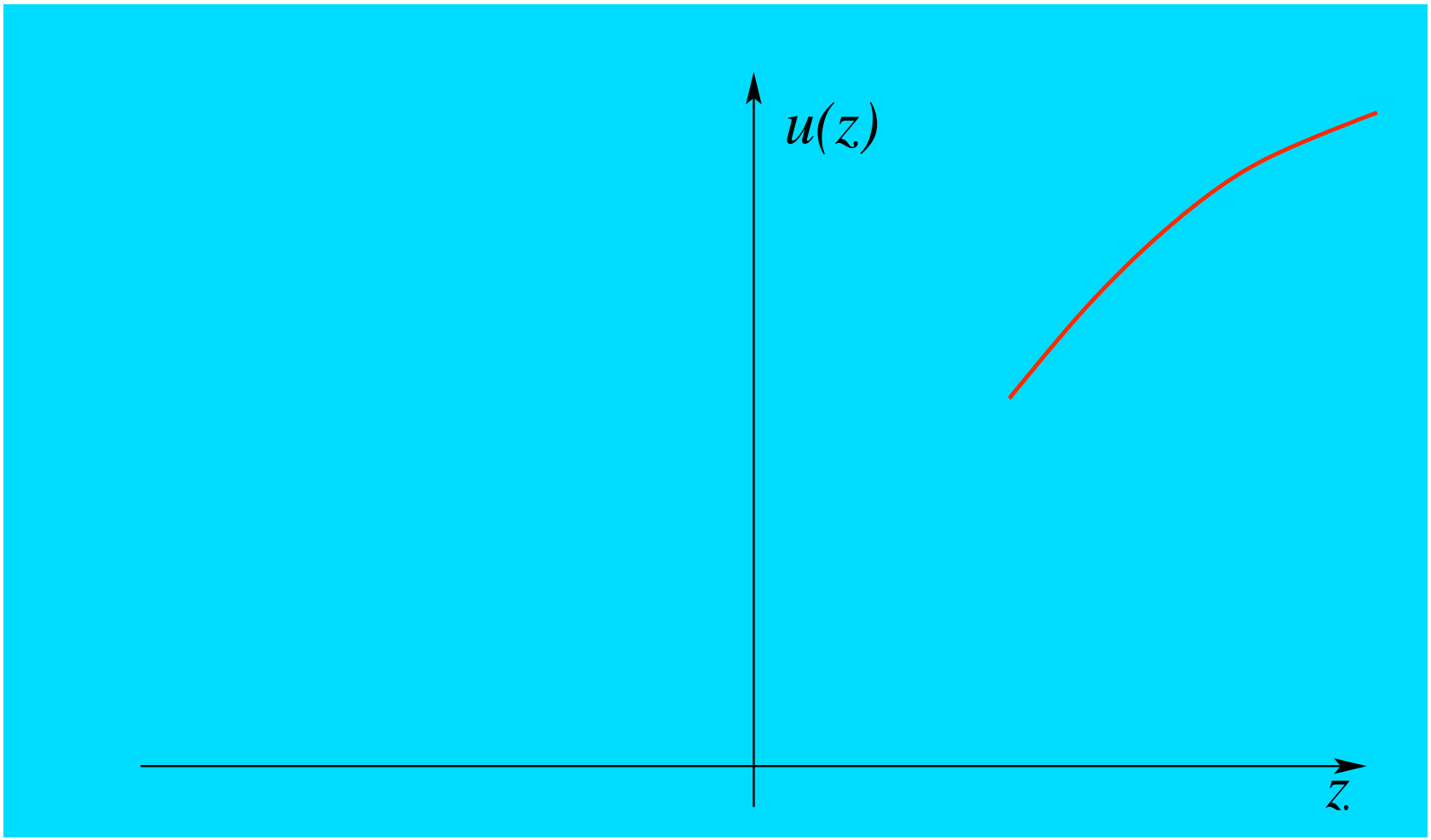
torus

double torus

$$\chi = 2 - 2h \ ; \quad g_s^{-\chi} \ .$$

String Equations

$$-\frac{1}{3}u'' + u^2 = z\ .$$



String Equations

Rather exciting:

Encodes complete perturbation theory of a string theory.

Operator insertions?

$$\mathcal{L} \longrightarrow \mathcal{L} + t_k O_k \ .$$

Corresponds to switching on other t_k in previous equations.

How does $u(z, t_k)$ behave?

$$\frac{\partial u}{\partial t_k} = R'_{k+1}[u] \ ; \qquad \text{e.g.,} \quad \frac{\partial u}{\partial t_1} = 6uu' - u''' \ .$$

The KdV Hierarchy

KdV Flows and String Equations

KdV flows organize the flows between the models.

These are the *(2k-1,2)* minimal models coupled to Liouville:
A certain "minimal string theory".

$$\frac{\partial F}{\partial t_k} = < O_k > \ , \quad u = < O_0 O_0 > \ .$$

The KdV Hierarchy gives relations between operator insertions...

Most powerful when combined with string equation:

$$\sum_{k=1}^{\infty} \left(k+\frac{1}{2}\right) t_k R_k = z \ .$$

$$\sum_{k=1}^{\infty} \left(k+\frac{1}{2}\right) t_k R_k' = 1 \ .$$

$$\sum_{k=1}^{\infty} \left(k+\frac{1}{2}\right) t_k \frac{\partial u}{\partial t_{k-1}} = 1 \ .$$

$$\sum_{k=1}^{\infty} \left(k+\frac{1}{2}\right) t_k \frac{\partial F}{\partial t_{k-1}} = \frac{\mu^2}{2} \ .$$

KdV Flows and String Equations

$$\left[\sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)t_k\frac{\partial}{\partial t_{k-1}}-\frac{\mu^2}{2}\right]Z=0\ .$$

$$L_{-1}\cdot Z=0\ .$$

Combine with recursion relations of KdV to get:

$$L_n\cdot Z=0\ ,\qquad n\geq -1\ ,$$

These are the "Virasoro Constraints"

$$L_0=\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)t_k\frac{\partial}{\partial t_k}+\frac{1}{16}\ ,$$

$$L_n=\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)t_k\frac{\partial}{\partial t_{k+n}}+4\nu^2\sum_{k=1}^{n}\left(k+\frac{1}{2}\right)t_k\frac{\partial^2}{\partial t_{k-1}\partial t_{n-k}}\ ,$$

These are recursion relations which show, e.g., the presence of only *k-1* O_k

Matrix Models

$$Z = \int dM \exp\{-N \mathrm{Tr}[V(M)]\} \ .$$

$$M: \quad N \times N \text{ Hermitian matrix}$$

$$V(M) = \sum_l g_l M^l$$

Get minimal string after "double scaling limit" (DSL):

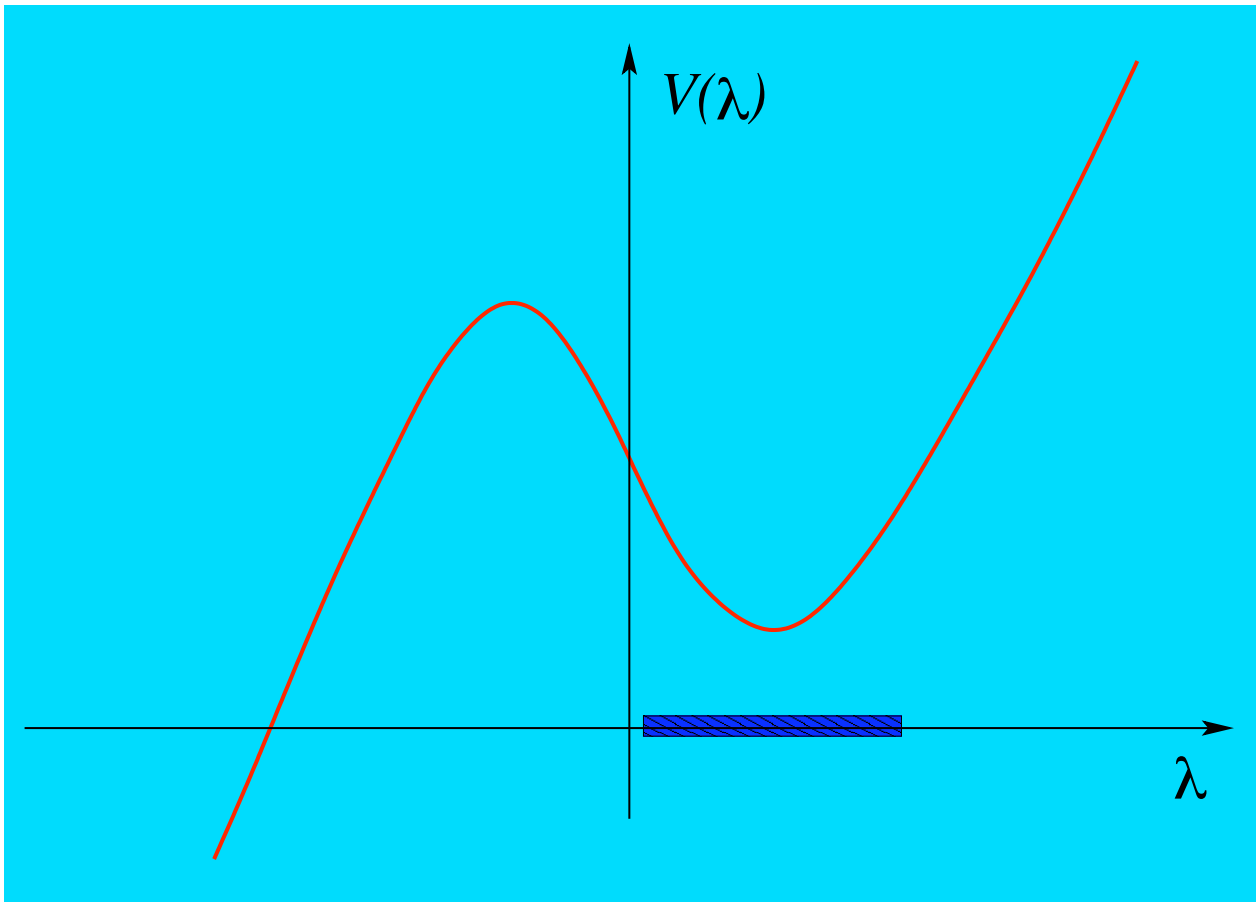
$$N \longrightarrow \infty \quad \text{as} \quad g_l \longrightarrow g_l^c \ .$$

Matrix Models

$$M = U\Lambda U^\dagger \ , \qquad \Lambda = \mathrm{diag}\{\lambda_1, \ldots \lambda_N\} \ .$$

$$dM = dU\,\Delta(\lambda)^2 \ , \quad \Delta(\lambda) \equiv \prod_{i\neq j}(\lambda_i - \lambda_j) \ .$$

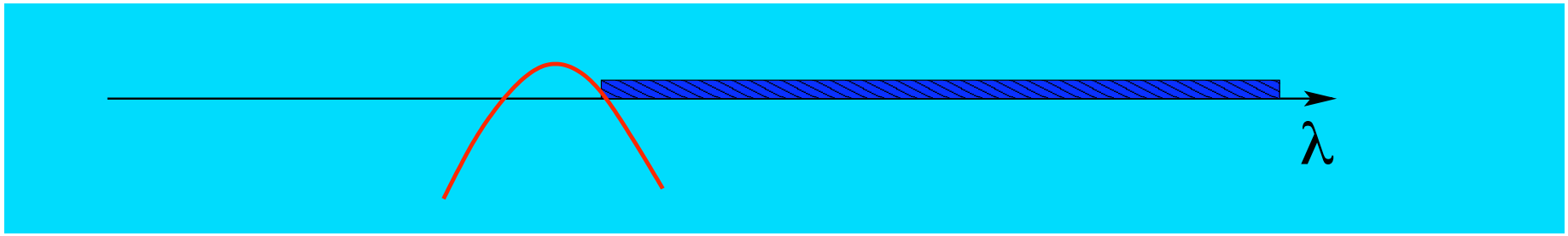
(van der Monde determinant)



DSL is when tune potential so that droplet is about to split, spilling to new config.

$$N \longrightarrow \infty \quad \text{as} \quad g_l \to g_l^c \ .$$

Eigenvalue Space

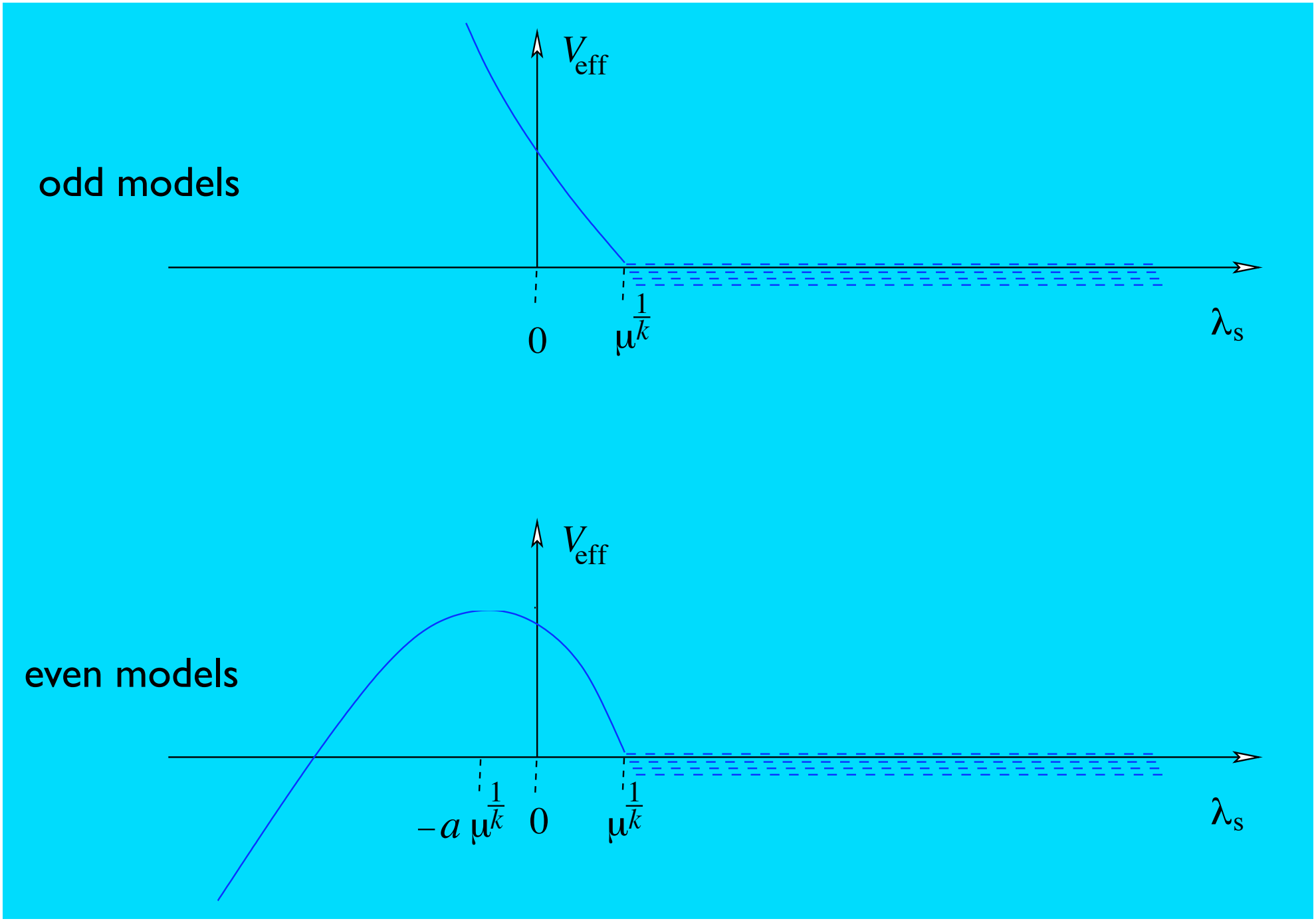


DSL focuses on tail of distribution near critical region.

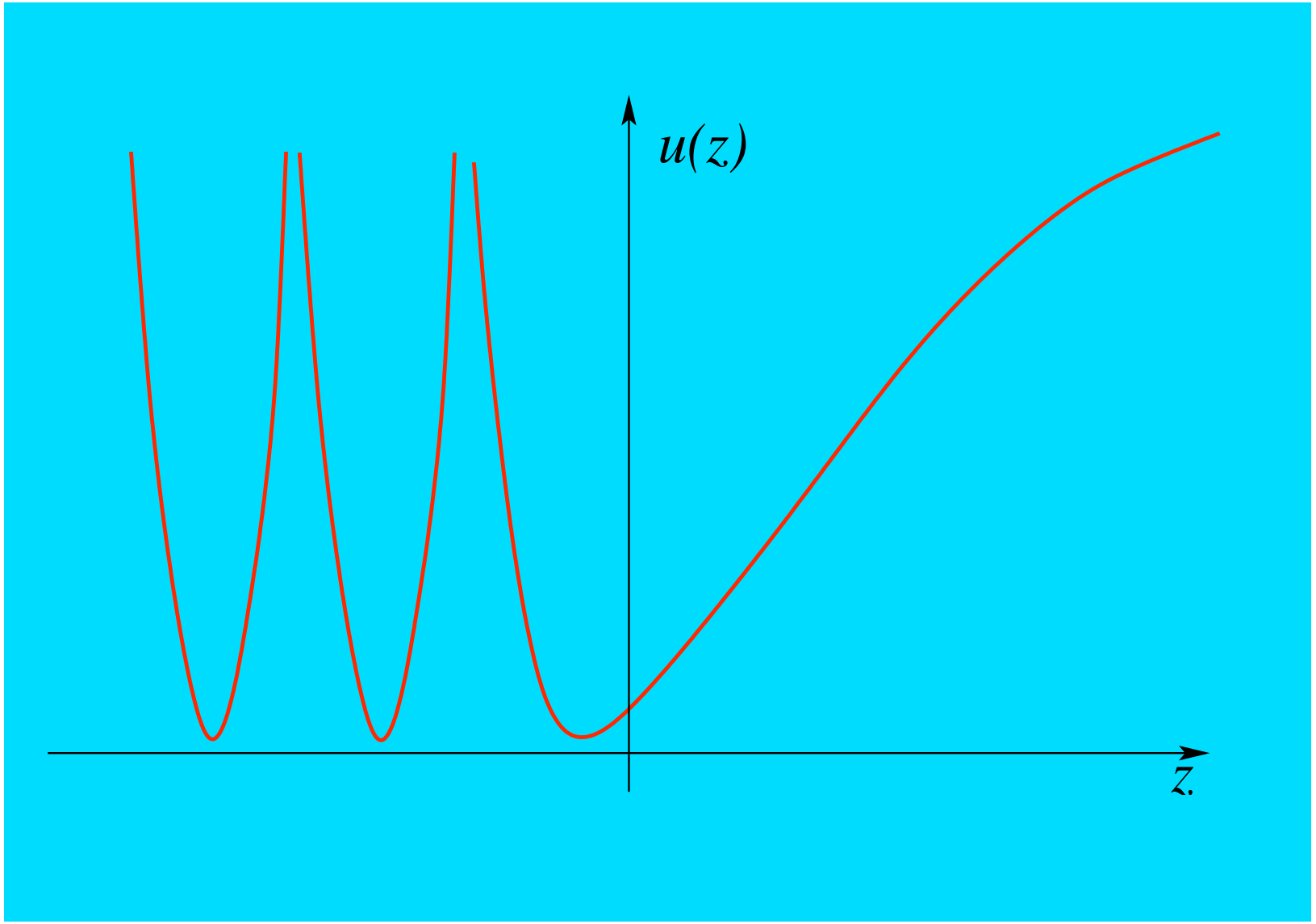
DSL extracts the universal physics of the matrix model.

Compute effective potential for one eigenvalue (at sphere level):

Eigenvalue Space



Non-Perturbative Physics



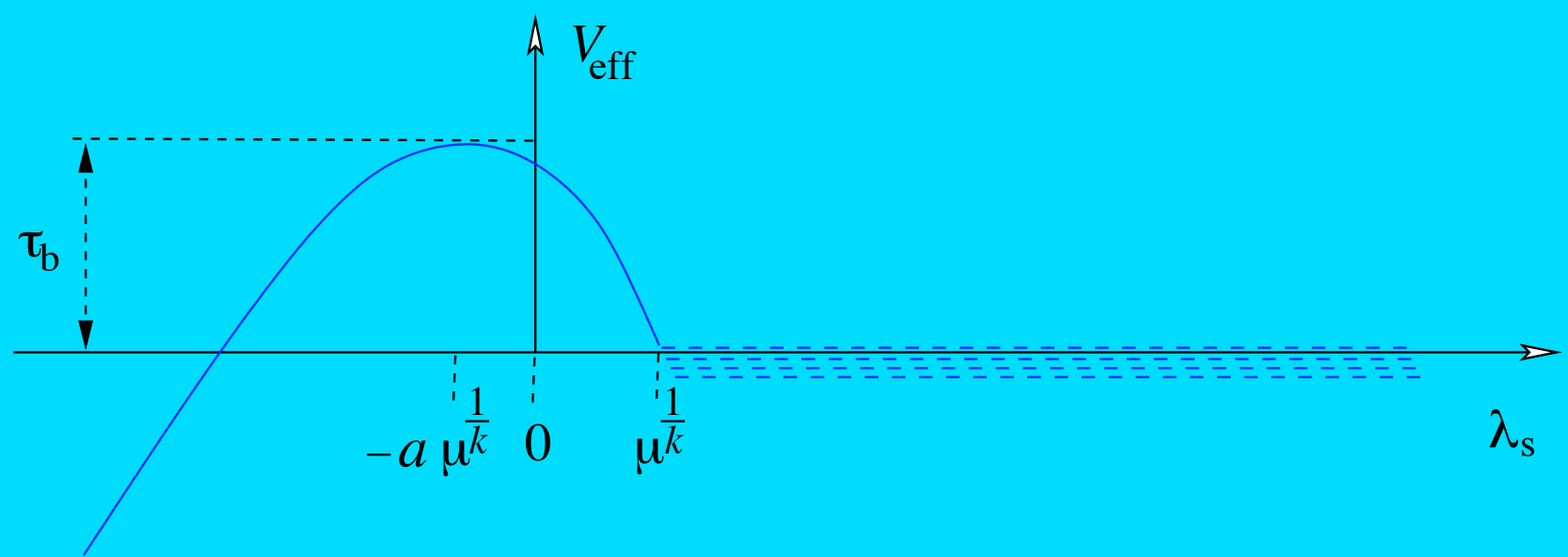
The non-perturbative physics is sick for even models.
No suitable real solution.

Instantons and D-branes

The even models have real instanton representing tunneling into abyss:

$$\tau_b^{(2)} = V_{eff}^{(2)}\left(-\frac{1}{2}\mu^{\frac{1}{2}}\right) = \frac{4\sqrt{6}}{5}\frac{1}{g_s}, \qquad g_s = \frac{\nu}{\mu^{\frac{5}{4}}}.$$

This instanton is a D-brane.



An Operator for Eigenvalue Space

Connection between $u(z)$ and eigenvalue space?

Operator:

$$\hat{\lambda}_s = -\mathcal{H} = -\left(-\nu^2 \frac{\partial^2}{\partial z^2} + u(z)\right) = Q \ .$$

So have:

$$Q\psi(z) = \lambda\psi(z) \ .$$

Expect continuous spectrum bounded from below.

Even models fail.

The "Modern" Picture

This is an Holographic correspondence, like AdS/CFT:

Have "non-critical" string. There are D0-branes.

Take N of these.

FZZ, T, MV...

Write down world-volume theory of open string modes.

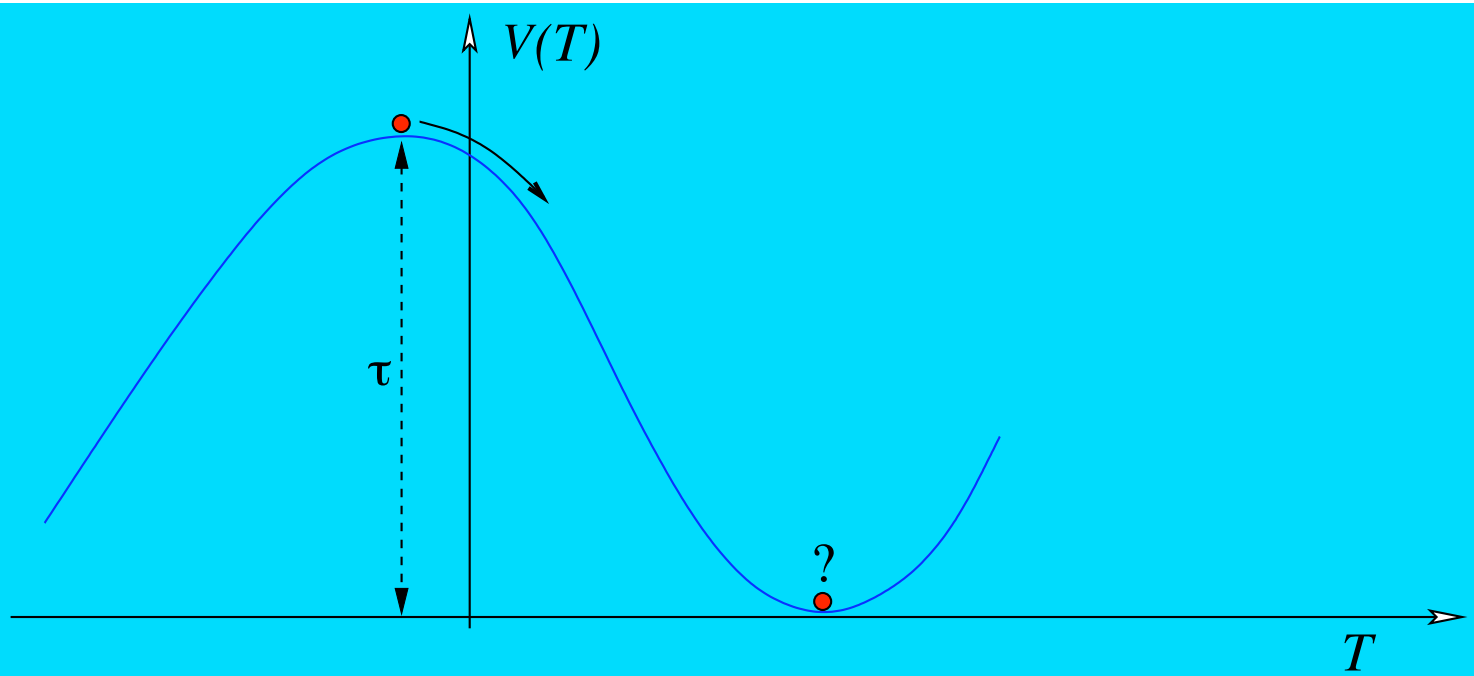
$$Z = \int dM \exp\{-N \mathrm{Tr}[V(M)]\} \ .$$

Holography: Recover physics of full string theory by looking at DSL.

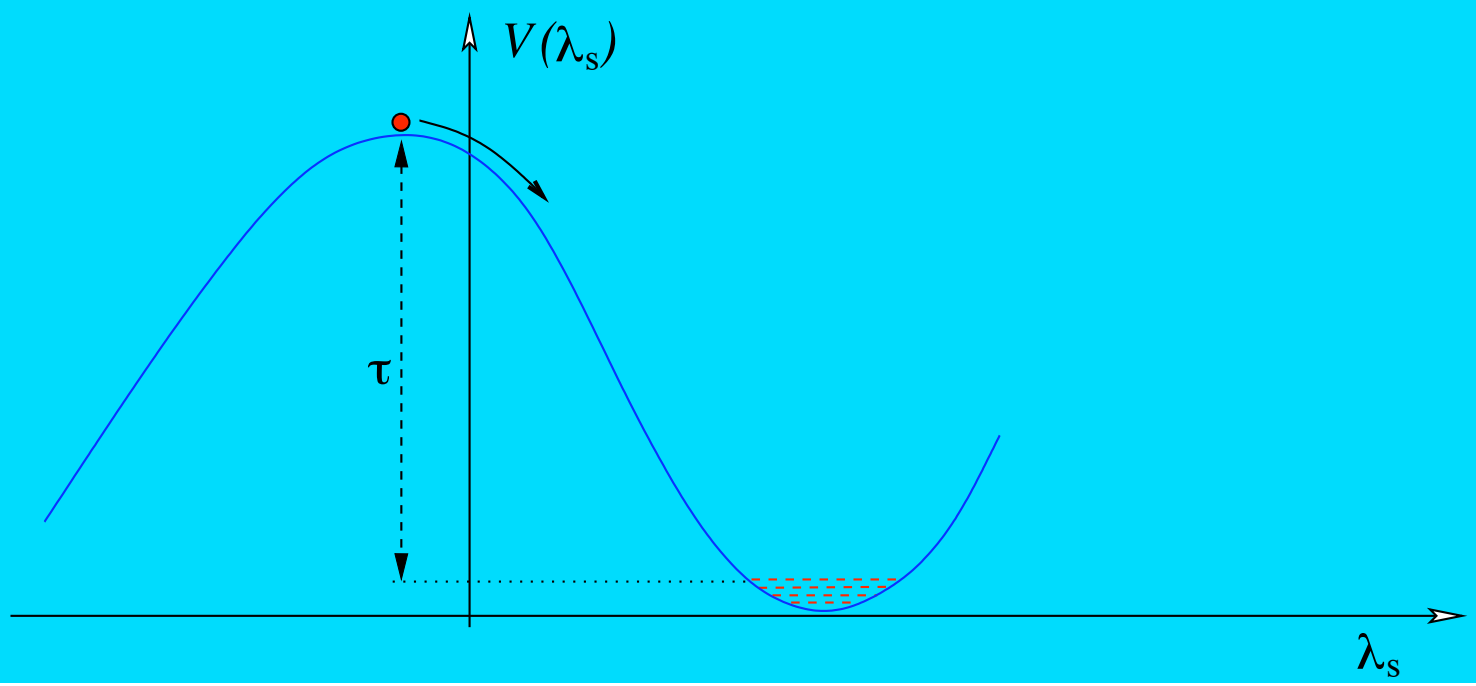
Earlier picture is just tachyon condensation in action!

Tachyon Condensation

Sen's Picture



The Matrix Model realisation



Better Non-Perturbative Physics

Can we do better? Yes, go back to 1991-

Dalley, cvj, Morris (DJM) discovered the equation we began with:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0 \ .$$

There are other solutions, e.g.:

$$u(z) \longrightarrow z^{\frac{1}{2}} \ , \ z \to +\infty \ ,$$

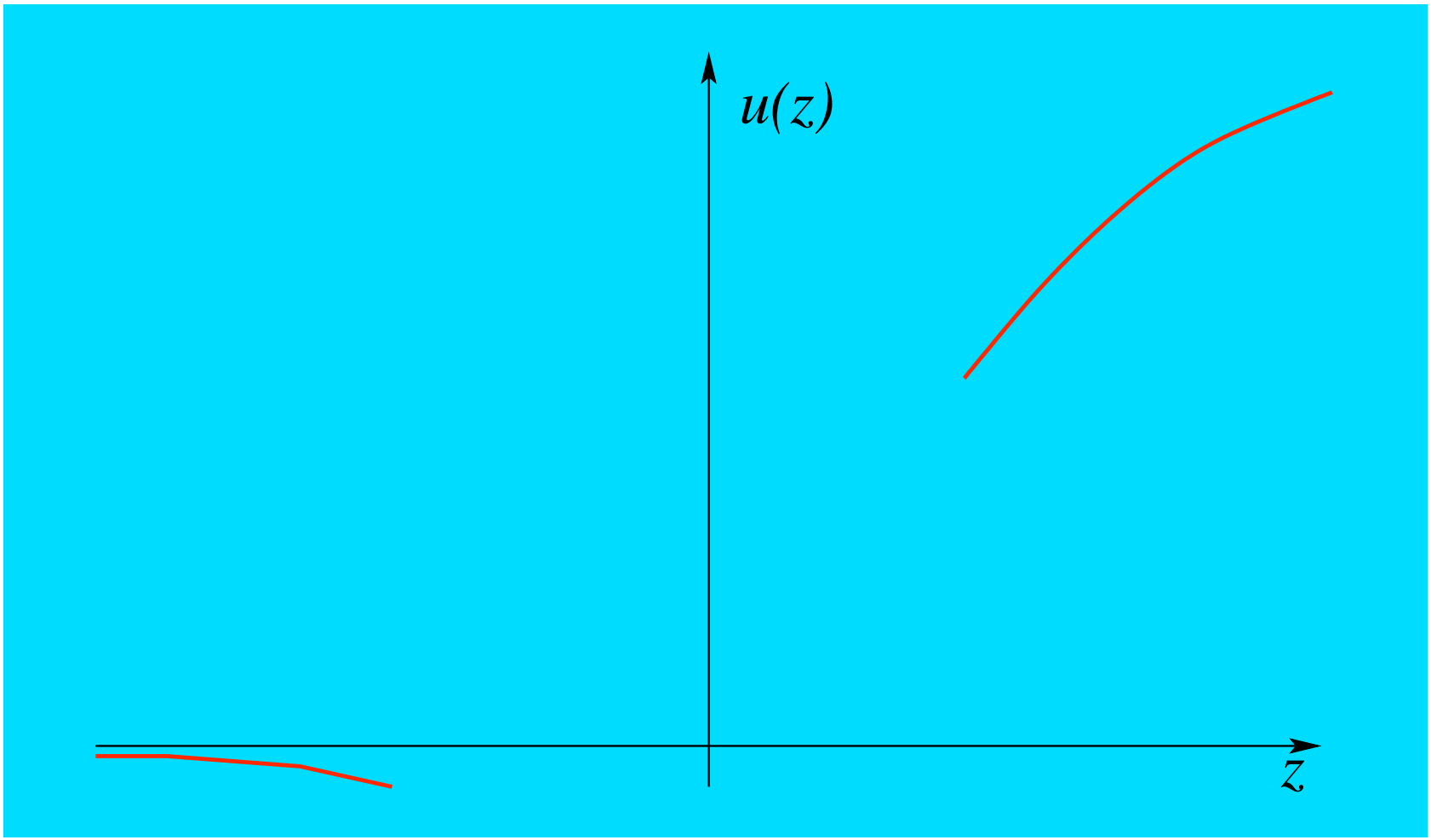
Same as pert. theory from before

but instead... $u(z) \longrightarrow 0 \ , \ z \to -\infty \ .$

New

Better Non-Perturbative Physics

There are really two perturbative regimes now.



Matrix Model Origins

Use (e.g.) a model of a complex matrix:

$$Z = \int dM \exp\left\{-N \mathrm{Tr}[V(M^\dagger M)]\right\} \ .$$

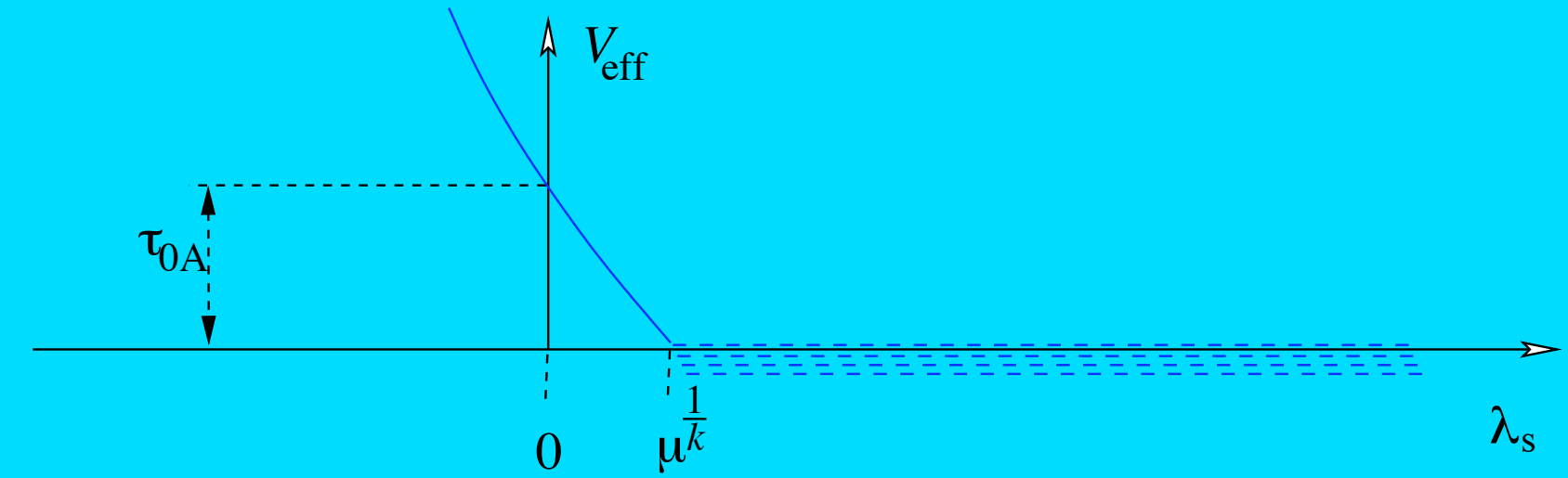
The natural eigenvalues, of $M^\dagger M$, are positive.

Simply put, there is a natural "wall" in the problem.

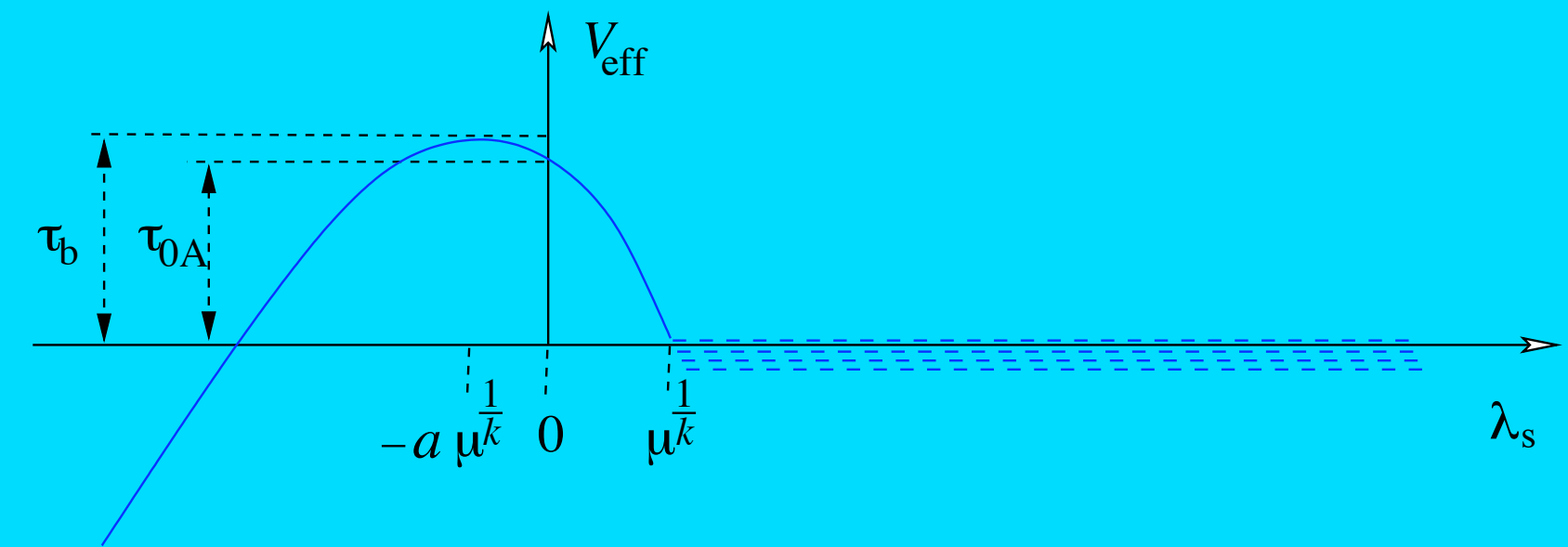
The eigenvalues cannot tunnel to the abyss anymore.

It does not exist.

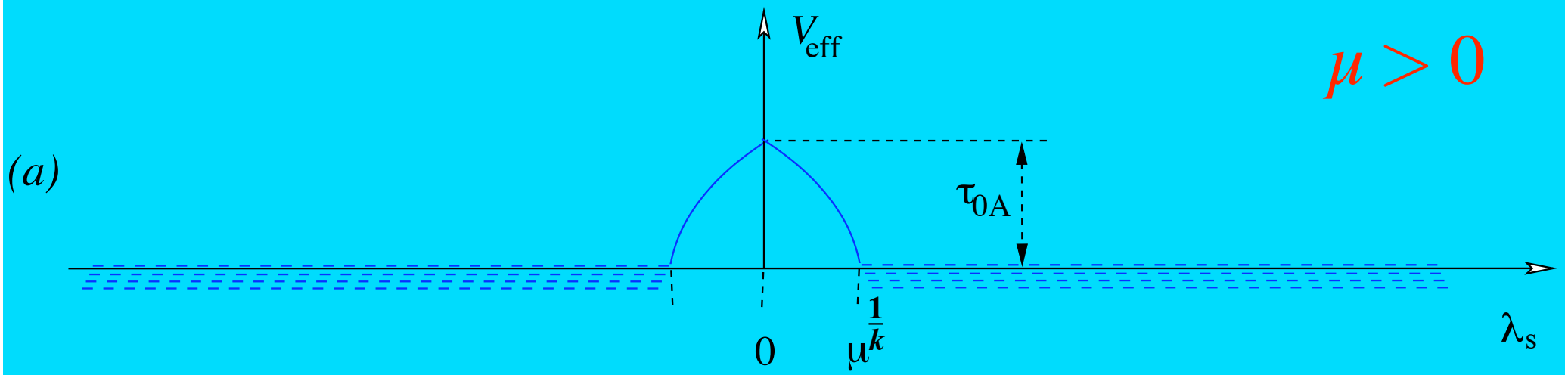
Eigenvalue Space Revisited



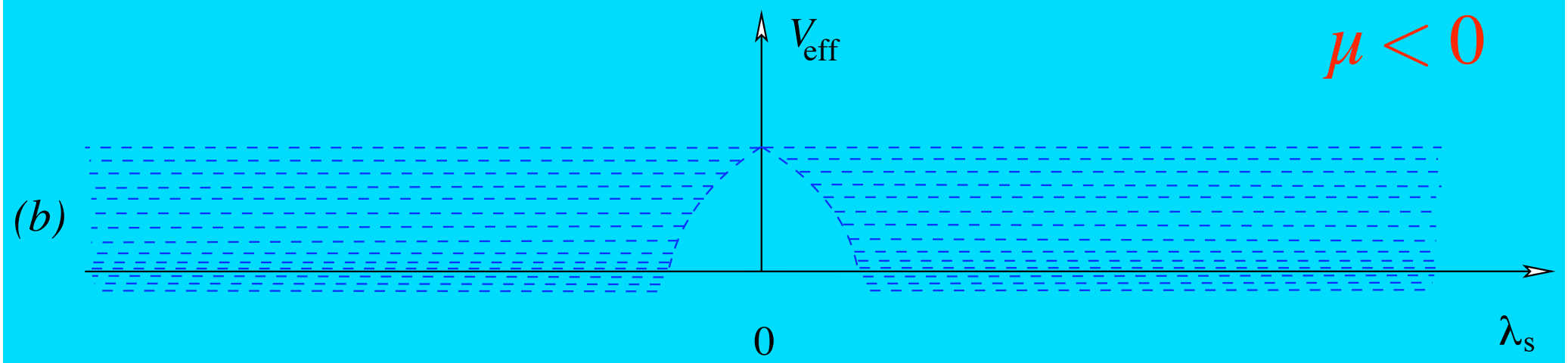
An analysis shows that there is a universal instanton at the wall: a D-brane?



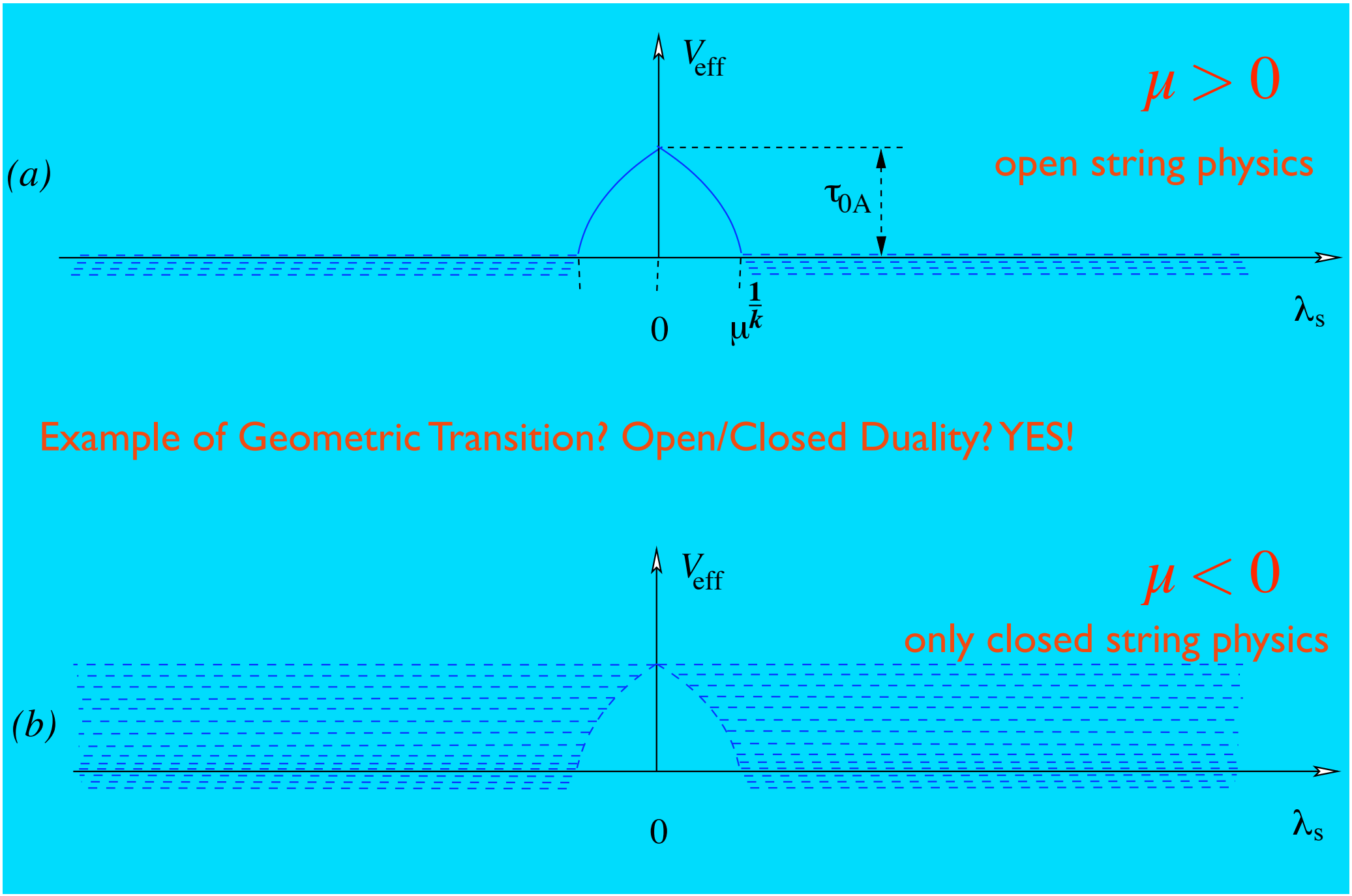
Eigenvalue Space Revisited



This picture is the covering space.... it makes stability more manifest... and clarifies the distinction between the two perturbative regimes....



Eigenvalue Space Revisited



Inserting Extra D-Branes

Extremely simple form for the result

Dalley, cvj, Morris, Watterstam (DJMW) 1993:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2 \,.$$

There's simply a constant on the right hand side.

$$F = \frac{4}{15}g_s^{-2} + \frac{8\Gamma}{5}g_s^{-1} + \frac{(6\Gamma^2+1)}{24}\log\mu - \frac{(4\Gamma^3+3\Gamma)}{48}g_s$$

$$- \frac{1}{5760}(180\Gamma^4 + 345\Gamma^2 + 28)g_s^2 + \cdots$$

$$= \sum_{h,b} C_{b,h} g_s^{2h-2+b}\Gamma^b \,.$$

sphere

disc

cylinder

torus

Rectangular Matrices

Can think of origin in a number of ways. One is as rectangular matrix model.

$$M \text{ is } N \times (N + \Gamma) \text{ matrix } .$$

$$M^\dagger M \text{ diagonalises to :}$$

$$\Lambda_\Gamma = \text{diag}\{\lambda_1, \ldots, \lambda_N, 0, \ldots, 0\} \ .$$

There are Γ extra zero eigenvalues.

Rectangle arises as from model of N D-branes with Γ extra ones supplying fundamental flavours.

The Other Regime

$$F = -\frac{(4\Gamma^2-1)}{4}\log\mu + (4\Gamma^2-1)(4\Gamma^2-9)(4\Gamma^2-25)\times$$
$$\left\{\frac{1}{960}g_s^4 + \frac{1}{30720}(48\Gamma^4 - 1240\Gamma^2 + 8371)g_s^8\right\} + \cdots$$

There seem to be missing orders...only even powers of Γ appear.

What does this regime mean?

Interpretation supplied by Klebanov, Maldacena and Seiberg (KMS):

The theory is type 0A minimal string:
The (4k,2) superconformal models + superLiouville.

The strange regime is closed string with just R-R flux.

This is a geometric transition!

Expectations and Questions

In forward pert. regime, Γ seems to represent some # of D-branes or quantized flux.

Expect them to correspond to the new instanton at the wall.

Take a closer look at full non-perturbative physics.

String equation seems to be happy with any value of Γ:

How does it know that positive integers are special ?

What happens at negative values?

Do the D-branes/fluxes care about the integrable KdV structure at all?

Virasoro Revisited

cvj 1993: Uncover role of Γ in the new Virasoro constraints:

Amazingly compact and suggestive result:

$$\tilde{L}_n \cdot Z_{\{\sigma,\Gamma\}} = 0 \ , \qquad \tilde{L}_n = L_n - (1+n)\frac{\Gamma^2}{4}\sigma^n - \sigma^{n+1}\frac{\partial}{\partial\sigma} \ .$$

Previous partition function can be written as wavefunction on coherent state of a twisted boson. (DVV)

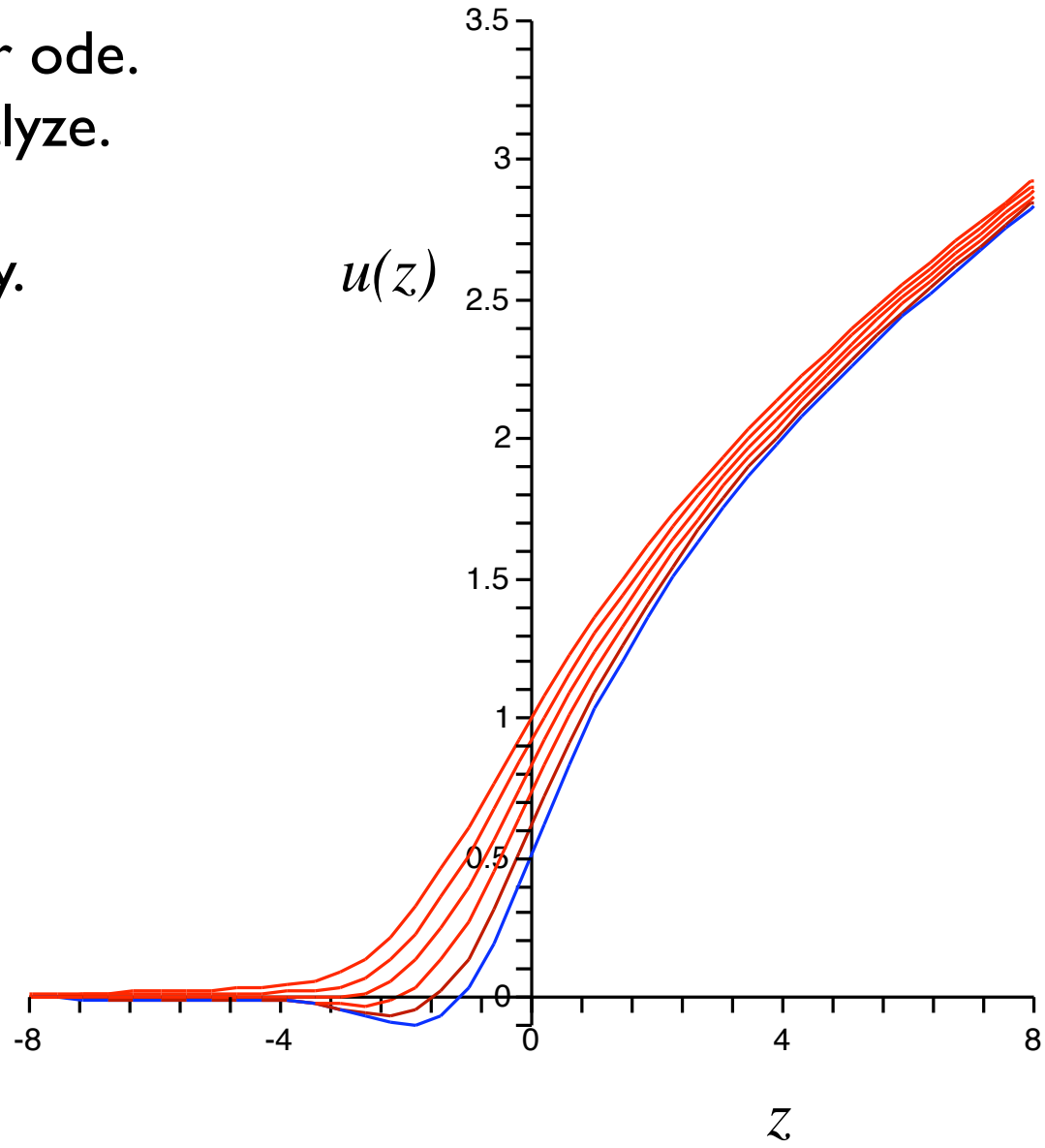
Above form shows that inserting the background is like inserting a vertex operator:

$$V_\Gamma =: e^{-\frac{\Gamma}{\sqrt{2}}\varphi(\sigma)} :$$

Fully Non-Perturbative Solutions

Highly non-linear ode.
Very hard to analyze.

Solve numerically.



This is for positive values. What about negative ones?

Fully Non-Perturbative Information

New analysis: J.e. Carlisle (Durham), cvj, and J.Pennington (USC)

Numerical Observations:

Can explore negative values of Γ.

The well gets deeper.

Hard to get to $\Gamma = -1$. Is this is a clue?

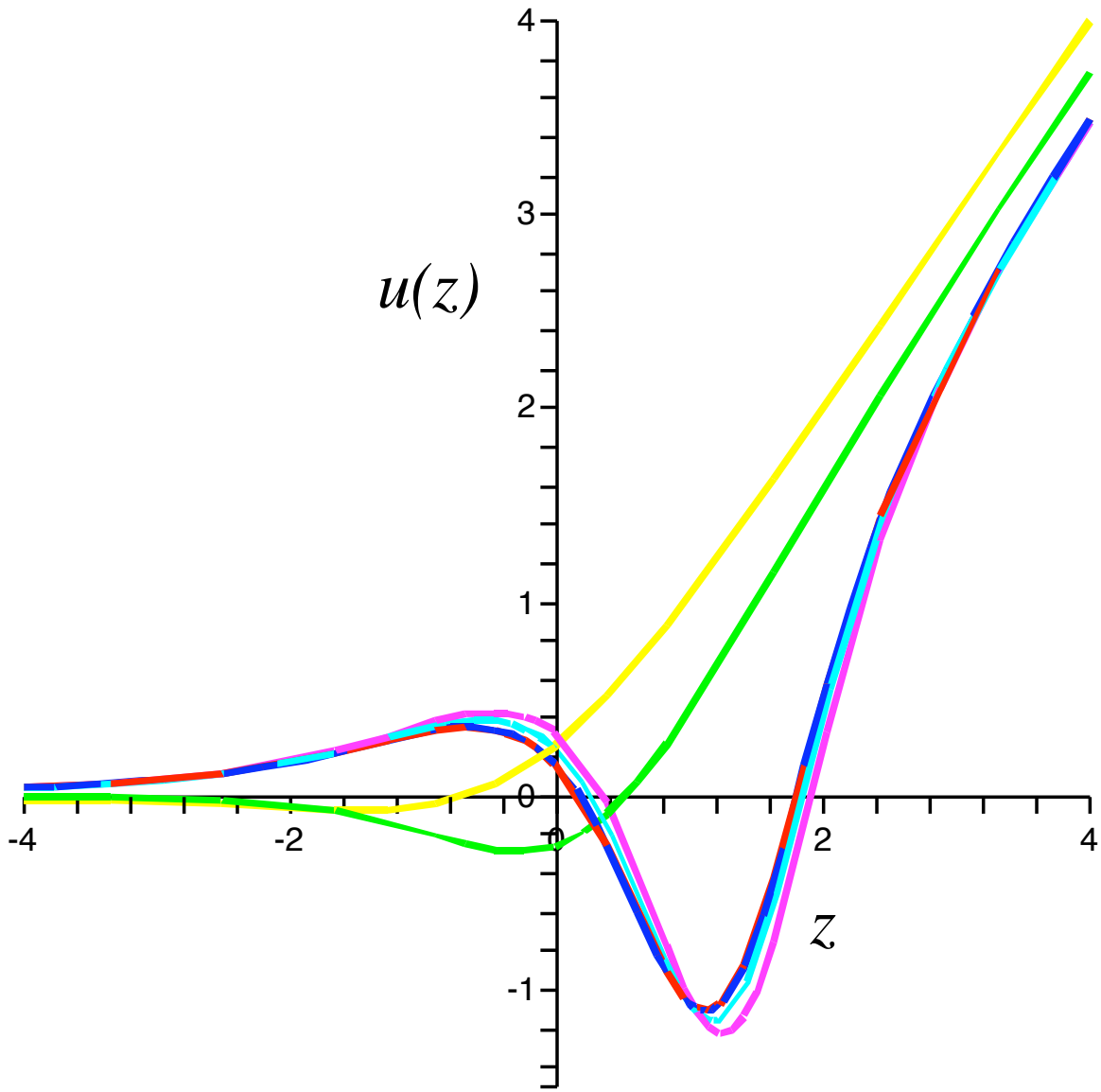
What is spectrum of $\mathcal{H} = -v^2 \frac{\partial^2}{\partial z^2} + u(z)$?

There is a continuum.

But the well is getting deeper....bound states may form!

Well gets narrower at same time....absence of bound states may pesist.

Fully Non-Perturbative Information



Fully Non-Perturbative Information

Analytical Observations:

Re-examined DJMW 1993 paper. There is a natural map:

$$u_\Gamma \longrightarrow u_{\Gamma\pm1} \ .$$

We were able to extract an explicit formula for this!

$$u_{\Gamma\pm1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Gamma\mathcal{R}' + 4\nu^2\Gamma^2}{4\mathcal{R}^2} \ .$$

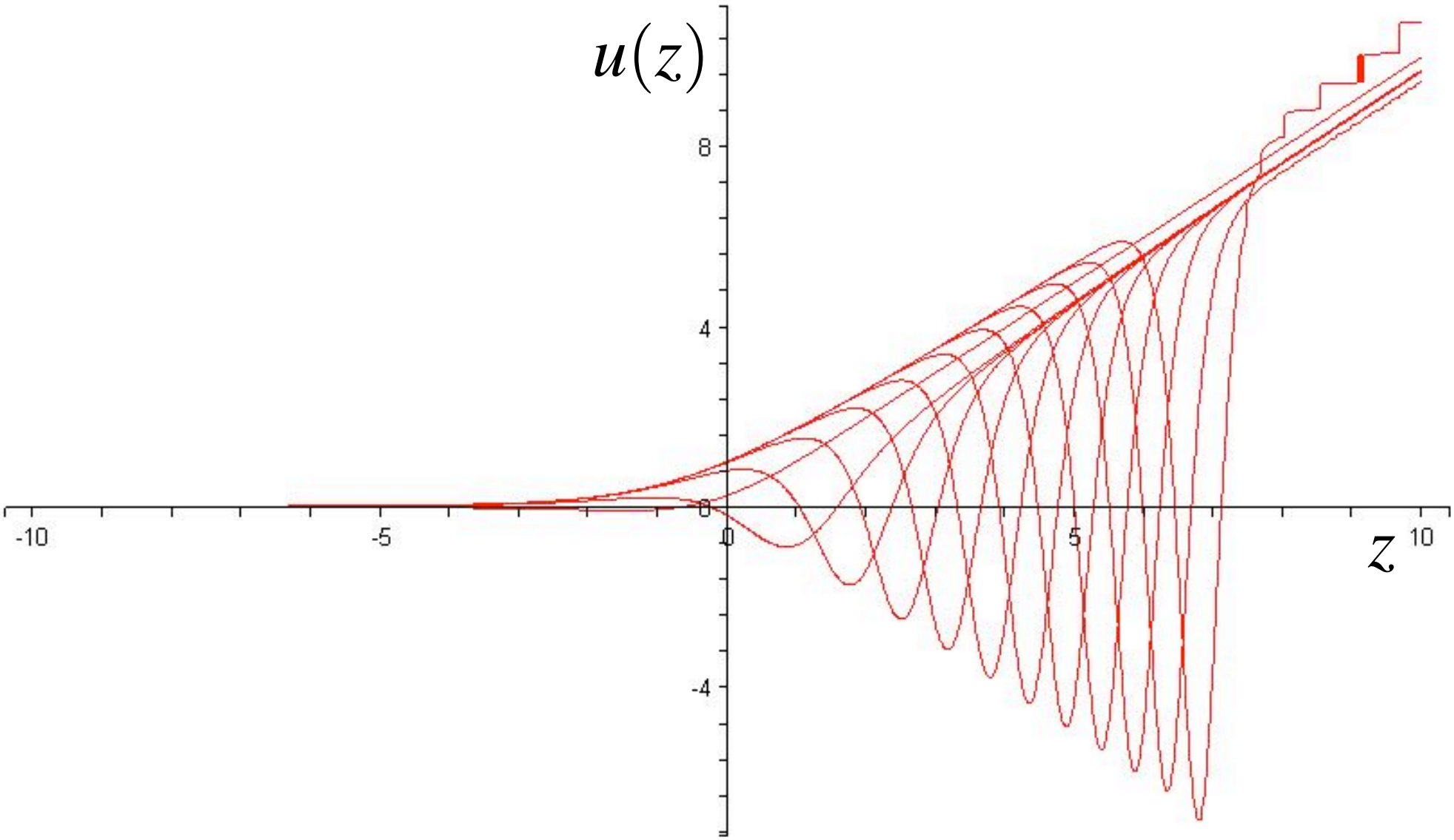
This allows better control over the $\Gamma < 0$ numerical explorations.

We observe that the well gets dramatically deeper as approach $\Gamma = -1$.

We observe that there were no bound states down to $\Gamma \simeq -0.999$.

Something interesting happens at $\Gamma = -1$.

Fully Non-Perturbative Information



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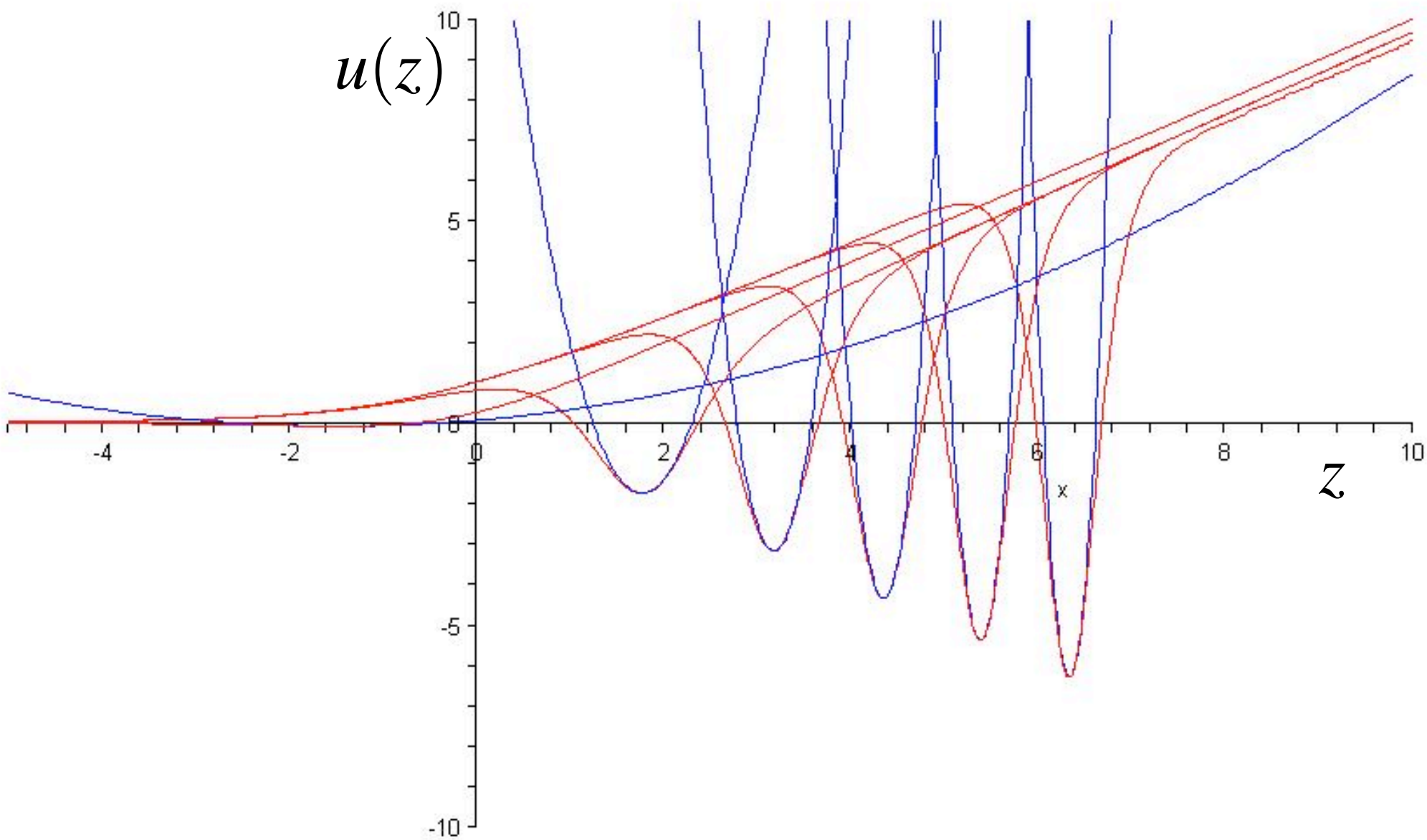

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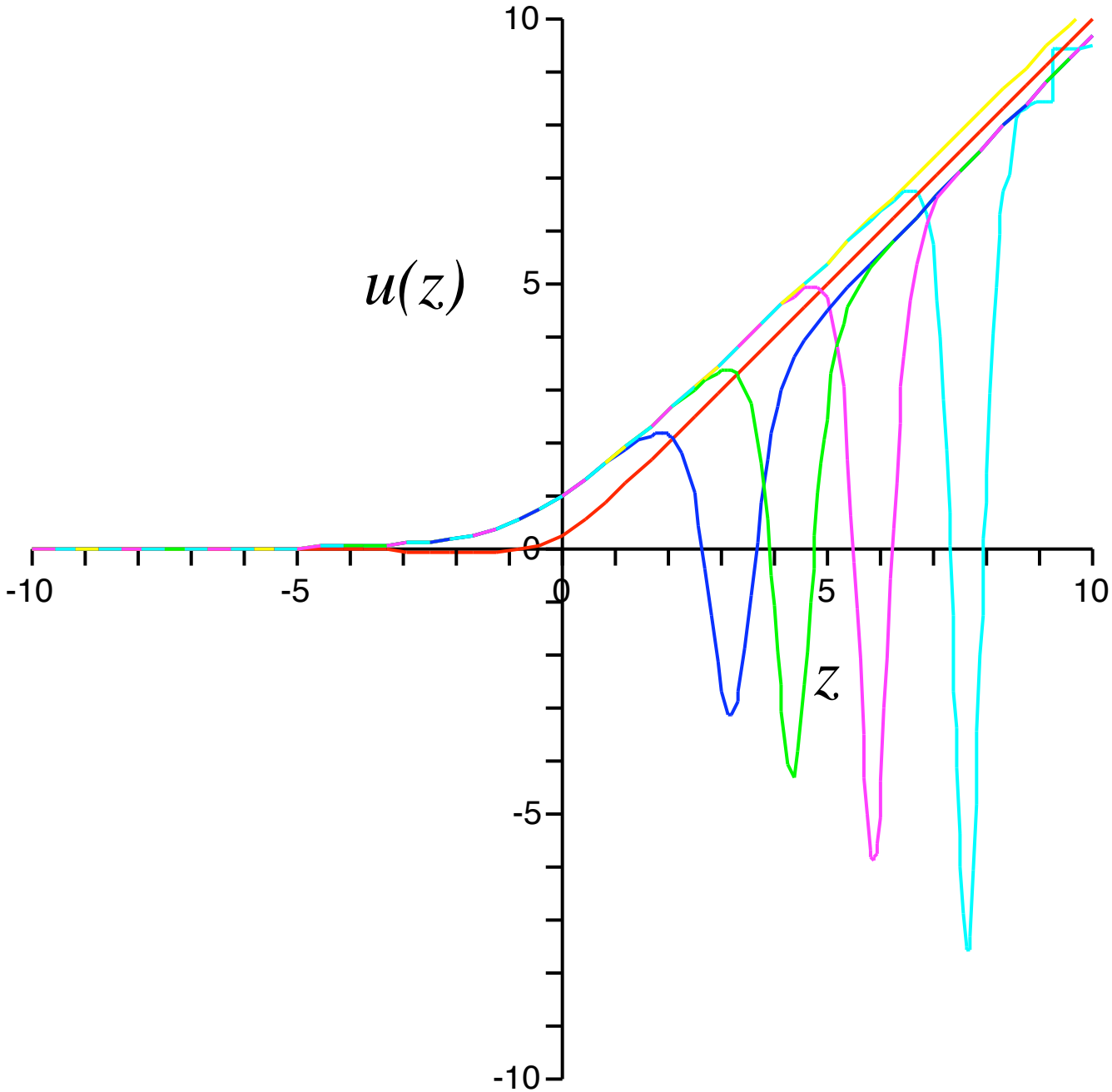

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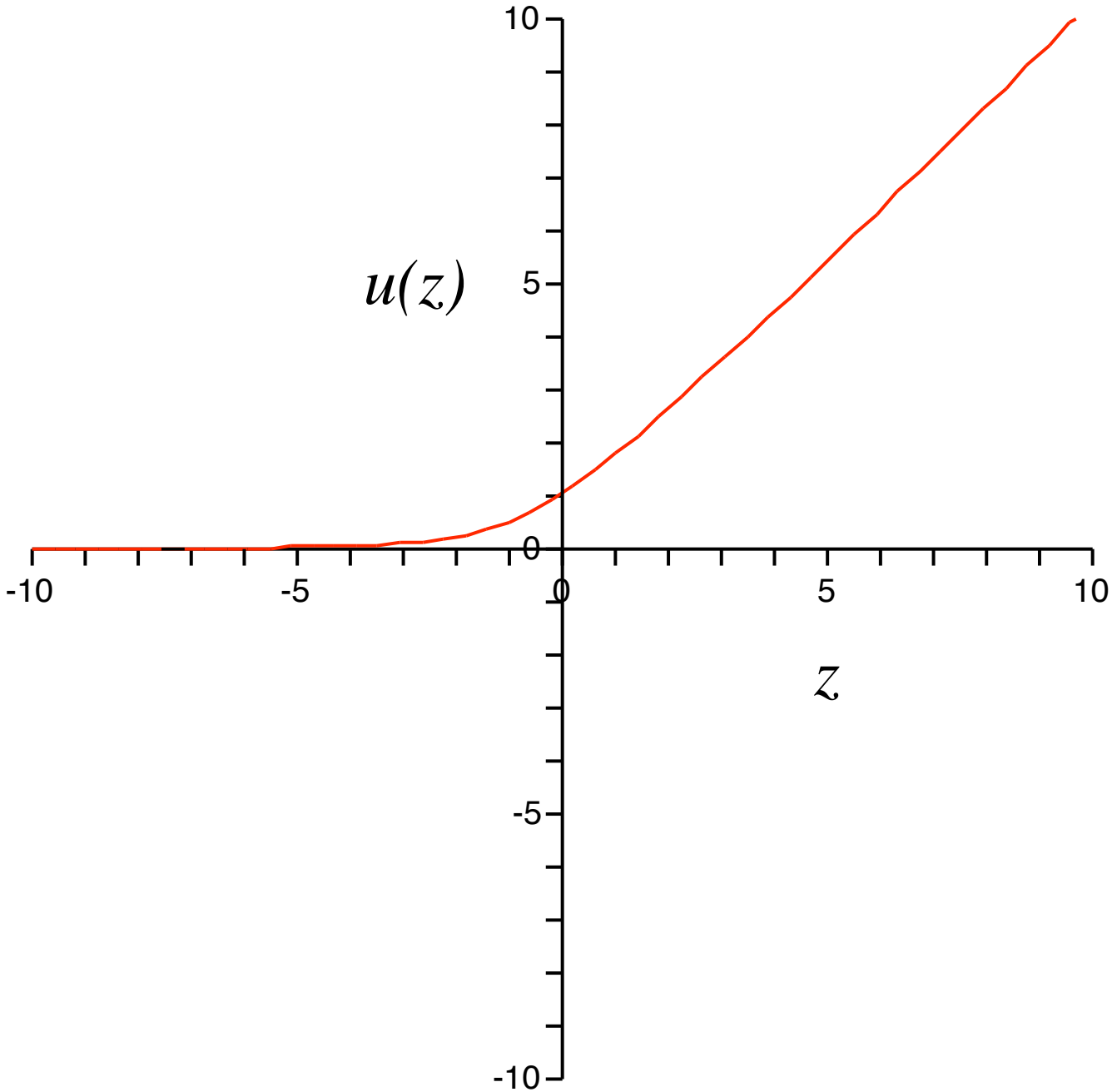

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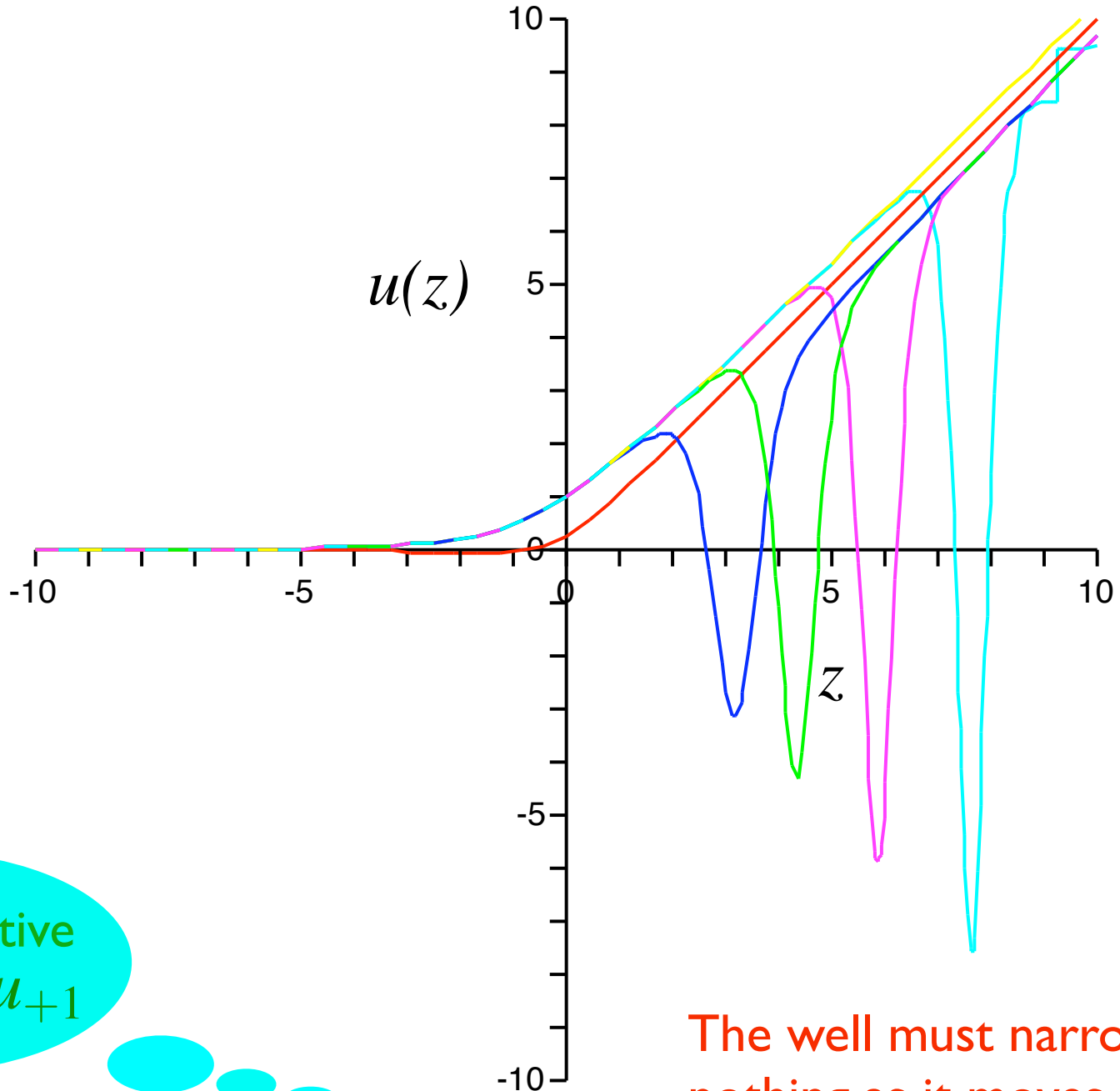

Highly suggestive that $u_{-1} = u_{+1}$

The well must narrow away to nothing as it moves to infinity.....

A Non-Perturbative Proof That $u_{-1} = u_{+1}$

$$u_{\Gamma\pm1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Gamma\mathcal{R}' + 4\nu^2\Gamma^2}{4\mathcal{R}^2} \,.$$

Use the $\Gamma = 0$ solution as a seed to get the $\Gamma = \pm 1$ cases!

$$u_{-1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}''}{4\mathcal{R}^2} = u_{+1}$$

Precisely the case of putting $\Gamma = 0$ gives either one of them.

Fully Non-Perturbative Information

Meaning:

These observations establish that the natural values are positive integers.

$u_\Gamma \longrightarrow u_{\Gamma \pm 1}$. And the case $\Gamma = -1$ returns us to the positive integers.

Fully non-perturbatively, the spectrum is continuous and bounded from below for these cases.

Any non-integer value is connected to cases with poles, i.e., $\Gamma < -1$

Bäcklund Transformations

What is meaning of the transformation?

Work with first derivative of the string equation:

$$u'\mathcal{R} + 2u\mathcal{R}' - \frac{1}{2}\mathcal{R}''' = 0 \ .$$

$u_\Gamma \longrightarrow u_{\Gamma\pm 1}$ is map between solutions of same equation.

This is generically a Bäcklund transformation.....

...but better, it is *the* KdV Bäcklund transformation!!

It is an explicit form of the known tranformation, specialized to self-similar (under scaling) solutions of the KdV hierarchy.

Bäcklund Transformations

What is meaning of the transformation?

KdV Bäcklund transformations usually relate solutions containing different numbers of solitons.....

They are bound states of the associated Hamiltonian, with energy $\hat{\lambda}$.

The velocity of these solitons is set by $v \simeq -\hat{\lambda}$.

In our case, there are no bound states, and we have Γ repeated zero eigenvalues $\hat{\lambda} = 0$.

Like threshold bound states?

$u_\Gamma \longrightarrow u_{\Gamma \pm 1}$ Changes the number of such "zero-velocity" solitons.

Summary

We've learned more about background D-branes of minimal string theory. These branes are charged and localized. ("ZZ Branes".)

We discussed several points of view, and focussed here on the string equations and their fully non-perturbative physics.

Remarkably, the string equations' properties are such that the cases of positive and integer Γ are singled out!

By uncovering the role of the celebrated Bäcklund transformations, we also learned that D-branes are intimately related to the solitons of the KdV hierarchy in a rather amusingly direct way.

The results and approach here are complementary to the continuum string approach. (See, e.g. Seiberg and Shih, *hep-th/0412315*)

Obvious conjecture can be stated for several other integrable hierarchies. They may define interesting string theories with branes, fluxes, etc, too!