

The structure of half BPS states in ADS/CFT

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Based on: hep-th/0306090, hep-th/0403110
hep-th/0411205 V. Balasubramanian, B. Feng, M Huang.
hep-th/0501078 with S. Vazquez

Also, Lin, Lunin, Maldacena, hep-th/0409174

Motivation

The AdS/CFT gives a dual description of string theory in negatively curved space time.

The most celebrated and studied case is the $\mathcal{N} = 4$ SYM.

The duality is a strong-weak coupling duality for the 't Hooft coupling. It is hard to make computations.

- We know a lot about closed strings in the CFT dual of AdS.
 - They are described by traces.
 - They seem to give rise to an integrable structure.
 - Well known limits: BMN, semiclassical strings, supergravity (BPS multiplets).

What do we know about D-branes in AdS/CFT?

- Some BPS states. (All supergravity solutions with 1/2 SUSY's)
- Low energy fluctuations for single D-brane states.
- Add infinite D-branes (defect CFT, flavor)

We don't know very much about the dynamics of branes on AdS.

Even qualitative aspects are missing!

- Operators that describe branes at angles?
- Gauge symmetry on compact D-branes?
- DBI effective action?

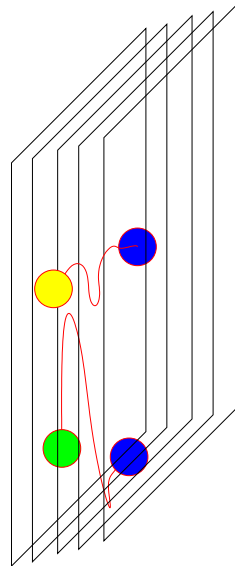
Plan for the talk

- Half BPS D-branes in $AdS_5 \times S^5$.
- Towards the dual description of half-BPS states.
- Free fermions and quantum droplets
- Eigenvalues and Combinatoric D-branes.
- Stacks of D-branes: New SUGRA solutions.
- Quantum droplets as spacetime bubbles
- Outlook

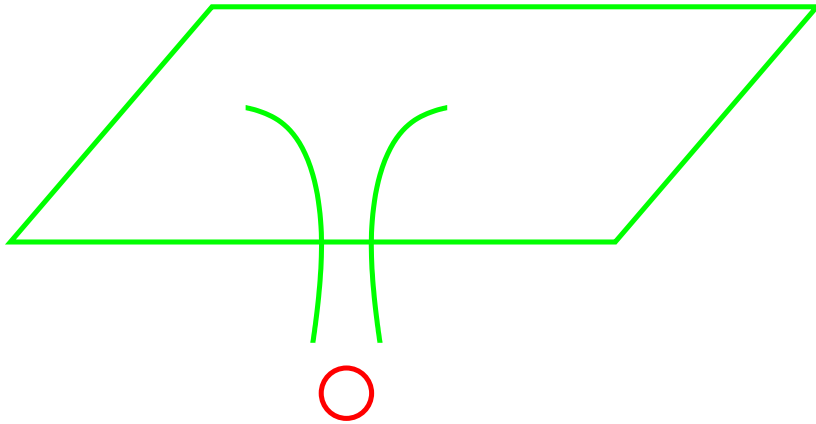
The AdS/CFT correspondence

Let us look at type IIB string theory in flat ten dimensions.

Let us consider now a stack of N parallel D3-branes located on top of each other, or separated just a little bit, but at a distance much smaller than the string scale, let us call it r .



We are interested in understanding the degrees of freedom of the D-brane stack which are very light. This is, we are looking for degrees of freedom whose energy is much smaller than the string scale.



Due to redshift at the horizon, closed strings near the horizon can have very small global energy even if they are massive: they decouple from the bulk of spacetime and they describe IR physics near the brane.

One can take the near horizon limit to understand the throat geometry : one obtains a IIB string theory on $AdS_5 \times S^5$ with N units of flux (number of D-branes)

AdS/CFT correspondence:

The quantum equivalence of these two forms of taking the IR dynamics on the collection of D-branes.

	SYM	$AdS_5 \times S^5$
Symmetry	$SU(2, 2 4)$	$SU(2, 2 4)$
Rank	$U(N)$	$\int_{S^5} F = N$
Coupling constant	g_{ym}^2	g_s
Radius	$g_{ym}^2 N$	R^4

OBSERVABLES

The $\mathcal{N} = 4$ SYM is a CFT. This means that using radial quantization one can find a relation between operators inserted at the origin, and states on an S^3 .

The idea is simple

$$ds^2 = r^2 \left(\left(\frac{dr}{r} \right)^2 + d\Omega_3^2 \right)$$

We can label all states on S^3 by operators inserted at the origin.

Witten, Gubser, Klebanov Polyakov: dictionary between states for strings on $AdS_5 \times S^5$ and single trace operators in $\mathcal{N} = 4$ SYM. Supergravity states are half BPS: they can be understood at weak coupling. They are descendants of $\text{tr}(Z^n)$ where Z is one complex scalar of the $\mathcal{N} = 4$ SYM.

D-branes on $AdS_5 \times S^5$: giant gravitons.

Are there interesting half BPS states in $AdS_5 \times S^5$?

- Fix N large, but do not take the infinite N limit
- Notice that traces are algebraically independent up to order N
- $Tr(Z^{N+1}) = \text{Polynomial in } Tr(Z^k)$
- This upper limit on the gravitons is called a stringy exclusion principle.

Susskind's idea: strings with a lot of energy grow transversely to their direction of motion. Also holographic arguments suggest that this is necessary to avoid making a black hole.

We could look for D-brane solutions which represent this growth.

Place a D3-brane wrapping a round S^3 in S^5 and give it a lot of angular momentum on the S^5 . One can find dynamically stable solutions of this kind. (McGreevy, Susskind, Toumbas)

Moreover, one can show that they preserve half of the supersymmetries.

The size of the S^3 inside the S^5 that the D-branes span has a maximum size.

The name for these branes is Giant gravitons

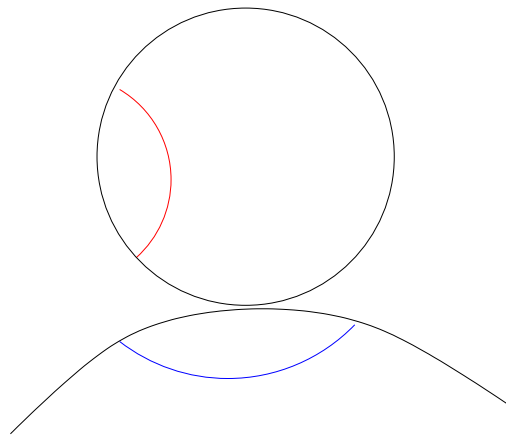
The effective action for the D-branes is of the DBI type

$$S \sim \int \sqrt{-\det(g_{ind})} + \int f^*(A_4)$$

The presence of the flux gives some magnetic field under which the D3-brane is coupled. There is a Lorentz force between the velocity and the magnetic field which prevents the D-brane from collapsing.

There is another giant graviton!

This one wraps an S^3 inside AdS_5 and has a lot of angular momentum along the S^5 . It is also dynamically stable and it also preserves half of the SUSY's. (Hashimoto , Hirano & Itzhaki, Grisaru, Myers & Tadjford.)



Towards the dual description of giant graviton states

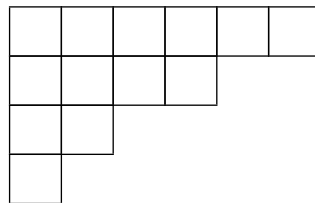
The giant gravitons have an energy of order N . They have to be built out of the same type of ingredients than other half BPS states. They have to be described as operators which are (descendants of) multi-traces of Z . Traces are not a good description of operators for these energies because they don't give rise to orthogonal states (not even at leading order in $1/N$).

In particular this means that we have to look for a better description of the operators.

This is provided by Schur polynomials.

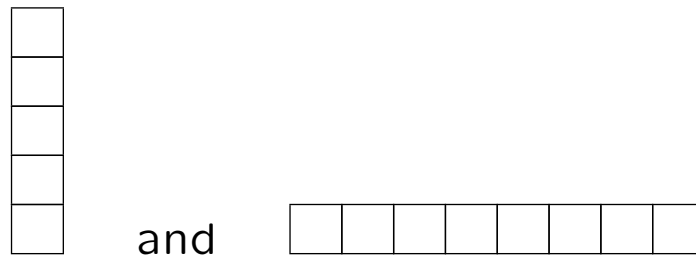
Z is a (operator valued) matrix in the adjoint of the group $U(N)$. One can consider Z as generic matrix of $GL(N, \mathbb{C})$. The trace of Z can be taken in a representation of the group which is **not the fundamental**.

Irreps are given by Young tableaux with columns of length less than or equal to N



These form a complete basis of half BPS states, one for each tableaux (Jevicki et al).

Of these tableaux, which represent the giant gravitons growing into S^5 and into AdS_5 ?



The first were conjecture by Balasubramanian et al. based on their better orthogonality properties. These are subdeterminants. The second set was conjectured by Jevicki et al, based on symmetry of the Young tableaux.

Free fermions and quantum droplets

If we use the [operator state correspondence](#) we can understand these states in the Fock space of the $\mathcal{N} = 4$ SYM theory compactified on S^3 . We use the letter Z now to indicate a particular raising operator for the s-wave of the field Z on S^3 .

All states are built out of this s-wave mode.

Because the states are BPS, we can integrate out all other fields, and there are no quantum corrections to the effective action of the s-wave mode of Z

What is the effective dynamics of this mode?

It is the dynamics of a (gauged) matrix valued harmonic oscillator. The mass term for the field Z arise from the conformal coupling of the scalars to the metric of the round S^3

$$S \sim \int dt \text{tr} \left[\frac{D_t Z^2}{2} - \frac{Z^2}{2} \right]$$

How do we solve the model?

Choose $A_0 = 0$, solve the system, and impose Gauss constraint.

Since we get a free system when $A = 0$, it is trivial to solve. Use the letter Z for the matrix valued raising operator associated to the dynamics. The full list of states is

$$\text{tr}(Z^{n_1}) \dots \text{tr}(Z^{n_k})$$

we can always choose the n_i so that

$$N \geq n_1 \geq \dots n_k$$

This also makes contact with the counting by Young tableaux. That is a different basis, but the counting of states per energy level is the same.

We have reduced the problem to a one matrix model.

We can also solve the problem by choosing a gauge where Z is diagonal.

Classically this gives us a **free harmonic oscillator per eigenvalue**. However:

- Permutations of eigenvalues are part of $SU(N)$ and are gauged. Therefore we obtain a system of **N bosons in the harmonic oscillator**.
- There is a change of variables to diagonal matrices from generic matrices. This involves a change of measure.
- The measure is the volume of the orbit of the matrix Z under gauge transformations.

This measure is well known: it is the square of the Vandermonde determinant.

The hamiltonian is therefore

$$\frac{1}{2} \left(\sum -\mu^{-2} \partial_{z_i} \mu^2 \partial_{z_i} + m z_i^2 \right)$$

$$\mu^2 = \prod_{i < j} (z_i - z_j)^2 = \Delta(Z)^2$$

$$\Delta(Z) = \det \begin{pmatrix} 1 & 1 & 1 & \dots \\ z_1 & z_2 & z_3 & \dots \\ z_1^2 & z_2^2 & z_3^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ z_1^{N-1} & z_2^{N-2} & z_3^{N-3} & \dots \end{pmatrix}$$

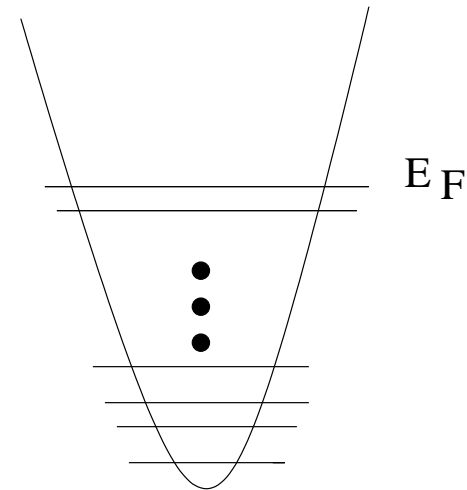
Consider describing the system in terms of the following wave-functions $\tilde{\psi}$

$$\psi = \Delta^{-1} \tilde{\psi}$$

The new Hamiltonian is free! (Brezin, Itzykson, Parisi, Zuber, 1978)

$$\tilde{H} = \frac{1}{2} \left(\sum -\partial_{z_i} \partial_{z_i} + m z_i^2 \right)$$

Notice that as far as the wave functions are concerned, we have a system of N free fermions in the Harmonic oscillator.



The vacuum is characterized by a Fermi energy

A complete set of wave functions can be described by the Slater determinants of the Harmonic oscillator energy eigenstates.

If we count the energy eigenvalues from the top eigenvalue down, the spectrum is described by a set of integers with the property $n_1 > n_2 > \dots > n_N$. The ground state is described by the integers $N - 1, N - 2, \dots, 0$.

We can associate a Young tableaux to these states as follows:

Write a tableaux with rows of length $n_i - (N - i)$. These are strictly decreasing.

It turns out that this description coincides exactly with the description based on group characters (Schur polynomials) that we had before.

From that point of view we have an identification:

Giant gravitons on AdS_5 correspond to raising the topmost eigenvalue by a very large amount: it is a fermion.

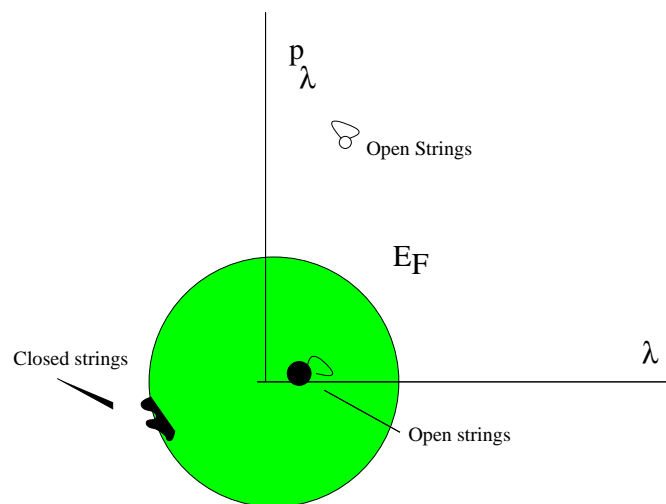
Giant gravitons on S^5 correspond to raising by one unit a lot of the eigenvalues: it is a hole in the Fermi sea.

In the past, it has been useful to consider matrix models in the phase space of the classical system. From this point of view we get a fermion droplet in two-dimensional phase space

A 2-D phase space can be associated to the degrees of freedom of a charged particle

in a constant magnetic field in the **lowest LANDAU LEVEL**.

A quantum hall system for free electrons.



How do we understand that these states are D-branes?

- Open strings end on them
- They have gauged dynamics on their worldvolume

Eigenvalues and combinatoric D-branes

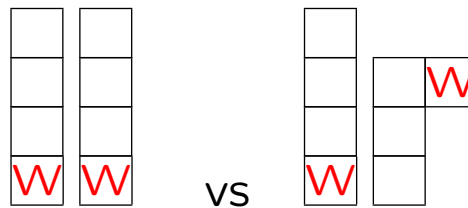
How do we build other gauge invariant states with the other fields in SYM?

We can consider Young-tableaux for giant gravitons and other fields on S^3 .
For example

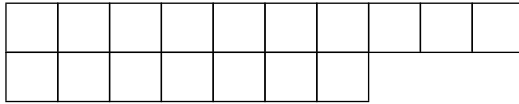


This can be interpreted as adding an open string to a giant graviton if W is a word in the fields (same as closed strings, but not put together in a trace).

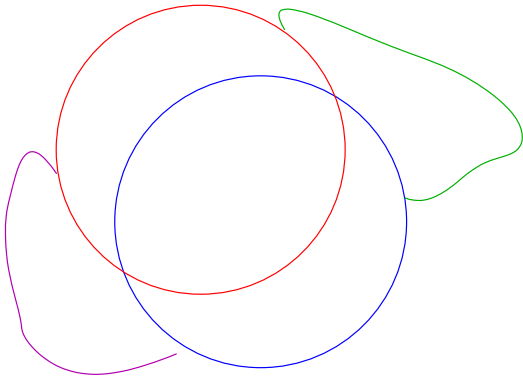
Caveat: We should keep track of both upper and lower indices: statistics of Z makes upper and lower indices transform the same way, but not for extra words.



Multiple giants:



How do we add strings stretching between giants? And how do we think of Gauss law?



Caricature: two giants with two strings of opposite orientations stretching between them

Two giants of different sizes. Each corresponds to an eigenvalue/hole having high energy.

For the case of eigenvalues we can treat them semiclassically.

$$Z \sim \begin{pmatrix} z_1 & 0 & \dots \\ 0 & z_2 & \dots \\ \vdots & \vdots & 0 \end{pmatrix}$$

This is just like Higgs mechanism: $U(N)$ is broken to $U(1)^2 \times U(N-2)$.

The phase of z_1 and z_2 are the position of the giant along S^5 . If phase and norm coincide then the two rows have equal length, and the symmetry is broken to $U(2) \times U(N-2)$. For this case we see that the gauge symmetry of the AdS giants is embedded in the original $SU(N)$ group.

A similar semiclassical description for holes is not known.

Gauss law and gauge invariance.

How do we string various strings between giants?

For giants along AdS we can think in terms of the spontaneous symmetry breaking. There are some bifundamentals which under $U(N-2) \times U(1)$, mark them with a different color. Remember that they also carry an upper or lower index with respect to the corresponding $U(1)$ charges (positive or negative charge: orientation)

Open strings stretching between giants will look like states:

Y word Y

acting on a semiclassical background for the eigenvalues.

This has charge under the $U(1) \times U(1)$, so it is not gauge invariant.

We can make it gauge invariant by adding another state

Y word Y

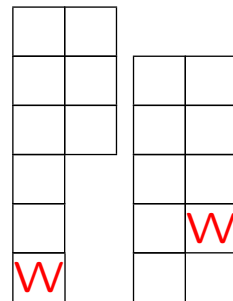
This should be interpreted as Gauss law: the D-brane is compact, so the total incoming string charge has to be balanced by the total outgoing string charge.

We also see the enhanced gauge symmetry for AdS giants when they come together.

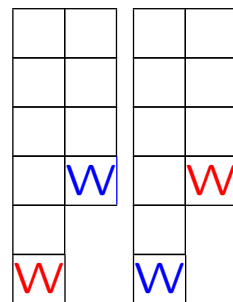
The gauge symmetry of AdS giants is embedded in the $U(N)$ gauge symmetry.

How do we see this for the S giants?

Use double Young tableaux:



This is a string stretching between two giants. Upper and lower indices don't transform the same way. We can not build a gauge invariant state from this. We need to add a second string that goes in the opposite direction:



We see Gauss law appearing again.

One can also see that when giants coincide there are "less states". The states are then in agreement with counting gauge invariant states on a $U(2)$ field theory.

One can also show that one reproduces the quadratic fluctuations of the DBI action for both types of D-branes : This is very non-trivial for the S^5 giants (only maximal giant is known).

Open string ending on MAXIMAL giant

- Why focus on maximal giant?
 1. In supergravity it is a static brane in global coordinates. Also it is geometrically the most symmetric configuration (diameter)
 2. Easy to take a Penrose limit on the worldvolume of the D-brane.
 3. It descends to dibaryons on orbifolds. (Most similar to baryons in QCD)

Explicit form of the giant operator

$$O_k \sim \epsilon^{\mu_1 \dots \mu_N} \epsilon_{\rho_1, \dots, \rho_N} (Z_{\mu_1}^{\rho_1} \dots Z_{\mu_k}^{\rho_k} \delta_{[\mu]}^{[\rho]})$$

And we use the simplifying notation

$$O_k \sim \epsilon \epsilon_{k,k} (Z, Z, \dots, Z)$$

To add a string, we put

$$O_{k,W} \sim \epsilon \epsilon_{k+1,k+1} (Z, Z, \dots, Z, W)$$

Notice: A giant is maximal if $N - k$ is small (of order 1, or up to \sqrt{N}).

Boundary conditions, I

In $\mathcal{O}_{k,W}$, W is not allowed to begin or end in Z .

$$\mathcal{O}_{k,W} \sim \mathcal{O}_{k+1,W'} + \mathcal{O}_{k+1} \text{tr}(W')$$

This is a Dirichlet like boundary condition for W .

Now we look at maximal giant.

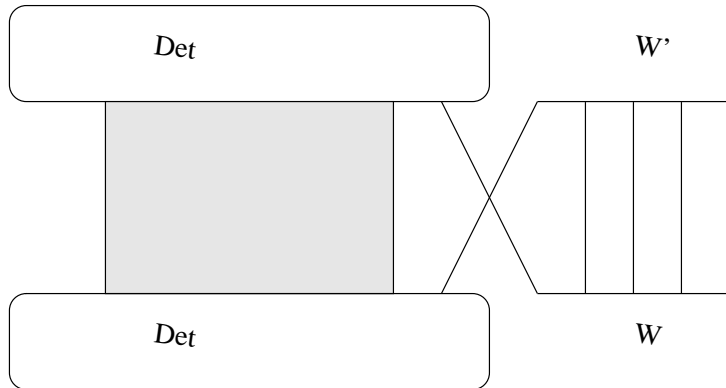
Boundary conditions, II

We need to split the problem into planar and non-planar diagrams. This is tricky in general.

Combinatorial Facts: For maximal giants the length of the spin chain is conserved by planar diagrams at one loop and it can not exchange Z with the giant by planar diagrams.

Spin chain facts: For the inside of the word W , the system one gets is the same $SU(2,2|4)$ integrable spin chain model of $\mathcal{N} = 4$ SYM theory.

Something special happens at the boundary



These also count as planar diagrams.

One can obtain the full open string spin chain model for $SO(6)$ sector easily.

Boundary conditions for Z, \bar{Z} are slightly different, but both "Dirichlet".

Ground state for spin chain of length L gets no planar one loop anomalous dimension, this lets us reproduce the DBI fluctuations and take a Penrose limit,

More importantly the open spin chain has integrable boundary conditions.

Quantum droplets as spacetime bubbles

Can we understand these D-branes we have studied in the same way that we understand the AdS/CFT?

Namely, can we consider all the supergravity solutions associated to these D-brane configurations?

Experience dictates that supergravity can describe coherent states of gravity waves and large collections of coinciding D-branes.

From the fermion droplet point of view these are described by two-dimensional drawings in two colors: We can not put the fermions exactly on top of each other, but we can condense them into droplets of various shapes and sizes.

Problem: Consider all 1/2 BPS solutions of supergravity with asymptotic $AdS_5 \times S^5$ boundary conditions.

This is exactly what Lin, Lunin, Maldacena considered recently.

One needs to consider solutions with a symmetry $SO(4) \times SO(4) \times \mathbb{R}$

This last translation symmetry is associated to the BPS bound.

The general shape of the solutions has to be of the form

$$ds^2 = g_{tt}(dt + V dx)^2 + g_3(dx dx) + \exp(G + H)d\Omega_3^2 + \exp(H - G)d\Omega'_3{}^2$$

and

$$F_5 \sim F_{\mu\nu}d\Omega_3 + F_{\mu\nu}^*d\Omega'_3$$

G, H , depend only on $x_{1,2,3}$. This ansatz just uses the bosonic symmetries of the system. Supersymmetry places a lot of additional constraints.

One can use spinor analysis to construct various conserved forms $\bar{\psi}\Gamma\psi$.

With these objects one finds that there is a special coordinate y , which is singled out by SUSY, and corresponds to the product of the radii of the spheres

$$\exp(H) = y$$

$$y \geq 0$$

so that when $y = 0$ one of the two spheres contracts. This depends on the values of e^G at that point..

All the final details are complicated to arrive at, however they can be written simply.

$$\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + y \exp(G) d\Omega_3^2 + y \exp(-G) d\tilde{\Omega}_3^2 \\
h^{-2} &= 2y \cosh(G) \\
y \partial_y V_i &= \epsilon_{ij} \partial_j z, \quad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij} \partial_y z \\
z &= \frac{1}{2} \tanh G
\end{aligned}$$

One can show that it all depends on z . And from the third line it follows that z satisfies

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y} \right) = 0$$

(A generalized Laplace-like equation)

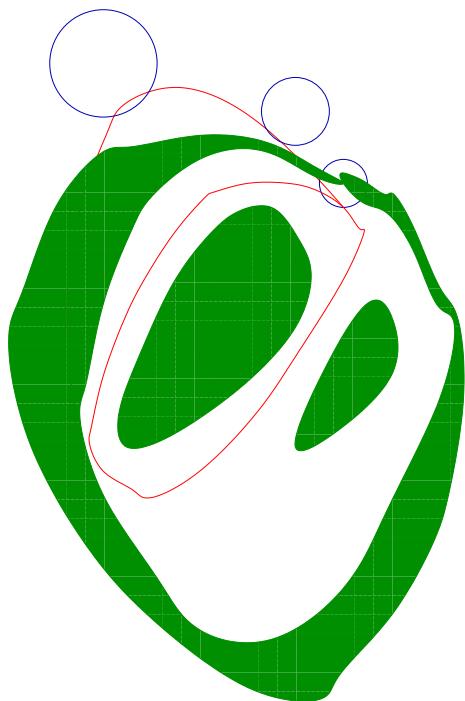
$y = 0$ is special. It corresponds to degenerations of the spheres. To have singularity free solutions z has to take special values. Other values lead to cone-singularities.

There are two special values of $z = \pm 1/2$ that correspond to only one of the spheres degenerating in such a way that the degeneration looks just like spherical coordinates. Given z in the boundary it can be rebuilt in the interior of the x_1, x_2, y geometry.

The boundary $y = 0$ is two-dimensional (x_1, x_2) , and it is painted in two colors

These configurations correspond exactly with the quantum plane description I gave before.

Topology



Notice that there are non-contractible spheres in the spacetime. These correspond to a topology change from the $AdS_5 \times S^5$ geometry.

SYM sum over topologies automatically. We see a physical realization of the "quantum foam" where each droplet is a (topological) bubble of spacetime.

One can even argue that for very small defects close to another droplet, they are better described as ordinary gravitons in the geometry. From the quantum hall point of view this is the point of view that the edge states are the best description of the low-energy dynamics.

Outlook

- We have seen that there is a lot of information in $1/2$ BPS states that is interesting.
- It can probably be generalized to $1/4$ BPS and $1/8$ BPS states.
- Is there a matrix model description of these states?
- Can one write the spectrum of fluctuations of the $1/2$ BPS geometries?
- Relations to confinement?
- How do we understand the droplet picture in terms of the 't Hooft expansion?
- Many more questions