An extended Hubbard model with ring exchange: a possible route to a non-Abelian topological phase

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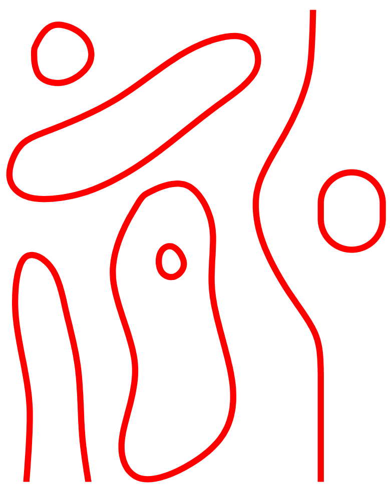
Microsoft Research

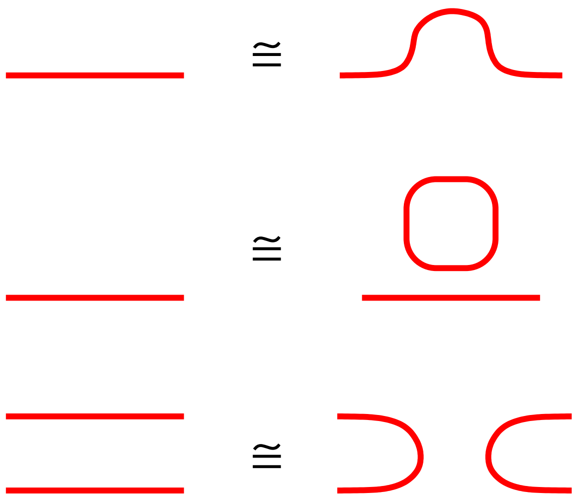
- Motivation: Quantum Computation

- A system with non-Abelian topological order can be used to construct a universal quantum computer with built-in fault tolerance – more in Mike Freedman's talk.

$\mathbf{Z}_2$ Topological Order

A low energy Hilbert space can be represented as a collection of loops. The ground state is a superposition of all loop configurations with the additional rules:





• The ground state wavefunction takes the same value on configurations which can be connected by the operations above. On the torus: 4-fold degenerate ground state – winding numbers modulo 2 as a result of the third relation.

Beyond Z_2: a Need for Richer Models

A drawback:

Unfortunately, these Z_2 models only support Abelian statistics of excitations.

In order to construct a universal quantum computer, we need a system which supports quasiparticles with non-Abelian statistics.

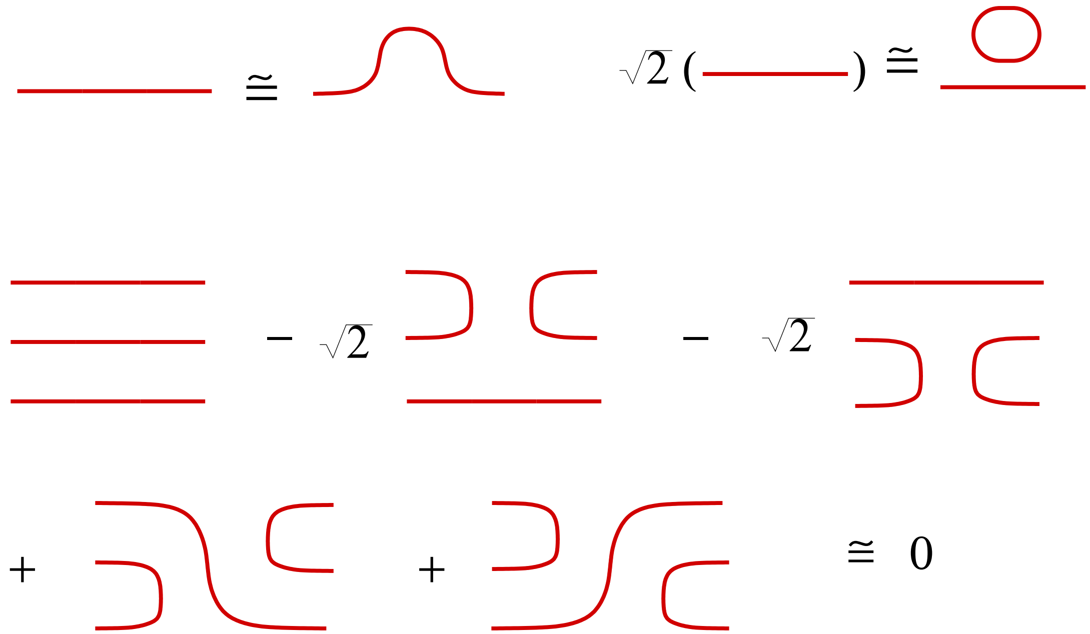
There is a family of models (and concomitant effective field theories) which generalise Z_2 models (M. Freedman '01).

They generically support *non-Abelian statistics*.

A Class of Models

Hilbert space: wavefunctions on spaces of curves representing spins, domain walls, dimers, etc..

These curves satisfy relations which are more complicated than the basic relation of a Z_2 model. There is a discrete family of possible consistent relations. They are the ideals of the Temperley-Lieb algebra.



• These models are parametrised by

$$d = \pm 2\, cos \frac{\pi}{k+2}$$ – a loop amplitude.

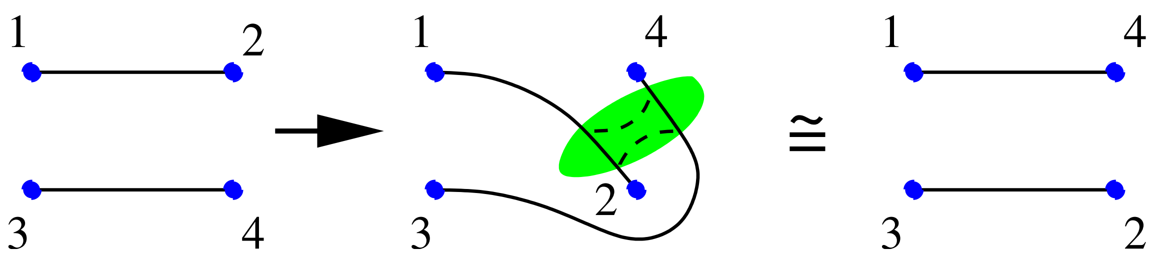
k	**d**
1	1
2	$\sqrt{2}$
3	$\left(1+\sqrt{5}\right)/2$
4	$\sqrt{3}$

• Skein relation on $k+1$ strands is required for finite dimensionality of Hilbert space.

• Related to doubled $SU(2)_k$ Chern-Simons Theory.

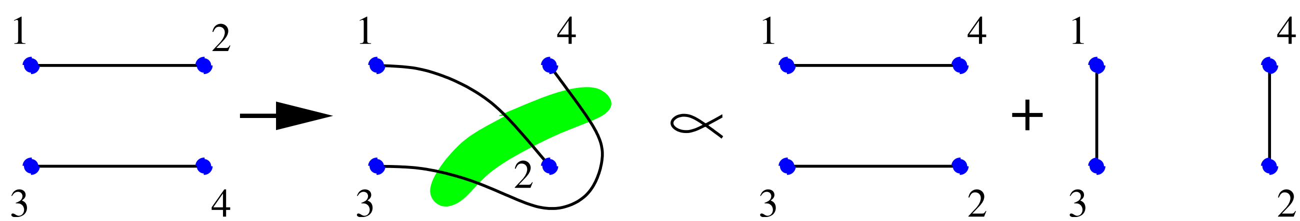
Abelian vs. non-Abelian Statistics

$k = 2$:



– Abelian

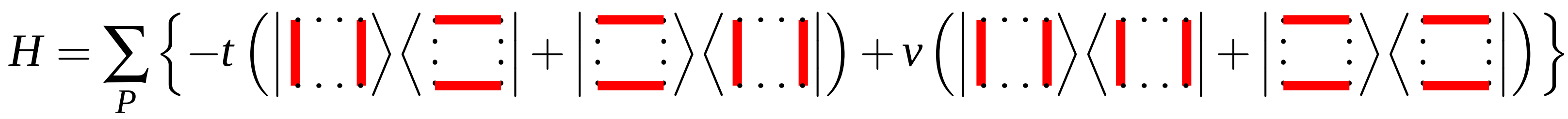
$k = 3$:



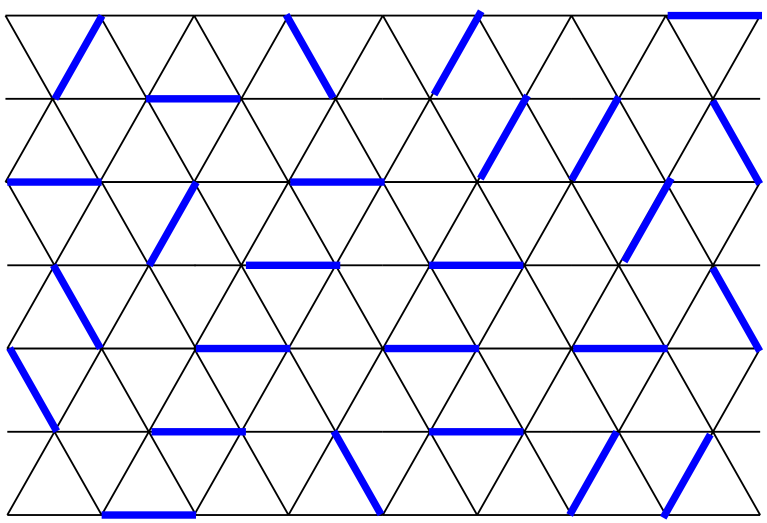
– non-Abelian

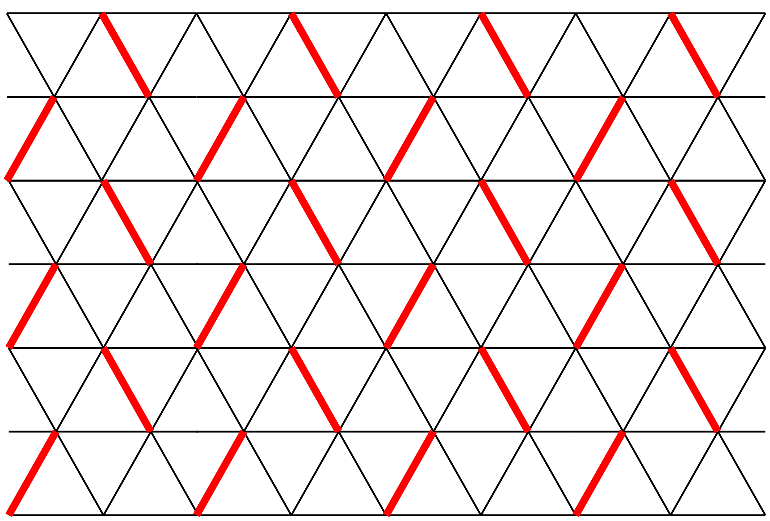
Quantum Dimers, RVBs

(Rokhsar and Kivelson; Chakraborty, Read and Sachdev; Moessner and Sondhi)



The ground state is a superposition of all possible dimer coverings:





Quantum Dimers, RVBs

(Rokhsar and Kivelson; Chakraborty, Read and Sachdev; Moessner and Sondhi)

$$H = \sum_P \left\{ -t \left(\left| \, \| \, \right\rangle \left\langle \, = \, \right| + \left| \, = \, \right\rangle \left\langle \, \| \, \right| \right) + v \left(\left| \, \| \, \right\rangle \left\langle \, \| \, \right| + \left| \, = \, \right\rangle \left\langle \, = \, \right| \right) \right\}$$

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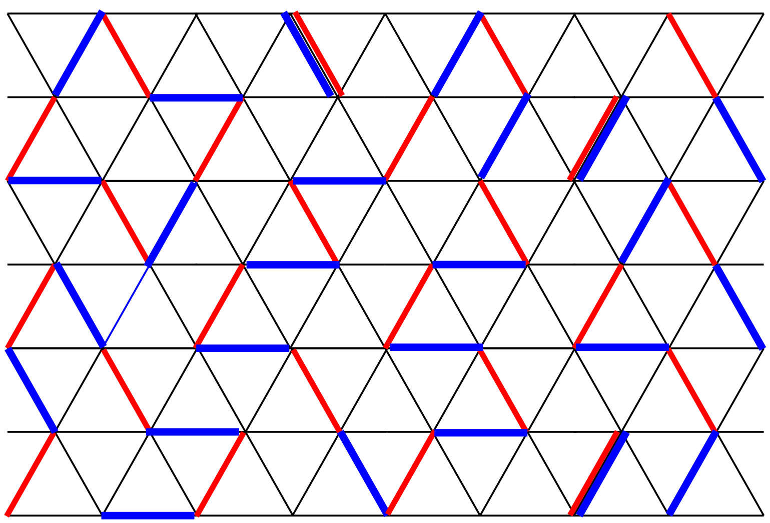
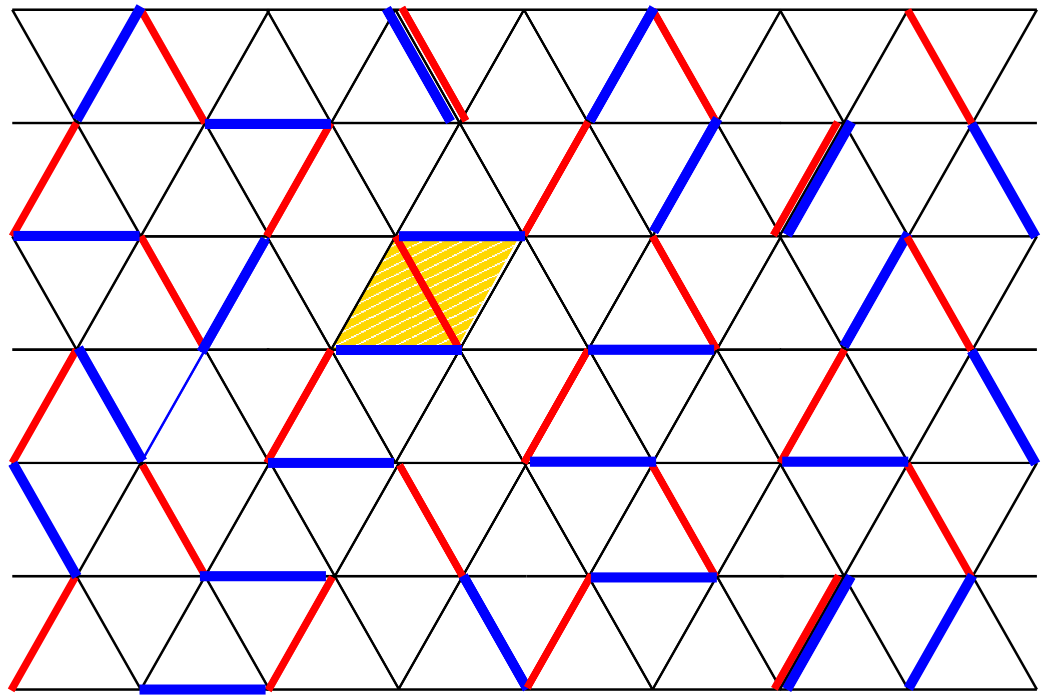


Elementary moves in a Quantum Dimer model.

Isotopy:



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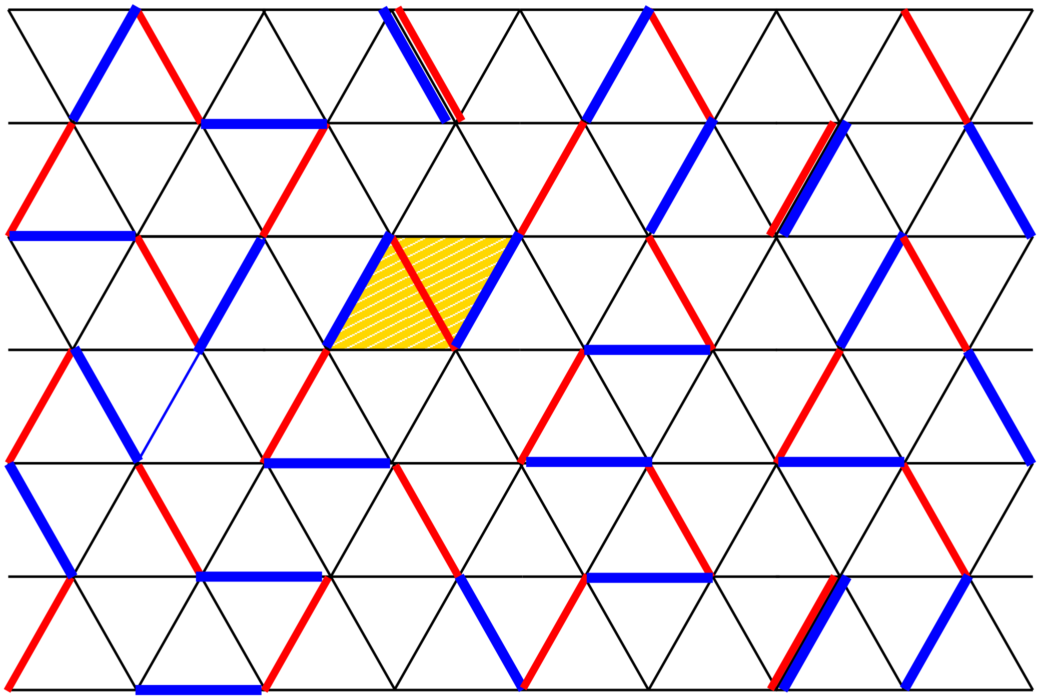

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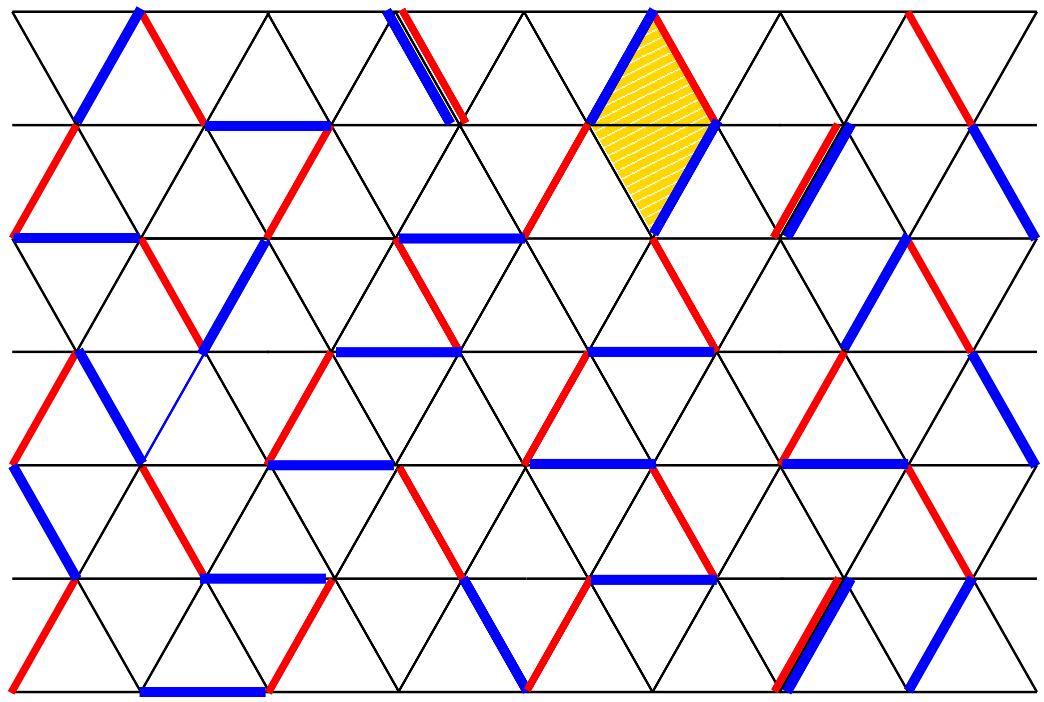
$d-$Isotopy:



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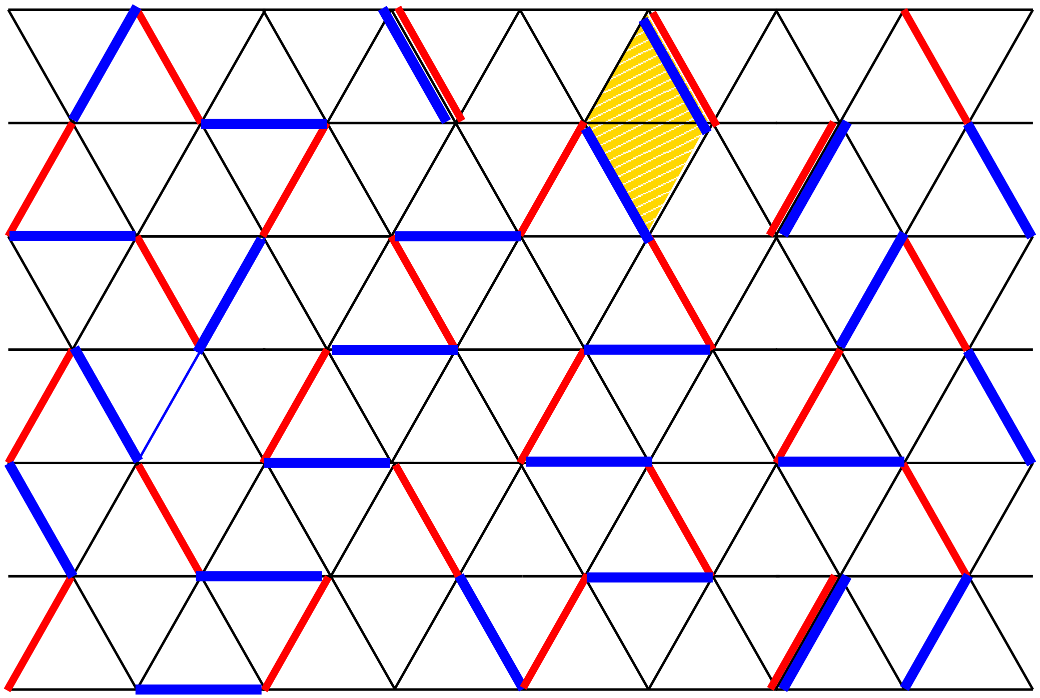
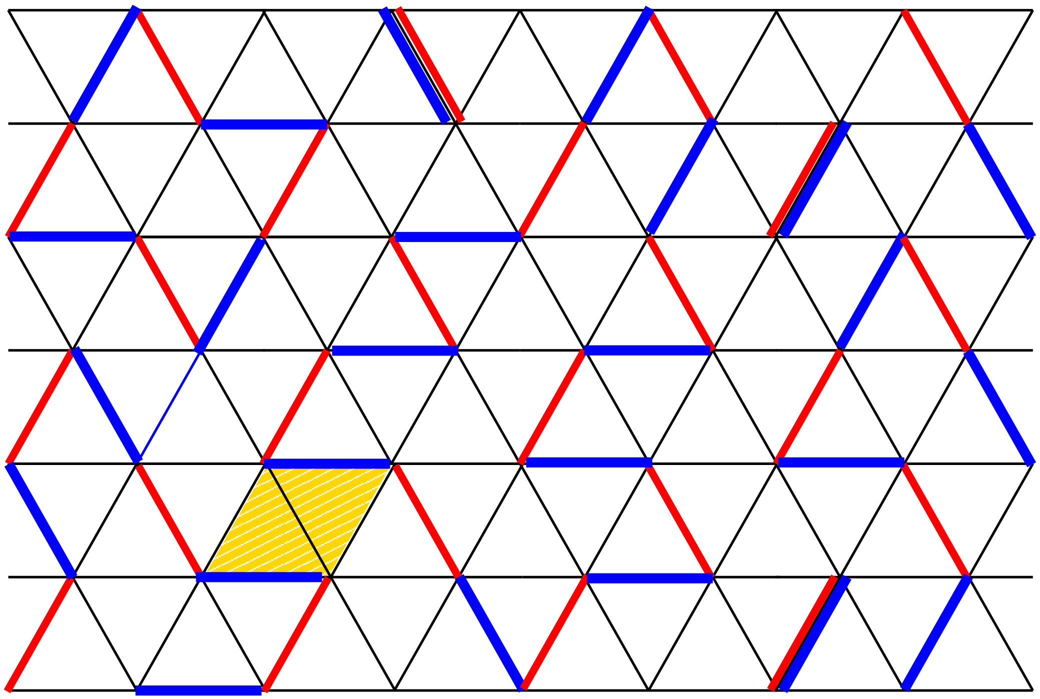


Elementary moves in a Quantum Dimer model.

“Surgery”:



Elementary moves in a Quantum Dimer model.

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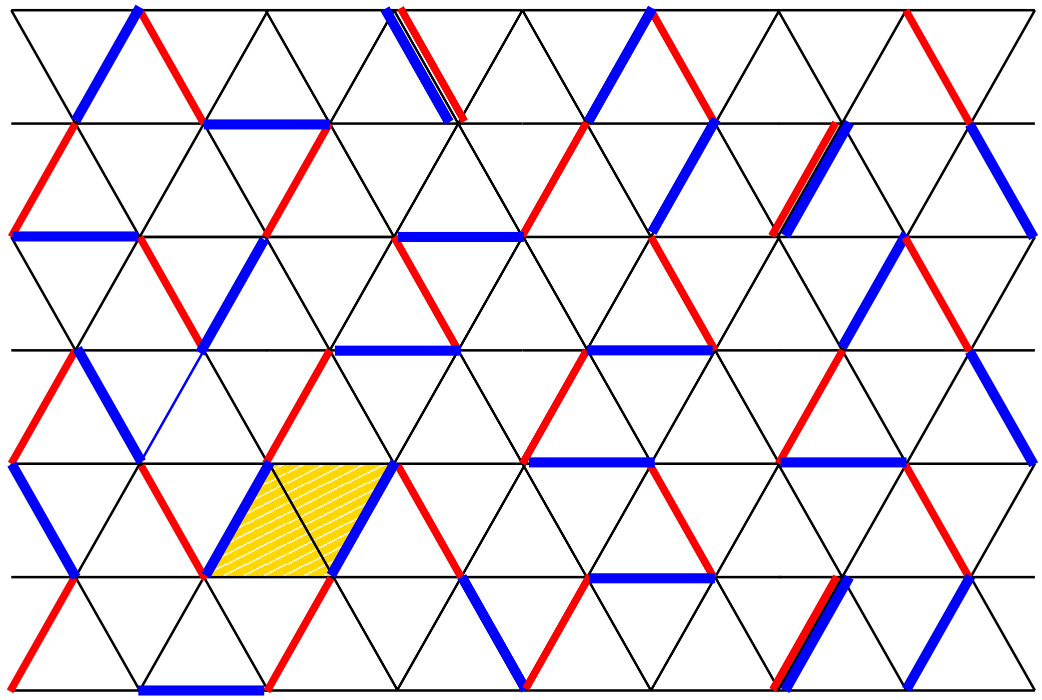
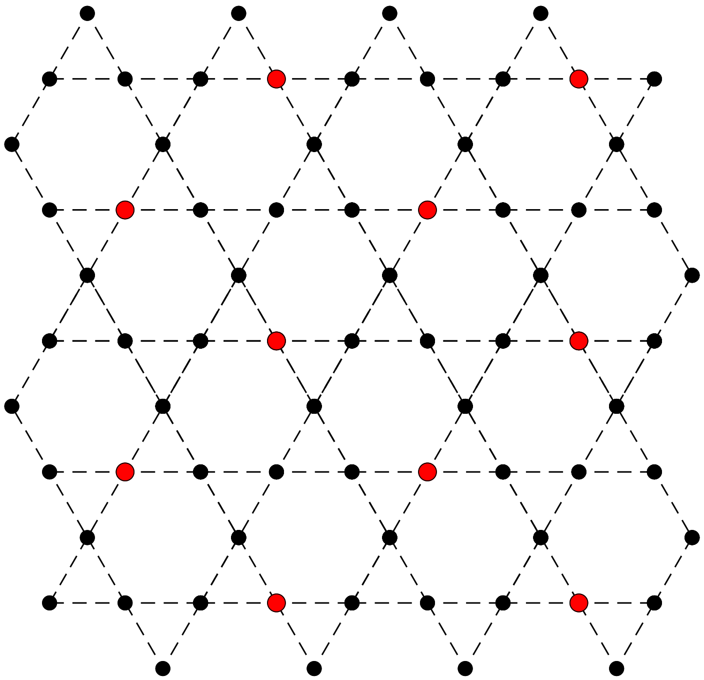
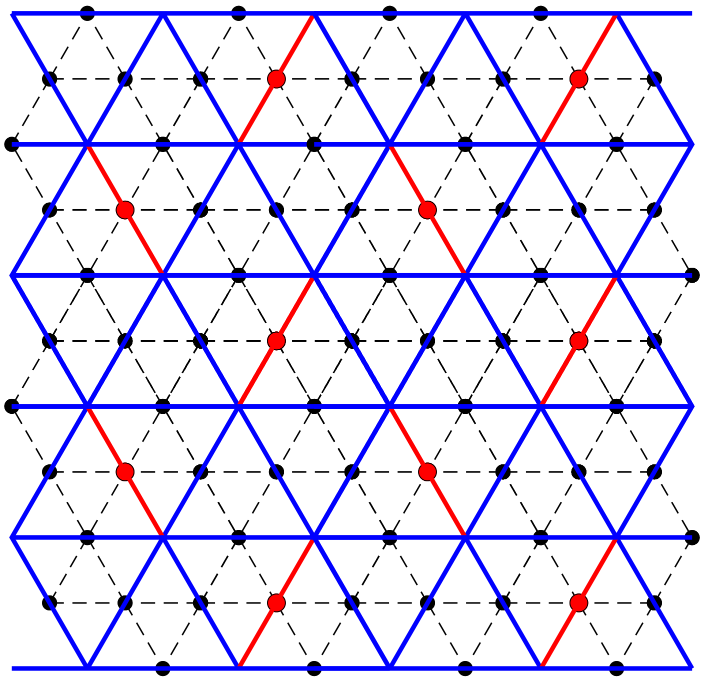


A microscopic Hamiltonian:
Extended Hubbard model on a Kagomé lattice



$$
\begin{aligned}
H &= -\sum_{\langle i,j\rangle} t_{ij}(c_i^\dagger c_j + c_j^\dagger c_i) + \sum_i \mu_i n_i \\
&+ U_0 \sum_i n_i^2 + U \sum_{(i,j)\in\hexagon} n_i n_j \\
&+ \sum_{(i,j)\in\bowtie,\notin\hexagon} V_{ij} n_i n_j + \text{Ring}
\end{aligned}
$$

A microscopic Hamiltonian:
Extended Hubbard model on a Kagomé lattice



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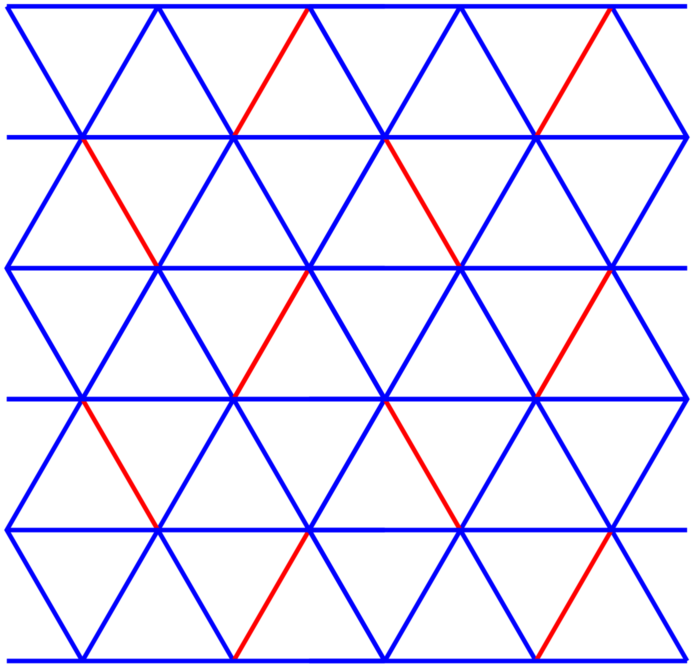
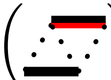


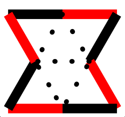
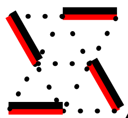
$$\begin{aligned} H &= -\sum_{\langle i,j \rangle} t_{ij} (c_i^\dagger c_j + c_j^\dagger c_i) + \sum_i \mu_i n_i \\ &+ U_0 \sum_i n_i^2 + U \sum_{(i,j) \in \hexagon} n_i n_j \\ &+ \sum_{(i,j) \in \bowtie, \notin \hexagon} V_{ij} n_i n_j + \text{Ring} \end{aligned}$$

Isotopy: $\Psi(X) - \Psi(X') = 0$ if $X \sim X'$

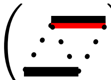
d-isotopy: $d\ \Psi(X) - \Psi(X \cup \bigcirc) = 0$

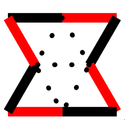
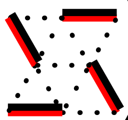
Lattice version:

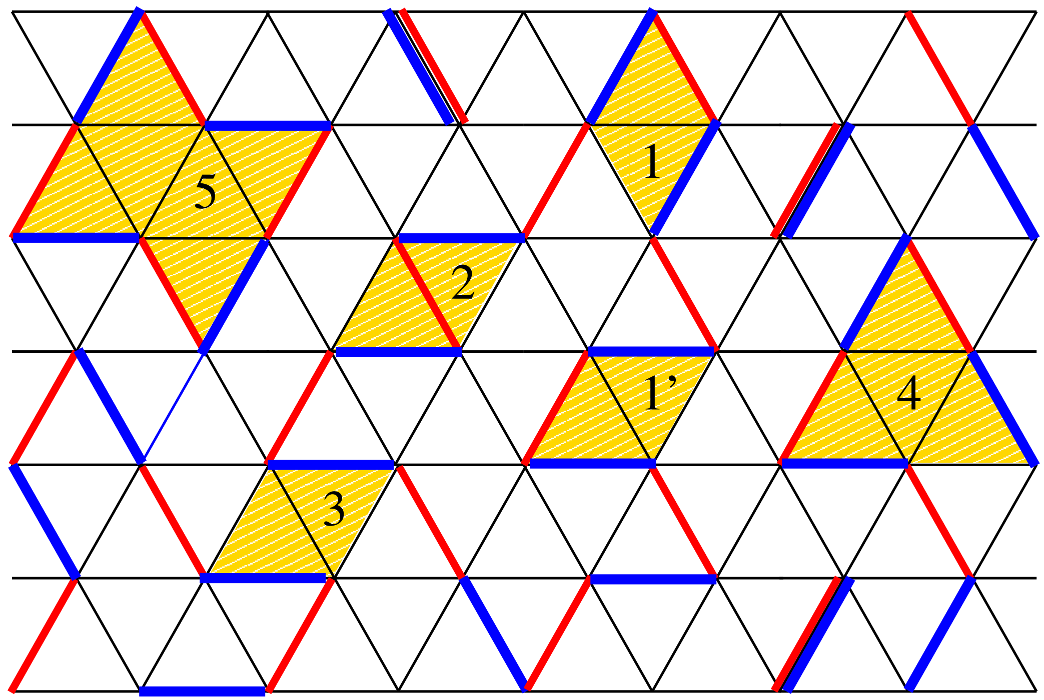
$d\,\Psi\Big($  $\Big) - \Psi\Big($  $\Big) = 0,$

$d^3\,\Psi\Big($  $\Big) - \Psi\Big($  $\Big) = 0$

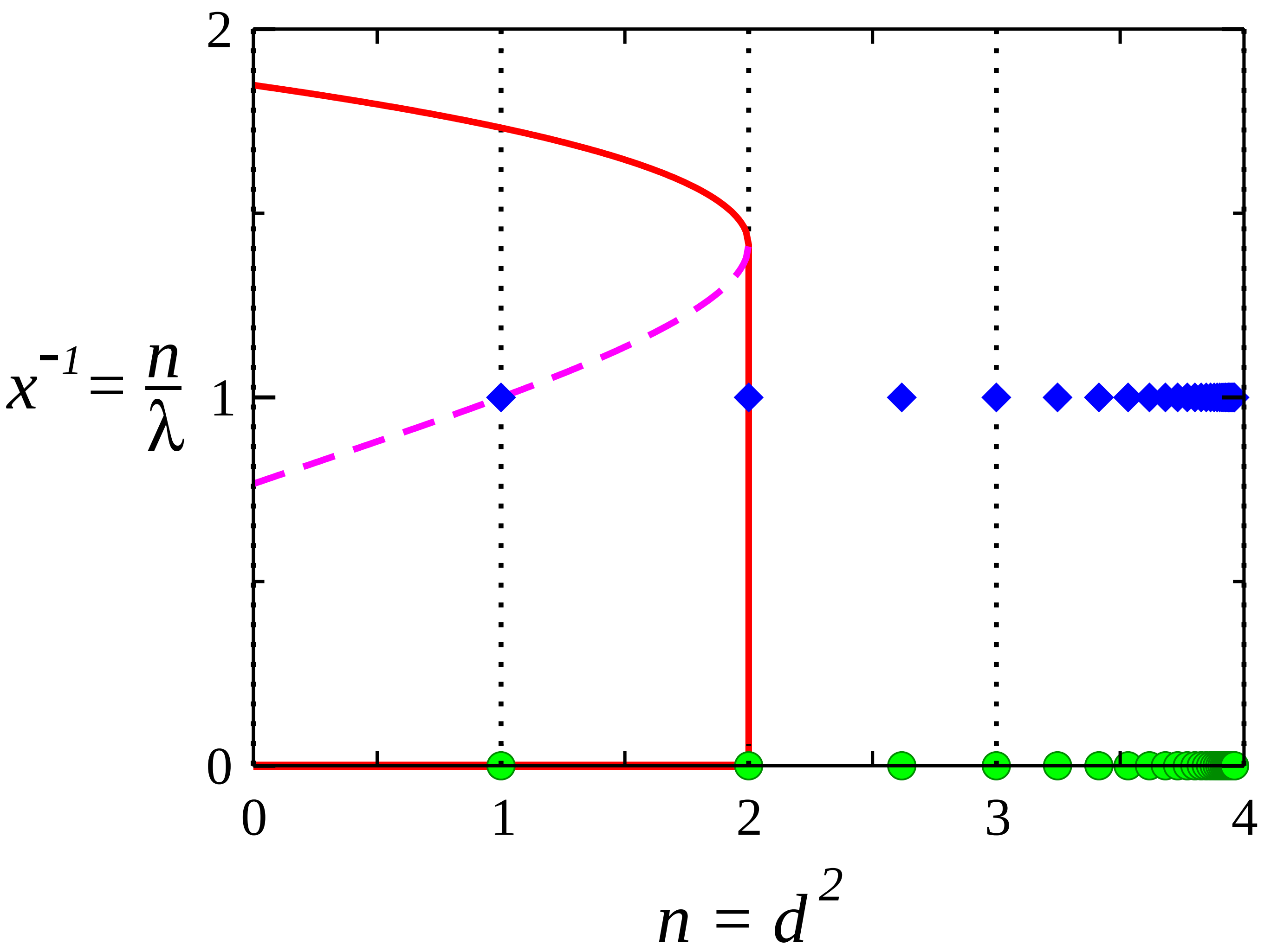
or, if we choose $a^4/b = d$,

$a\,\Psi\Big($  $\Big) - \Psi\Big($  $\Big) = 0,$

$b\,\Psi\Big($  $\Big) - \Psi\Big($  $\Big) = 0$



Phase diagram for the $O(n)$ model (Blöte, Nienhuis):



Conclusion

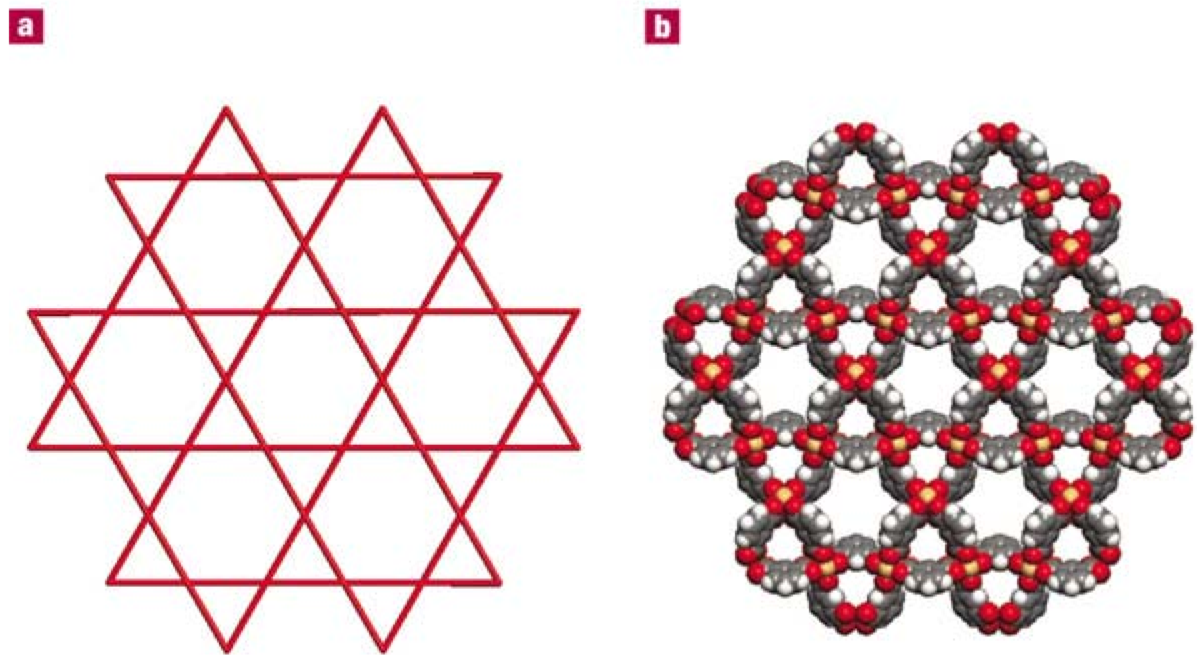
• We have constructed a microscopic model whose ground state manifold is the $d-$isotopy space, a necessary precondition for SU$(2)_k \times \overline{\text{SU}(2)_k}$ non-abelian topological order.

• Near the "special" values, $d = 2\cos\pi/(k+2)$, this space is expected to collapse to a stable topological phase with anyonic excitations closely related to SU(2) Chern-Simons theory at level k.

• Can we find or construct such materials?

Frustrated magnets? Nanostructures? Ultra-cold atoms in optical lattices? Josephson junction arrays?

Nature Materials **1,** 91–92 (2002)
A molecular toolkit for magnetism
JERRY L. ATWOOD



From theory to reality. a, A planar representation of a Kagomé lattice, and **b**, a space-filling representation of the lattice synthesized by Zaworotko and colleagues[6] by self-assembling copper(II) cations with 1,3-benzenedicaboxylate anions. Copper atoms are shown in orange, carboxylate groups in red and benzene rings in grey.