Symmetry Breaking for Quantum Shapes

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Overview

- -states in gauge theories and gravity
- Quantum Theory on non-simply connected configuration spaces
- Symmetry breaking
- Molecular and nuclear examples
- Future directions and Field theories

Traditional view of Yang-Mills θ-states

$$\int_{\square\times\mathbb{R}} dV\,\left(\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{\hbar\theta}{16\pi^2}F^{*}_{\mu\nu}F^{\mu\nu}\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$D_\mu M = \partial_\mu M + [A_\mu, M]$$

$$A_\mu \to A_\mu + U^{-1} D_\mu U$$

Traditional view of Yang-Mills θ-states

$F^{*}_{\mu\nu} F^{\mu\nu}$ is a topological term which is related to the winding number of map between $\partial(\square \times \mathbb{R})$ and the gauge group G.

Hamiltonian approach

$$\int_{\Box} d\Box \left(\frac{-\hbar^2}{2} \frac{\Box^2}{\Box A_a^2} + \frac{1}{2} B_a^2 \right) \Box \, (A_a) = E \Box \, (A_a)$$

$$D \cdot \frac{\Box}{\Box A_a} \Box_{phys}(A_a) = 0$$

$$A_\mu \rightarrow A_\mu + U^{-1} D_\mu U$$

there are gauge transformations which are not generated by the constraint, the so called large gauge transformations.

General gauge transformations commute with the hamiltonian so they change the wave function by at most a phase.

Hamiltonian

For a 3-sphere the spatial gauge transformations are maps between the 3-sphere and gauge group G. The large ones form a group which is $\pi_3 G = \mathbb{Z}$.

Given a large gauge transformation g_n the unitary operator $U_{g_n}|phys>= e^{-in\theta}|phys>$

Hamiltonian

Define a new state $\Box$ in the following way

$$\Box = e^{iW(A_a)} \Box_{phys}$$

$$\int_{\Box} d\Box \left(\frac{1}{2} \left(\frac{\hbar}{i} \frac{\Box}{\Box A_a} - \frac{\hbar \Box}{8 \Box^2} B \right)^2 + \frac{1}{2} B_a^2 \right) \Box (A_a) = E \Box (A_a)$$

The action which gives this hamiltonian is

$$\int_{\Box \times \mathbb{R}} dV \left(\frac{1}{2g^2} F_{\mu\Box} F^{\mu\Box} - \frac{\hbar \Box}{16 \Box^2} F^*_{\mu\Box} F^{\mu\Box} \right)$$

θ-states in terms of Hamiltonian

The Hamiltonian is defined on the configuration space $\mathcal{C}$ over the spatial surface □.

$$\mathcal{C} = \mathcal{A}/\mathcal{G}$$

where $\mathcal{A}$ is the vector space of all spatial vector potentials on □ and $\mathcal{G}$ is the group of spatial gauge transformations.

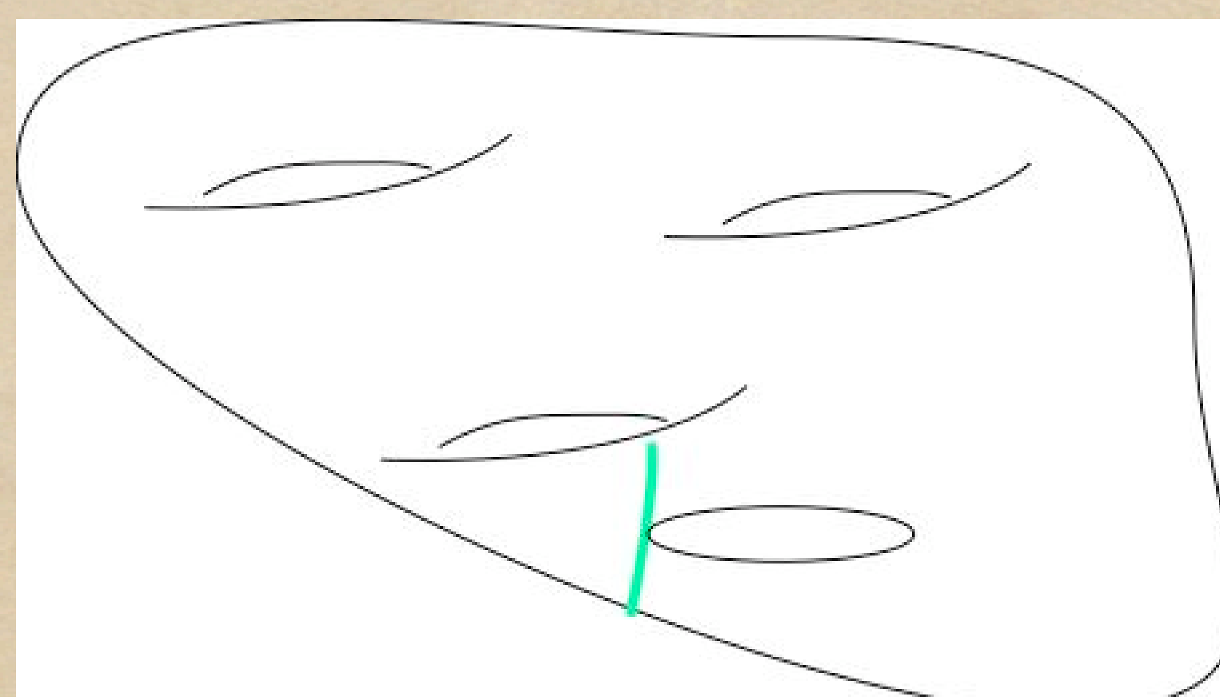
$$\pi_1 \mathcal{C} = \pi_1(\mathcal{A}/\mathcal{G}) = \pi_0 \mathcal{G} = [\square, \mathrm{G}] = \mathbb{Z}$$

assuming □ is 3-sphere. $\pi_1 \mathcal{C}$ is always abelian for gauge theories.

Math

Given any space X with $\pi_1 X$ non-trivial, then there exists a space $\tilde{X}$ such that there is an onto map $p : \tilde{X} \to X$ such that $p(\tilde{x}) = [\tilde{x}]$.

$x_1 \in [\tilde{x}]$ if and only if $\tilde{x} = g x_1$ where $g \in \pi_1 X$.
$\tilde{X}$ is called the universal cover of X.



Examples

1. Let T^n be the n-torus, then the universal cover is $\mathbb{R}^n$

2. For $SO(3)$ the universal cover is $SU(2)$ where $p : SU(2) \to SO(3)$ is given by $p(q)x = qxq^{-1}$ where $x \in \mathbb{R}^3$. Represent, $x = ai + bj + ck$, $q = f1 + mi + nj + sk$, $ij = k$, $i^2 = j^2 = -1$, $q^* = f1 - mi - nj - sk$. $qq^* = 1$ iff $q \in SU(2)$

Quantization on non-simply connected configuration spaces

Quantizing on any non-simply connected space one can use the same techniques as quantizing on a space with a symmetry by working in the covering space.

Recall for any space with symmetry the wave functions are irreducible unitary representations of the symmetry group.

Basic repsentation theory

Let Q be some closed coordinate space on which a group acts, and □:Q C. Assume G is finite. Given □, g□ gives another function on Q. Fix some □ and act on it with all g□ G.

B={$□_1$, $□_2$, ... , $□_f$ } where f ≤ |G|. $\hat{g}□_i = □_k g_{ki} □_k$

Pick $□_i$ ‘s othronormal.

$$g_{ik} = \int □_i^* \hat{g} □_k dV$$

Reps

This set of matrices is called a representation of G. B is a basis and f = dim B is the dimension of the representation. The g_{ik}'s are the unitary reps.

Character of a group element is $\chi(g) = tr(\hat{g})$

Once can show that $\mathcal{H} = \bigoplus \mathcal{H}^{\ell}$ with $\mathcal{H}^{\ell} = \bigoplus \mathcal{H}^{\ell}{}_{\beta}$

ℓ labels irreducible representations on G and β is the multiple occurrences of the representations.

Reps

$$\square = \square_{\square} \square_{l} \square_{l}^{\square}$$

$$\square_{l}^{\square} = \frac{f_{\square}}{|G|} \square_{g} tr(\hat{g}^{\square *}) \hat{g} \square$$

$$\square_{l}^{\square} = \frac{f_{\square}}{|G|} \square_{g} \square^{\square}(\hat{g})^{*} \hat{g} \square$$

Quantum Gravity Example

$$\mathcal{C} = Riem(\Sigma)/Diff(\Sigma)$$

For general 3-manifolds

$$\pi_1 \mathcal{C} = \pi_1 (Riem(\Sigma)/Diff(\Sigma))$$

is non-abelian unlike the gauge theory case.

Additionally, there are quantum states which are ferimonic!

D. Witt J. Math Phys. (1986),
J. Friedman and D. Witt Topology (1986).

Quantum Shapes

Look at a rigid rotator the configuration space is SO(3). $\pi_1 \mathcal{C} = \mathbb{Z}_2$

2. For $SO(3)$ the universal cover is $SU(2)$ where $p : SU(2) \to SO(3)$ is given by $p(q)x = qxq^{-1}$ where $x \in \mathbb{R}^3$. Represent, $x = ai + bj + ck$, $q = f1 + mi + nj + sk$, $ij = k$, $i^2 = j^2 = -1$, $q^* = f1 - mi - nj - sk$. $qq^* = 1$ iff $q \in SU(2)$

If the rotator has a symmetry group then H, then H □ SO(3). Hence, the configuration space for a symmetric rotator is

$$\mathcal{C} = \mathrm{SO(3)/H}.$$

such a rotator is called a shape.

Quntum Shape

$$\pi_1 \mathcal{C} = \pi_1(SO(3)/H) = \pi_1((SU(2)/\mathbb{Z}_2)/(H^*/\mathbb{Z}_2))$$

$$\pi_1 \mathcal{C} = \pi_1(SU(2)/H^*) = H^*$$

Groups

Finite subgroups of $SU(2)$ are the $\mathbb{Z}_m$, D^*_{4m}, T^*, O^*, I^*.

The groups are the cyclic, group the binary dihedral group, binary tetrahedral group, binary octahedral group, and binary icosahedral group.

The order of the groups is m, 4m, 24, 48, and 120.

Quantum Shapes Violating P and T

The mechanism is much the same as the one leading to P and T violation in QCD in the presence of the theta term(for □ □ 0, □): P and T change the UIR of H□ to its complex conjugate.

In QCD, the analogous result is that P and T change the UIR n $e^{in□}$ of Z to its complex conjugate, Z being the fundamental group of the gluon field configuration space.

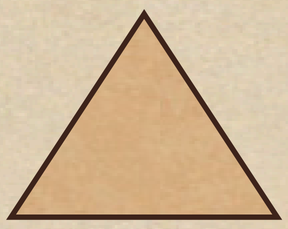
It merits emphasis that P and T violation being discussed here is quantum mechanical. The left-right distinction found here is not the same as the distinction between isomeric nuclei. It cannot be seen by a classical physicist. In a similar way, the QCD □ has no classical consequence and affects only quantum theory.

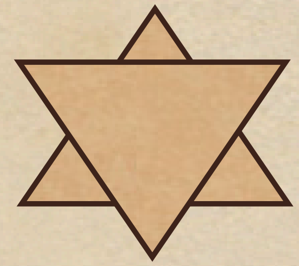
Molecules as quantum shapes

When the Born-Oppenheimer approximation or some version thereof applies molecules are examples of quantum shapes.

Binary dihedral groups give example symmetry violations.

C_2H_6 is an example of such a molecule.





In molecular physics, there is no known microscopic source of P or T violation. For this reason, it was speculated that in a more exact treatment, there must exist mechanisms mixing states mapped to each other by P and T .

A. P. Balachandran, A. Simoni, and D. Witt. Int. J. Mod. Phys. A, 7:2087, 1992.

Nuclear

Skyrmion Spin from B-O approximation

The spectral flow arguments can be used to strongly suggests
that there are quarks moving in the background of the Skyrme field U.
The system thus naturally lends itself to a separation into the “fast” quark degrees of freedom, and the “slow” Skyrme degrees of freedom. It is therefore plausible that the simplest attempt to quantize this system would be via the Born-Oppenheimer approximation or some modified version.

The configuration space is
$\mathcal{C}$ = (SO(3) $\times$ SO(3))/H. This is equivalent
written as a quotient of the covering group

$\mathcal{C}$ = (SU(2)$\times$SU(2))/K,

Conclusion

Look for quantum systems for such effects.

Extend some of the ideas back field theories