An extended Hubbard model with ring exchange: a possible route to a non-Abelian topological phase

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• Motivation: Quantum Computation

• A system with non-Abelian topological order can be used to construct a universal quantum computer with built-in fault tolerance – more in Mike Freedman’s talk.
**$\mathbb{Z}_2$ Topological Order**

A low energy Hilbert space can be represented as a collection of loops. The ground state is a superposition of all loop configurations with the additional rules:

- The ground state wavefunction takes the same value on configurations which can be connected by the operations above. On the torus: 4-fold degenerate ground state – winding numbers modulo 2 as a result of the third relation.
Beyond $\mathbb{Z}_2$: a Need for Richer Models

A drawback:

Unfortunately, these $\mathbb{Z}_2$ models only support Abelian statistics of excitations.

In order to construct a universal quantum computer, we need a system which supports quasiparticles with non-Abelian statistics.

There is a family of models (and concomitant effective field theories) which generalise $\mathbb{Z}_2$ models (M. Freedman ’01).

They generically support *non-Abelian statistics*. 
A Class of Models

Hilbert space: wavefunctions on spaces of curves representing spins, domain walls, dimers, etc..

These curves satisfy relations which are more complicated than the basic relation of a $\mathbb{Z}_2$ model. There is a discrete family of possible consistent relations. They are the ideals of the Temperley-Lieb algebra.

\[
\begin{align*}
\cdots & \cong \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve1.png}
\end{array} \\
\sqrt{2} \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve2.png}
\end{array} \right) & \cong \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve3.png}
\end{array} \\
\cdots & \cong \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve4.png}
\end{array} \\
\sqrt{2} \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve5.png}
\end{array} \right) & \cong \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve6.png}
\end{array} \\
+ \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve7.png}
\end{array} & + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{curve8.png}
\end{array} \cong 0
\end{align*}
\]
• These models are parametrised by

\[ d = \pm 2 \cos \frac{\pi}{k+2} \] – a loop amplitude.

<table>
<thead>
<tr>
<th>k</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>\sqrt{2}</td>
</tr>
<tr>
<td>3</td>
<td>(1 + \sqrt{5}) / 2</td>
</tr>
<tr>
<td>4</td>
<td>\sqrt{3}</td>
</tr>
</tbody>
</table>

• Skein relation on \( k + 1 \) strands is required for finite dimensionality of Hilbert space.

• Related to doubled \( SU(2)_k \) Chern-Simons Theory.
Abelian vs. non-Abelian Statistics

$k = 2$:

\[
\begin{array}{c}
1 & 2 \\
3 & 4
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
1 & 4 \\
3 & 2
\end{array}
\quad \Rightarrow 
\begin{array}{c}
1 & 2 \\
3 & 4
\end{array}
\]

– Abelian

$k = 3$:

\[
\begin{array}{c}
1 & 2 \\
3 & 4
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
1 & 4 \\
3 & 2
\end{array}
\quad \infty 
\begin{array}{c}
1 & 4 \\
1 & 2
\end{array}
\]

– non-Abelian
Quantum Dimers, RVBs
(Rokhsar and Kivelson; Chakraborty, Read and Sachdev; Moessner and Sondhi)

\[ H = \sum_P \left\{ -t \left( |\uparrow\uparrow\rangle \langle \downarrow\downarrow| + |\downarrow\downarrow\rangle \langle \uparrow\uparrow| \right) + v \left( |\uparrow\downarrow\rangle \langle \downarrow\uparrow| + |\downarrow\uparrow\rangle \langle \uparrow\downarrow| \right) \right\} \]

The ground state is a superposition of all possible dimer coverings:
Quantum Dimers, RVBs
(Rokhsar and Kivelson; Chakraborty, Read and Sachdev; Moessner and Sondhi)

\[ H = \sum_{P} \left\{ -t \left( | \begin{array}{c} 1 \end{array} \rangle \langle \begin{array}{c} 1 \end{array} | + | \begin{array}{c} 2 \end{array} \rangle \langle \begin{array}{c} 2 \end{array} | \right) + v \left( | \begin{array}{c} 1 \end{array} \rangle \langle \begin{array}{c} 1 \end{array} | + | \begin{array}{c} 2 \end{array} \rangle \langle \begin{array}{c} 2 \end{array} | \right) \right\} \]

The ground state is a superposition of all possible dimer coverings:
Elementary moves in a Quantum Dimer model.

Isotopy:
Elementary moves in a Quantum Dimer model.

Isotopy:
Elementary moves in a Quantum Dimer model.

$\sim$-Isotopy:
Elementary moves in a Quantum Dimer model.

$d-$Isotopy:
Elementary moves in a Quantum Dimer model.

“Surgery”:  

[Diagram of a triangular lattice with highlighted regions indicating the 'Surgery' process.]
Elementary moves in a Quantum Dimer model.

“Surgery”:
A microscopic Hamiltonian:
Extended Hubbard model on a Kagomé lattice

\[ H = - \sum_{\langle i,j \rangle} t_{ij} (c_i^+ c_j + c_j^+ c_i) + \sum_i \mu_i n_i \]
\[ + U_0 \sum_i n_i^2 + U \sum_{(i,j) \in \hexagon} n_i n_j \]
\[ + \sum_{(i,j) \in \bigtriangleup, \notin \hexagon} V_{ij} n_i n_j + \text{Ring} \]
A microscopic Hamiltonian:
Extended Hubbard model on a Kagomé lattice

\[ H = - \sum_{\langle i,j \rangle} t_{ij} (c_{i}^\dagger c_{j} + c_{j}^\dagger c_{i}) + \sum_{i} \mu_{i} n_{i} \]
\[ + U_{0} \sum_{i} n_{i}^{2} + U \sum_{(i,j) \in \bigtriangleup} n_{i} n_{j} \]
\[ + \sum_{(i,j) \in \bigtriangleup, \notin \bigtriangleup} V_{ij} n_{i} n_{j} + \text{Ring} \]
A microscopic Hamiltonian: Extended Hubbard model on a Kagomé lattice

\[
H = - \sum_{\langle i,j \rangle} t_{ij} (c_i^\dagger c_j + c_j^\dagger c_i) + \sum_i \mu_i n_i \\
+ U_0 \sum_i n_i^2 + U \sum_{(i,j) \in \bigtriangleup} n_i n_j \\
+ \sum_{(i,j) \in \bigtriangleup, \not\in \bigtriangleup} V_{ij} n_i n_j + \text{Ring}
\]
Isotopy: $\Psi(X) - \Psi(X') = 0$ if $X \sim X'$

d-isotopy: $d \Psi(X) - \Psi(X \cup \bigcirc) = 0$

Lattice version:

$$d \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) - \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = 0,$$

$$d^3 \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) - \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = 0$$

or, if we choose $a^4/b = d$,

$$a \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) - \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = 0,$$

$$b \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) - \Psi \left( \begin{array}{c} 
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = 0$$
Phase diagram for the $O(n)$ model (Blöte, Nienhuis):

$$x^{-1} = \frac{n}{\lambda}$$
Conclusion

• We have constructed a microscopic model whose ground state manifold is the $d$–isotopy space, a necessary precondition for $\text{SU}(2)^k \times \text{SU}(2)^k$ non-abelian topological order.

• Near the “special” values, $d = 2 \cos \pi/(k + 2)$, this space is expected to collapse to a stable topological phase with anyonic excitations closely related to $\text{SU}(2)$ Chern-Simons theory at level $k$.

• Can we find or construct such materials?
Frustrated magnets? Nanostructures? Ultra-cold atoms in optical lattices? Josephson junction arrays?
From theory to reality. **a**, A planar representation of a Kagomé lattice, and **b**, a space-filling representation of the lattice synthesized by Zaworotko and colleagues by self-assembling copper(II) cations with 1,3-benzenedicaboxylate anions. Copper atoms are shown in orange, carboxylate groups in red and benzene rings in grey.