First-Principles Study of Integer Quantum Hall Transitions in Mesoscopic Samples

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Experimental data:


Current I injected in 1, extracted in 4 → two pairs of $R_H$ and $R_L$: 

$$R_{14,62}^H = \frac{V_6 - V_2}{I}; \quad R_{14,53}^H = \frac{V_5 - V_3}{I};$$

$$R_{14,23}^L = \frac{V_2 - V_3}{I}; \quad R_{14,65}^L = \frac{V_6 - V_5}{I};$$
Each QHE transition $n \rightarrow n+1$ has 3 distinct regimes:

i) Low $\nu$ (high B): $R_L + R_H = h/n e^2$;

ii) middle – uncorr. fluctuations;

iii) high $\nu$ (low B): $R_L$ fluctuates but $R_H = h/e^2(n+1)$;

For $0 \rightarrow 1$ transition, only (iii) visible, (i) and (ii) replaced by transition to the insulator.

Other interesting features:

$$R_{14,62}^H + R_{14,23}^L =$$

$$R_{14,53}^H + R_{14,65}^L =$$

$$R_{14,63} = R_{63,63}$$

+ interesting symm. $B \rightarrow -B$
Theory – numerical simulations:

Response function of the system is the 6x6 conductance matrix $g$:

$$ I_\alpha = \sum_{\beta=1}^{6} g_{\alpha \beta} V_\beta $$

If $g$ is known, solve for $V_1, \ldots, V_6$ from

$$ \begin{pmatrix} -I \\ 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} = g \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_5 \\ V_6 \end{pmatrix} \rightarrow \text{compute directly } R_{14,63}^H = \frac{V_6 - V_3}{I}, \text{ etc.} $$

Multi-probe Landauer formula [equiv. to Kubo formula, e.g. H. Baranger and D. Stone, PRB 40 8169 (1989)]

$$ g_{\alpha, \beta \neq \alpha} = \frac{e^2}{h} \sum_{i,j} |t_{\alpha i, \beta j}|^2 = \frac{e^2}{h} p_{\beta \rightarrow \alpha} \sum_{\beta=1}^{6} g_{\alpha \beta} = 0 \rightarrow g_{\alpha \alpha} = \sum_{\beta \neq \alpha} g_{\alpha \beta} $$

transmission amplitude that an electron with $E_F$ injected in channel $j$ of contact $\beta$ will emerge in channel $i$ of contact $\alpha$.

Total probability that electron injected in $\beta$ emerges in $\alpha$
→ 2µm x 4µm sample (~$10^4$ states in the LLL; we assume no LL mixing)

→ disorder potential = sum of random gaussians scatterers

→ add confinement potential (we cut the corners)

→ lead = collection of semi-infinite 1D hopping chains, each representing a different channel

→ we work at fix $B$, vary $E_F$ (the electron density, the filling factor $\nu$)

→ we solve the Lippman-Schwinger equation exactly → find all transmission amplitudes

→ we end up with $g(E_F)$ and $\nu(E_F) \rightarrow g(\nu)$ for $0 < \nu < 1$ (only LLL).
1) For $\nu < 0.35$, $g$ is symmetric: $g_{\alpha\beta} = g_{\beta\alpha}$

2) For $\nu \rightarrow 1$, $g(\nu) \rightarrow g^0_{\alpha\beta} = e^2/h (-\delta_{\alpha\beta} + \delta_{\alpha+1,\beta} + \delta_{\alpha6}\delta_{\beta1})$

$$ \hat{I} = n\hat{g}^0 \cdot \hat{V} \rightarrow R_H = \frac{h}{ne^2}; R_L = 0 \leftrightarrow \text{QH plateau}$$

Variations in $g(\nu)$ from $g^0 \rightarrow$ fluctuations in $R_H, R_L$. 

Edge currents
our numerical simulations recapture all the experimental results. What is happening?
General structure of $g$: ($e^2/h=1$)

There are two types of contributions to $g$: a symmetric part ("resistors", tunneling) and a chiral part.

e.g.: resistor $R$ between 1 and 2 is described by

Each such term involves only two contacts;
we denote unit resistor by $r(\alpha,\beta)$]

\[
\frac{1}{R} \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} \rightarrow \begin{cases}
I_1 = \frac{V_2 - V_1}{R} \\
I_2 = -I_1
\end{cases}
\]

A chiral current is a closed loop involving more than 2 contacts, in "chiral" order.

E.g: $2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 2 = r(2,3,5,6)$ ; with this notation, $g^0 = r(1,2,3,4,5,6)$

\[
r(2,3,5,6) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\]
Medium filling factor $\nu$: edge states connecting neighboring contacts become established $\Rightarrow$ chiral loops start to form. One expects direct competition between both types of transport.

Large filling factor $\nu$: most transport due to chiral currents. Tunneling contributes to back-scattering (Jain-Kivelson model).

Low filling factor $\nu$: all states are localized $\Rightarrow$ electrons can move between different contacts only through direct tunneling $\Rightarrow$

$p_{1\rightarrow 2} \sim p_{2\rightarrow 1} = |t_{12}|^2 + \mathcal{O}(t^2)$, i.e. $g_{12} \sim g_{21}$.

A chiral current $r(1,2,6) = 2|t_{12}t_{26}t_{61}|$ is also established.

This is why for small $\nu$, $g$ is a symmetric matrix.
For each pair $\alpha, \beta$: $\min(g_{\alpha\beta}, g_{\beta\alpha})$ is symmetric term ($\alpha\beta$ resistor); remaining term contributes to one or more chiral currents.

Consider the general form:

$$\hat{g} = n\hat{g}^0 + r_{12}\hat{r}(1,2) + r_{26}\hat{r}(2,6) + r_{16}\hat{r}(1,6) +$$
$$+ r_{34}\hat{r}(3,4) + r_{45}\hat{r}(4,5) + r_{35}\hat{r}(3,5) +$$
$$+ c_1\hat{r}(1,2,6) + c_2\hat{r}(2,3,5,6) + c_3\hat{r}(3,4,5) +$$
$$+ c_4\hat{r}(1,2,3,5,6) + c_5\hat{r}(2,3,4,5,6) + c_0\hat{g}^0$$

$$R^H + R^L = R_{14,63} = R_{63,63} = \frac{h}{e^2} \cdot \frac{1}{n + c_0 + c_2 + c_4 + c_5}$$

Conclusions:

a) $R_{14,63} = R_{63,63}$ for all filling factors $\nu$.

b) At small $\nu$, all c’s = $O(t^2) \rightarrow R^H + R^L = h/ne^2$

Largest neglected terms: $r_{23}$ and $r_{56} \sim 10^{-4}$
Large filling factor $\nu$:
\[
\hat{g} = n\hat{g}^{0} + (1-t_1-t_2-t_3)\hat{g}^{0} \\
+ t_2[\hat{r}(1, 2, 6) + \hat{r}(3, 4, 5)] \\
+t_3\hat{r}(1, 2, 3, 5, 6) + t_1\hat{r}(2, 3, 4, 5, 6)
\]
\[
\begin{align*}
R_{14,62}^H &= R_{14,53}^H = \frac{h}{e^2(n+1)} \\
R_{14,23}^L &= R_{14,65}^L = \frac{h}{e^2(n+1)} \cdot \frac{t_2}{n+1-t_2}
\end{align*}
\]

Summary:

- Low $\nu$, $R^H + R^L = h/ne^2$ holds until edge state is established between 2-3 or 5-6, i.e. the localization length becomes comparable to distance between 2-3 contacts.

- High $\nu$, $R^H = h/(n+1)e^2$ while $R^L$ fluctuates: only if localization length comparable to 3-5 distance.

$\Rightarrow$ The central regime corresponds to the “critical region”