Stochastic Conformal Maps:

Fractal Geometry in 2D Critical Phenomena

P. Wiegmann
University of Chicago

Pictures are taken from papers of Werner and Bauer and Bernard.

- **Stochastic Loewner Evolution**

- **2D Critical Phenomena (CFT) =**

Stochastic Growth process

- **O. Schramm, 2000-**
- **G. Lawler, W. Werner, S. Rhode 2001-2002**

- **M. Bauer and D. Bernard, Oct 2002**

- **J. Cardy** 1992
- **B. Duplantier** 1999

- **Iterative conformal maps**
- **M. Hastings and Levitov 1996** - Diffusion Limited aggregations

2D Critical Phenomena:

- Equilibrium Statistical Mechanics
  (Ising, Potts ,... models)
- Geometrical Critical Phenomena
  Percolation, self-avoiding walks, growth ,...

Two sorts of questions

- Critical exponents
- Correlation functions $\sim$

boundary related questions

Three complementary languages:

- Statistical mechanics ($\sim$ combinatorics)
  Example: $\boldsymbol{Q}$-Potts model

$$\boldsymbol{Z} = \sum_{\text{closed loops}} \boldsymbol{Q}^{\text{Length of a loop}}$$

- Percolation: $Q \to 1$

Eucledian QFT (space+time)

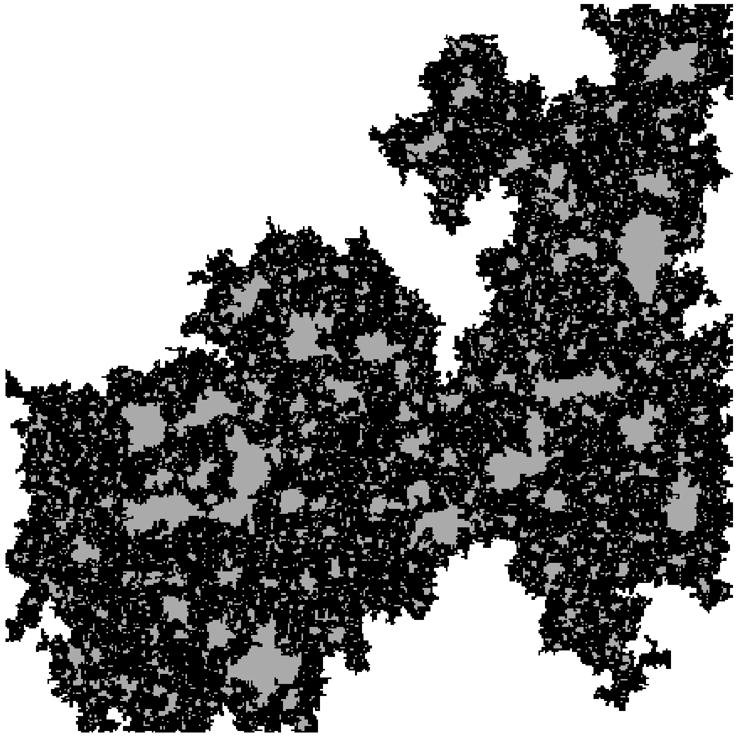
- Correlation functions

$$< \mathbf{0}|\phi(\mathbf{1})...\phi(n)|\mathbf{0} >$$

$|0>$ is the lowest eigenstate of the transfer matrix.

A problem of identitification of operators

Stochastic (Fractal) Geometry of critical clusters



. Part of a (big) critical percolation cluster on the square lattice

Conformal Field Theory

- Scaling in 2D =Local Conformal Invariance=
  trace of stress energy tensor of QFT vanishes locally

$$T_a^a(z) = 0 \quad \text{mod (conformal anomaly)}$$

- **States** $\leftrightarrow$ **operators** transformed according to irreducible reprs of Virasoro algebra.
  Holomorphic component of a stress energy tensor

$$T(z) = \sum_n z^{-n-2} L_n$$

$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m}$$

- **Critical exponents** (conformal dimensions of operators) are obtained from the value of **Central charge**

$$\boxed{-2 < c < 1}$$

  and a weight of representation.

- **Correlation functions** obey **hypergeometric** differential equations (*Ward Identities* of conformal invariance).

Geometrical aspects of Critical Phenomena

Two major developments :

- Crossing Probability
  (Cardy, 1992)

- Conformal measure of critical clusters
  (Duplantier, 1998)

Percolation: Is there left to right crossing?

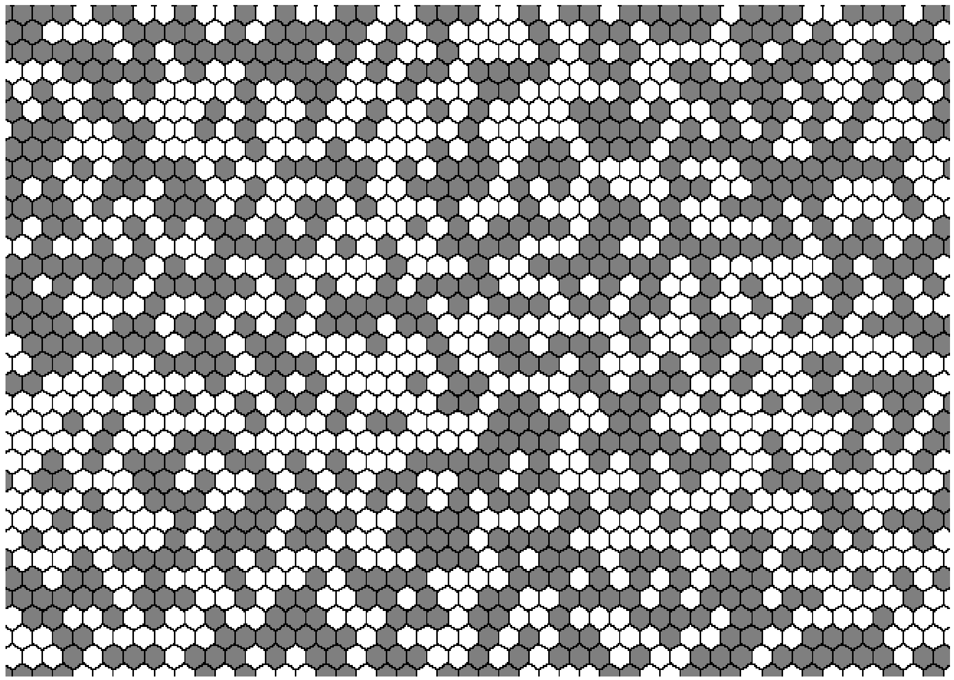


Fig. 10.1. Is there a left to right crossing of white hexagons?

And now?

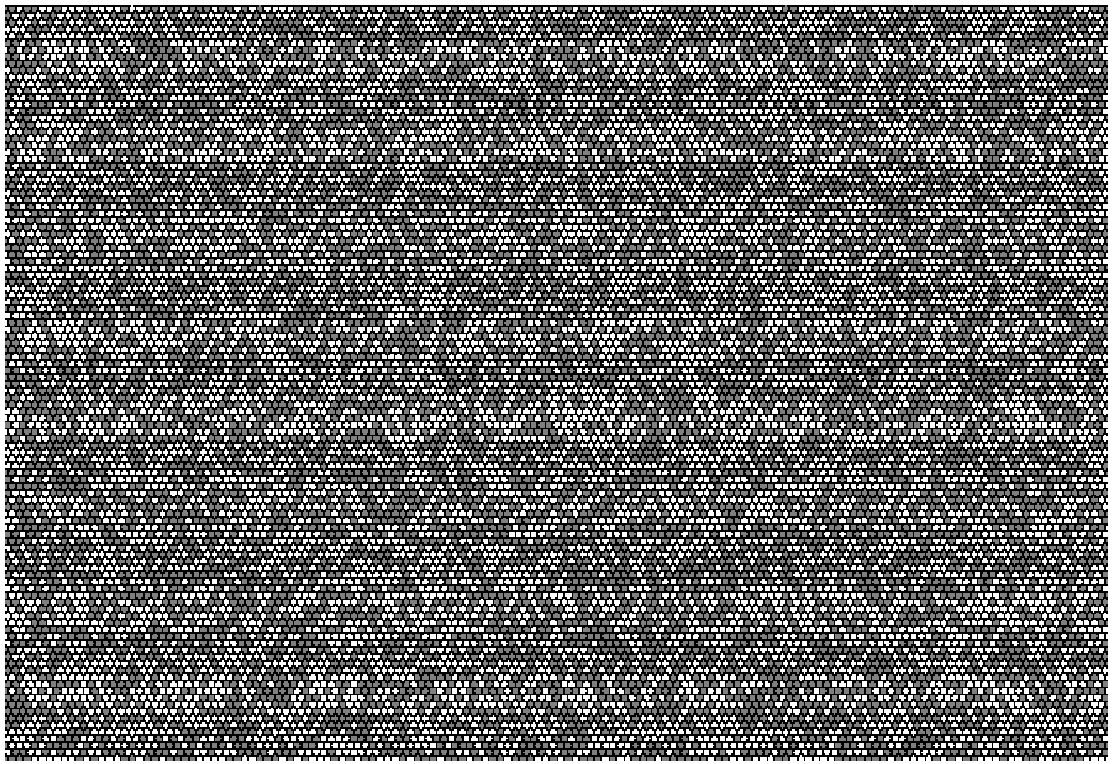


Fig. 10.2. And now?

Crossing Probability

$$\frac{L'}{L} = \frac{K(1-k^2)}{2K(k^2)}$$

$$x = \frac{(1-k)^2}{1+k)^2}$$

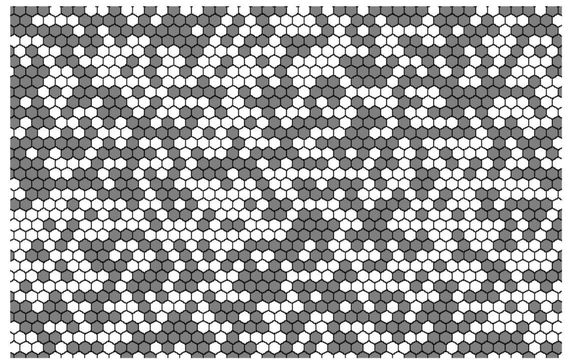
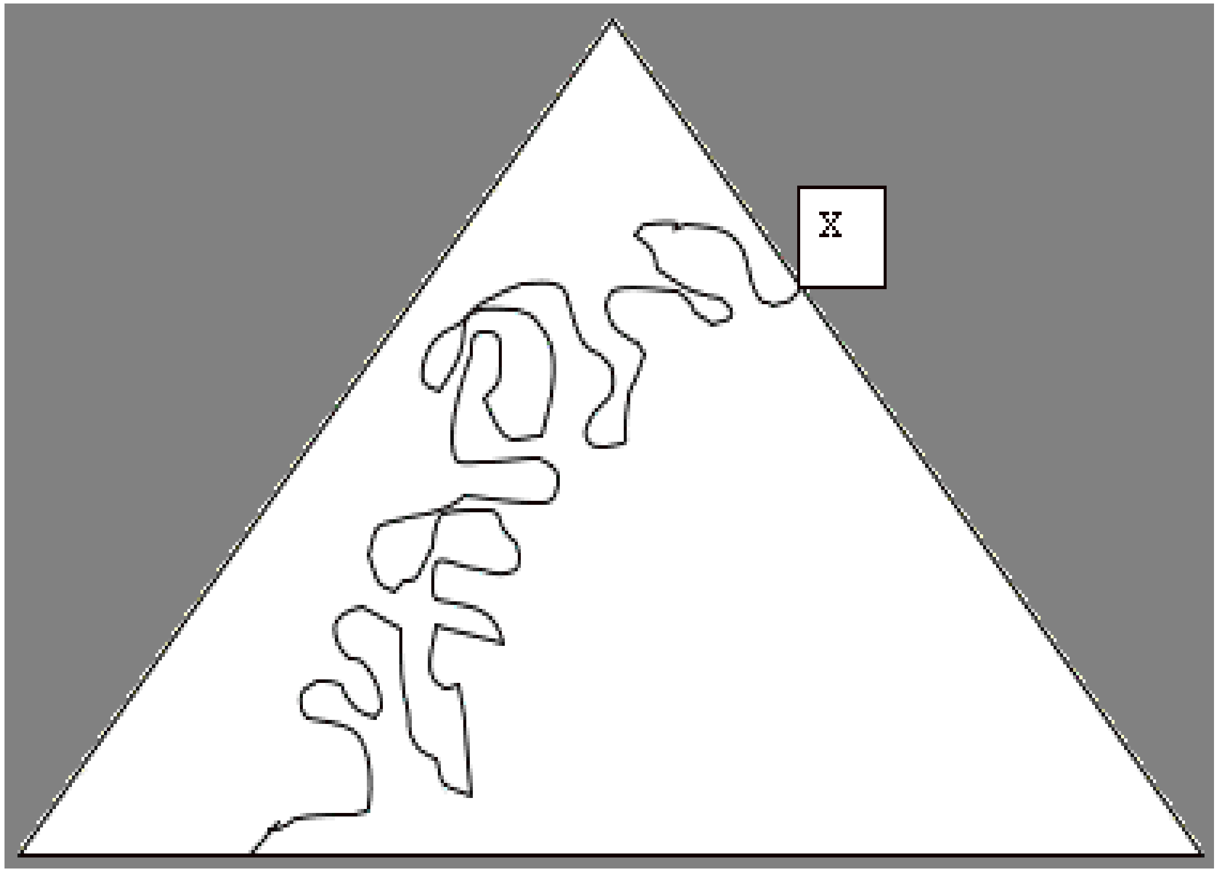


Fig. 10.1. Is there a left to right crossing of white hexagons?

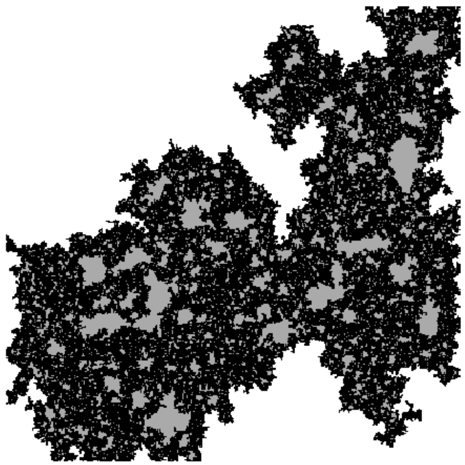
$$\boxed{x(1-x)\frac{d^2P}{dx^2} + \frac{2}{3}(1-2x)\frac{dP}{dx} = 0}$$

$$P(x) = \frac{3\Gamma(2/3)}{\Gamma(1/3)} x^{1/3} F(1/3, 2/3, 4/3; x)$$



Crossing probability is simply $P(x) = 1$

Conformal measure of critical clusters



. Part of a (big) critical percolation cluster on the square lattice

- $w(z)$ - a conformal map of a critical cluster onto a unit disk
- $w'(z)$ - conformal measure or electric field created by a charged cluster

$$< \left(\text{Electric field}(z) \right)^{\delta} > = < |w'(z)|^{\delta} > \sim \left(\frac{z}{R} \right)^{\Delta(\delta, c)}$$

B. Duplantier (1999)

$$< \left(\text{Electric field}(z) \right)^{\delta} >=< |w'(z)|^{\delta} >\sim \left(\frac{z}{R} \right)^{\Delta(\delta,c)}$$

$$\Delta(\Delta - \gamma_{str}) = (1 - \delta\gamma_{str})\delta$$

$$\gamma_{str} = \frac{1}{12}(c - 1 + \sqrt{(1-c)(25-c)})$$

Interface as a stochastic growth process

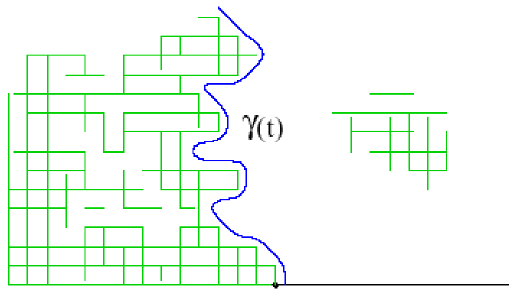


Figure 2: *A FK-cluster configuration in the Potts models. The SLE_κ trace $\gamma(t)$ is the boundary of the FK-cluster connected to the negative real axis.*

An interface as a random self-avoiding walk

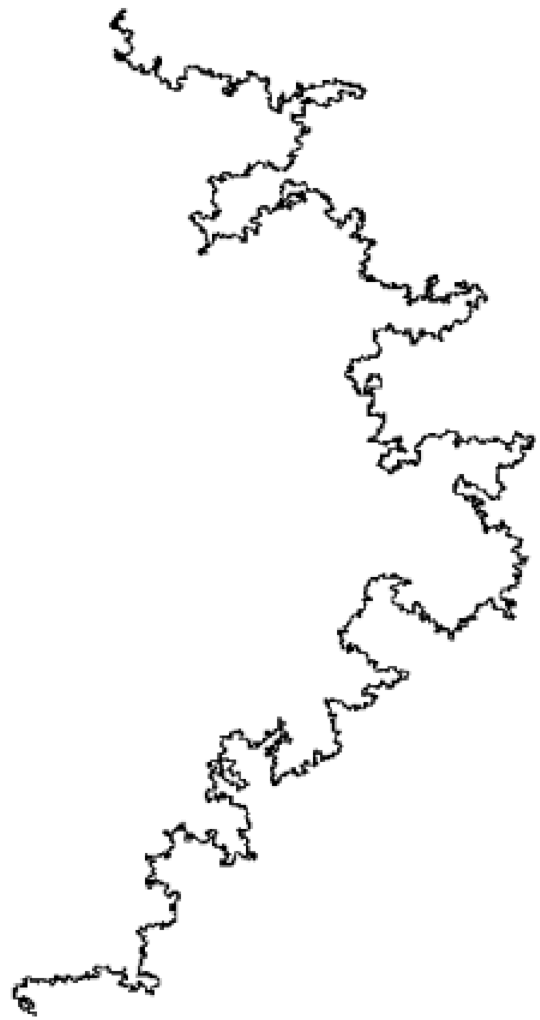


Fig. 4.1. Sample of the beginning of a half-plane walk (conjectured to converge to chordal $SLE_{8/3}$).

Evolution of Conformal Maps:

Hadamard formula

- A map of the cluster onto an exterior of a unit disk

$$w(z) = \frac{z}{r} + \mathcal{O}(\frac{1}{z}), \qquad r - \text{conformal radius}$$

- Hadamard formula

$$\boldsymbol{\frac{\delta w(z)}{w(z)} = \frac{w(z) + e^{i\theta}}{w(z) - e^{i\theta}} \cdot \frac{\delta r}{r}}$$

$$\frac{\delta r}{r} = \delta(\text{Area})|w'(z(e^{i\theta}))|^2.$$

Iterative Conformal maps

$$\log \frac{w_{n+1}(z)}{w_n(z)} = \frac{w_n(z) + e^{i\theta_n}}{w_n(z) - e^{i\theta_n}}\epsilon, \qquad \frac{r_{n+1}}{r_n} = \epsilon$$

A sequence $\underbrace{\theta_1,\ \theta_2,\ \theta_3 \ldots}$ on **mathematical** plane

$$\Downarrow$$

A sequence $\overbrace{D_1,\ D_2,\ D_3 \ldots}$ of domains on **physical** plane

Loewner's equation

- On a plane

$$w_{n+1}(z) - w_n(z) = \epsilon \frac{2}{w_n(z) - \theta_n}$$

- In continuum limit

$$\begin{cases} \frac{dw(z,t)}{dt} = \frac{2}{w(z,t) - \theta(t)}, \\ w(z, t = 0) = z. \end{cases}$$

Stochastic Loewner equation

$$\begin{cases} \frac{dw(z,t)}{dt} & = & \frac{2}{w(z,t)-\theta(t)}, \\ \\ w(z,t=0) & = & z. \end{cases}$$

Wienner process (white noise)

$$\eta = \frac{d\theta}{dt}$$

$$< \eta(t)\eta(0) >= \kappa\delta(t)$$

Stochastic Conformal Maps generate a self-avoiding path statistically equivalent to hulls of clusters of 2D critical phenomena (Schramm, 2000)

SLE

- $0 < \kappa < 4$ - path is simple
- $4 < \kappa < 8$ - path touches itself (but never crosses)
- $\kappa > 8$ - path fills the space

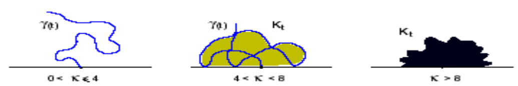


Figure 1: *The different possible configurations of the SLE_κ traces depending on κ.*

Fractal dimension of the path (hull): $\boldsymbol{\Delta = 1 + \frac{\kappa}{8}}$

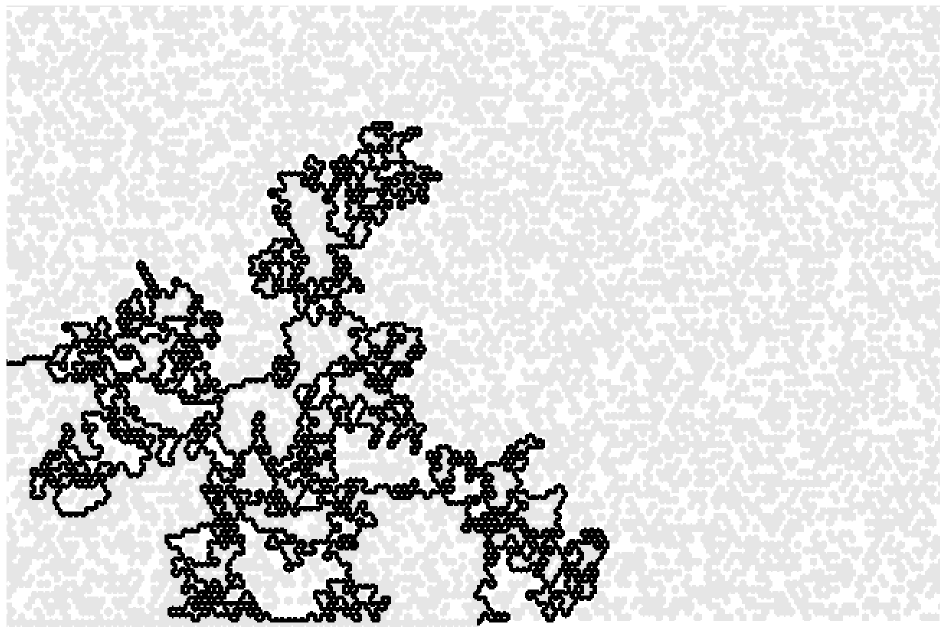


Fig. 1.5. The exploration process, proved to converge to SLE_6 (see Chapter 10)

SLE=CFT

- *Stochastic Loewner evolution describes* **all** *2D critical phenomena at* $-2 < c < 1$

- Relation between κ- strength of noise and central charge

$$c = 1 - 3\frac{(4-\kappa)^2}{2\kappa}$$

- Duality: hull - trace

$$\kappa \longrightarrow 16/\kappa$$

- Fractal dimension

$$\Delta = 1 + \frac{\kappa}{8}$$

Examples

- percolation: $\kappa = 6, \quad c = 0, \quad \Delta = 1 + 3/4$
- Self-avoiding walks: $\kappa = 16/6, \quad c = 0, \quad \Delta = 1 + 1/3$
- Free fermions: $\kappa = 4, \quad c = 1, \quad \Delta = 3/2$
- Ising model $\kappa = 3, \quad c = 1/2, \quad \Delta = 1 + 2/3$
- Q-Potts model (Ising model $Q = 2$)

$$\boxed{Q = 4\cos^2(4\pi/\kappa)}$$

SLE is Langevin dynamics

$$\begin{cases} \dfrac{dw(z,t)}{dt} & = & \dfrac{2}{w(z,t)-\theta(t)}, \\ w(z,t) & \rightarrow & w(z,t)+\theta(t) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{dw(z,t)}{dt} & = & \dfrac{2}{w(z,t)} + \dot{\theta}(t), \\ \langle \dot{\theta}(t)\dot{\theta}(0) \rangle & = & \kappa\delta(t) \end{cases}$$

Langevin $\rightarrow$ Fokker-Plank Equation

- Langevin Equation

$$\dot{\phi} = -\frac{\delta S}{\delta \phi} + \eta$$

- Fokker-Plank equation/ Feynman-Kac formula

$$\dot{\mathcal{P}} = H_{FP} \cdot \mathcal{P}$$

$$H_{FP} = -\left(\frac{\kappa}{2}\frac{\delta}{\delta \phi} - \frac{\delta S}{\delta \phi}\right)\frac{\delta}{\delta \phi}$$

Fokker-Plank equation for SLE

$$\mathcal{P}(t, w(z)) = \langle \delta\Big(w(z,t) - w(z)\Big) \rangle$$

$$\frac{d}{dt}\mathcal{P}(t, w(z)) = H_{FP}\mathcal{P}(t, w(z))$$

$$H_{FP} = -\left(\frac{\kappa}{2}\frac{\partial}{\partial w} - \frac{2}{w}\right)\frac{\partial}{\partial w}$$

Fokker-Plank equation for SLE = Conformal Ward Identities for correlation functions of CFT

Conformal measure - $h(t,z) = < w'(z)^\delta >$

- The Feynman-Kac formula

$$\partial_t h = \frac{\kappa}{2} h'' + \frac{2}{z} h' - \frac{2\delta}{z^2} h$$

- Conformal invariance:

$$h(t,z) = h(\frac{z}{\sqrt{t}})$$

- hypergeometric equation

$$h'' + (\frac{4}{\kappa z} + \frac{4z}{\kappa}) h' - \frac{4\delta}{\kappa z^2} h = 0$$

- exponent: $h \sim z^\Delta$

$$\Delta(\delta,\kappa) = \frac{1}{2\kappa}(\kappa - 4 + \sqrt{(\kappa-4)^2 + 16\delta\kappa})$$

Virasoro algebra

- Diffeomorphisms

$$w \longrightarrow w + \epsilon w^n, \quad z(w) \longrightarrow z(w) + l_n z(w)$$

- Classical Virasoro algebra $\quad l_n = w^{n+1}\frac{\partial}{\partial w}$

$$[l_n,\, l_m] = (n-m)l_{n+m}$$

- Probability distribution

$$\mathcal{P} \longrightarrow \mathcal{P} + L_n\mathcal{P}$$

- Extended Virasoro algebra

$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}$$

What is c ?

Fokker-Plank equation and Virasoro algebra

$$\begin{cases} H_{FP} = \frac{\kappa}{2} L_{-1}^2 - 2L_{-2} \\ \frac{d}{dt}\mathcal{P} = H_{FP}\mathcal{P} \end{cases}$$

- Equilibrium state - $H_{FP}|0>=0$
- Conformal invariance - $L_0|0>=h|0>$
- Normalization - $L_n|0>=0, \quad n>0.$

$$\Downarrow$$

$$\boxed{c = 1 - 3\frac{(4-\kappa)^2}{2\kappa}}$$

Dimensions

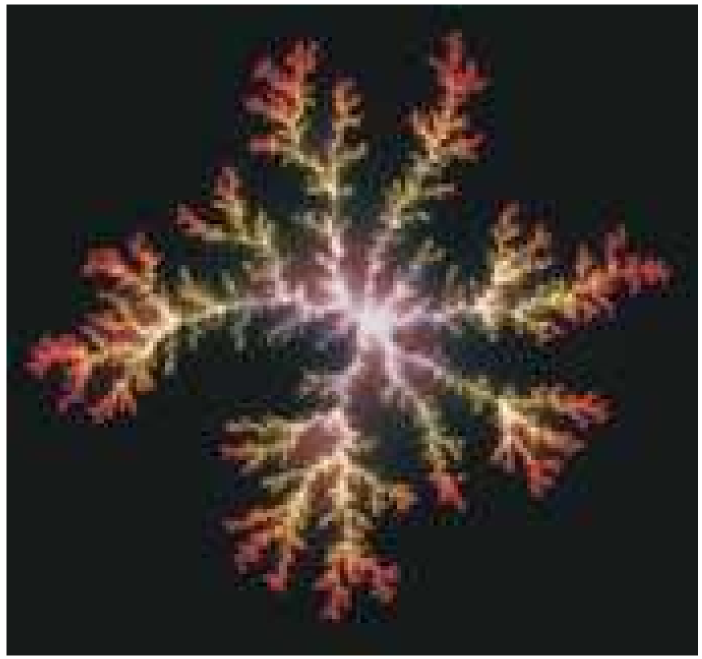
- Dimensions (from Kac table):

$$h_{r,s}(c(\kappa)) = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}$$

- Identification of CFT Operators with geometrical objects:
  SLE-trace/hull $\Rightarrow$ Boundary Operator $(r, s) = (2, 3)$

Stochastic growth:

Diffusion Limit of aggregation (DLA)



An aggregate grows by particles diffusing and sticking to the aggregate.

Stochastic Hadamard formula

- $\boldsymbol{\theta}_n$ are random and uncorrelated (Poisson distribution)

$$\log \frac{w_{n+1}(z)}{w_n(z)} = \epsilon_n |w'(e^{i\theta_n})\frac{w_n(z) + e^{i\theta_n}}{w_n(z) - e^{i\theta_n}}$$

$$P[\theta_n] = \frac{1}{2\pi}$$

- This process generates a branching tree (Hastings and Levitov)

- Fractal geometry $\Longrightarrow$ Analytical aspects of conformal map;
- CFT Ward Identity is SLE-Fokker-Plank equation in the equilibrium regime;
- Identification of CFT Operators and SLE processes;
- CFT as stochastic deformation of Riemann surfaces in moduli space;
- Other fractals, like DLA?
- Relation between stochastic conformal maps and Random Matrix theory