A ~~Some~~ Trivial Theorem~~s~~ in Quantum Mechanics

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What is
Quantum Information
about?

Attitude! (mostly)

Classical:

$$\Delta x \, \Delta p = 0$$

Quantum:

$$\Delta x \, \Delta p \geq \hbar/2$$

Classical:

$$\Delta x \, \Delta p = 0$$

Quantum:

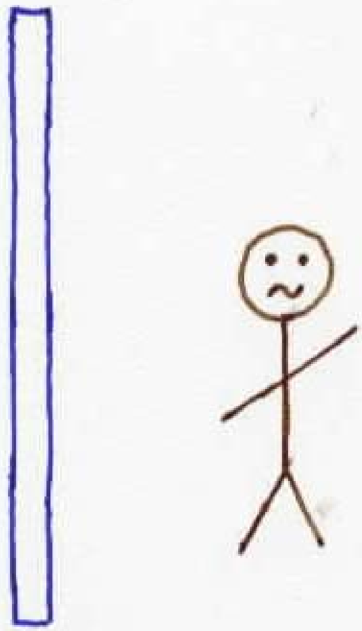
$$\Delta x \, \Delta p \geqslant \hbar/2$$

As if the world were short-changing us.

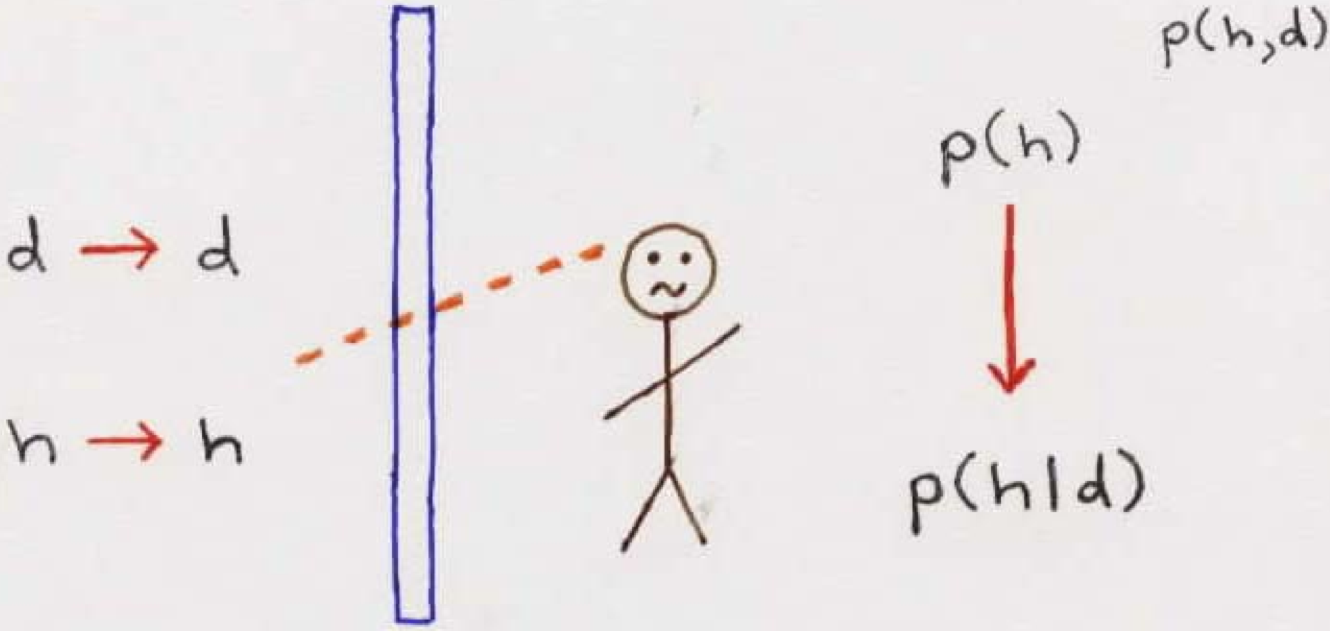
The Weatherman

$p(h,d)$

$p(h)$



The Weatherman



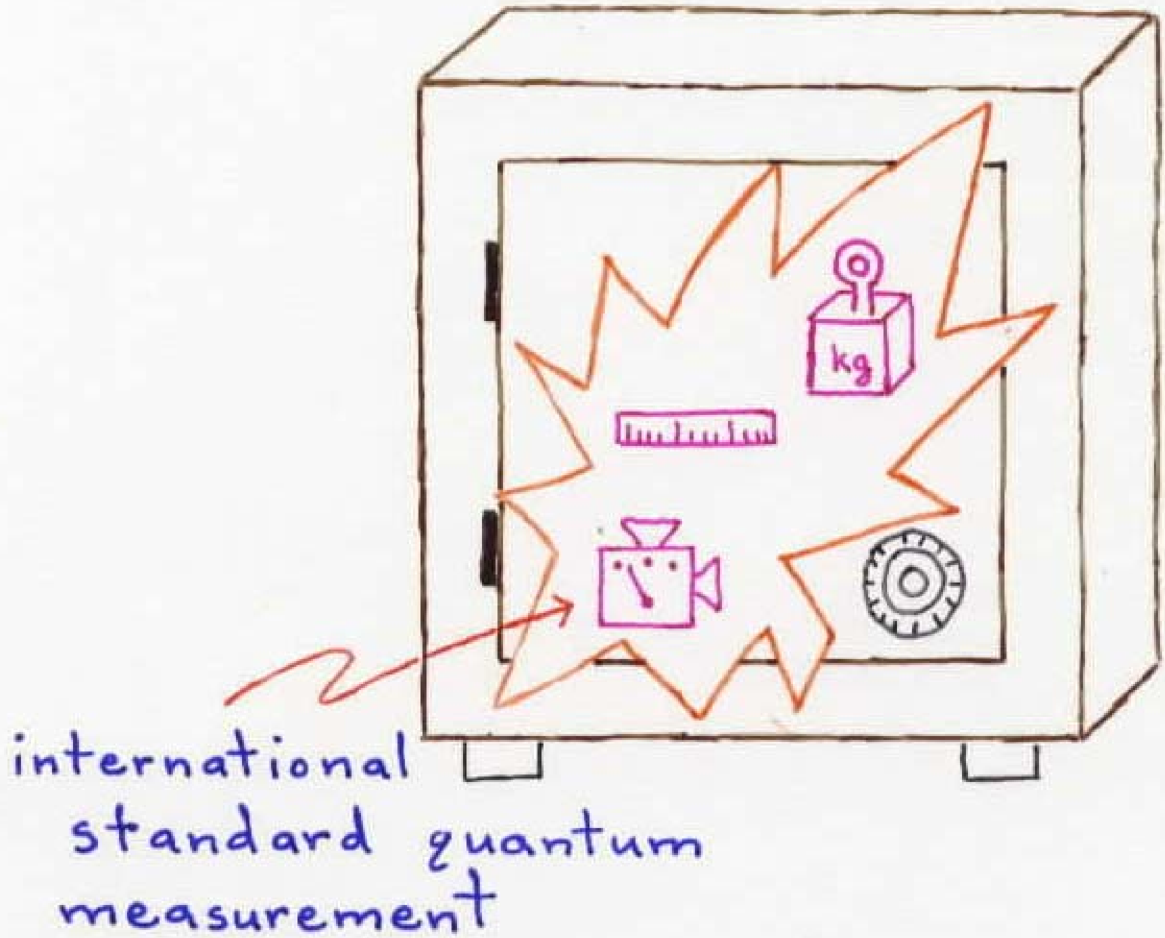
Is this transition a mystery physics should contend with?

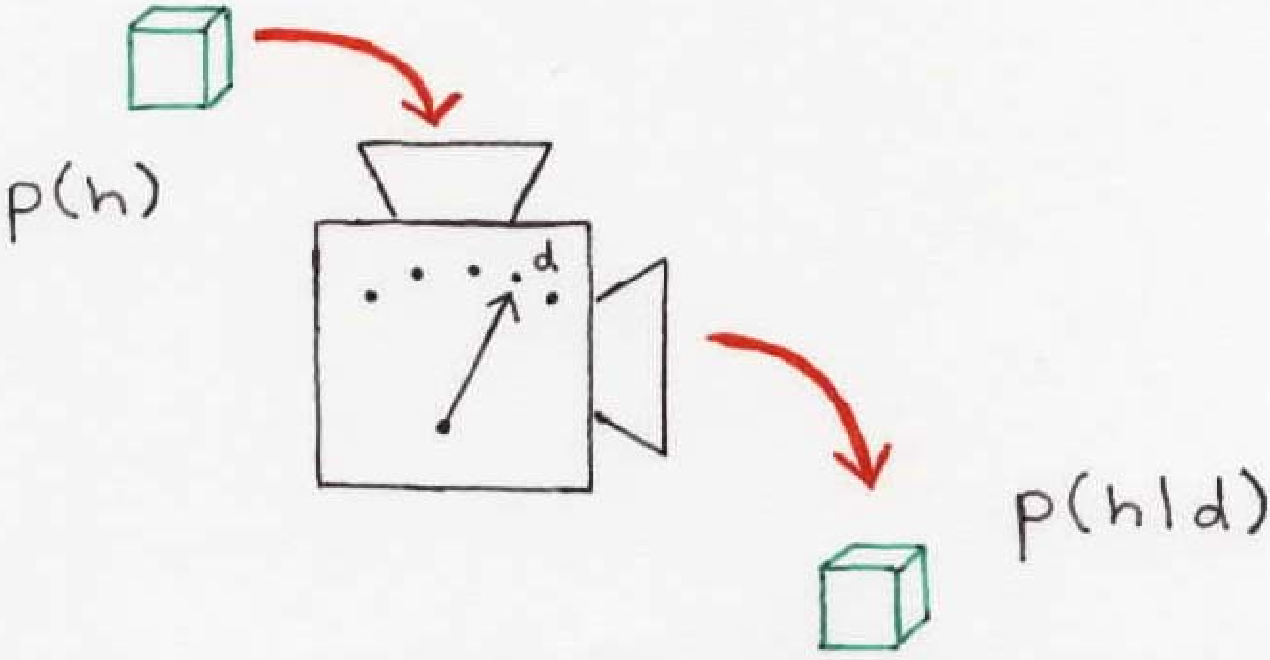
Even so, what does it have to do with the weather?

~~Hidden~~

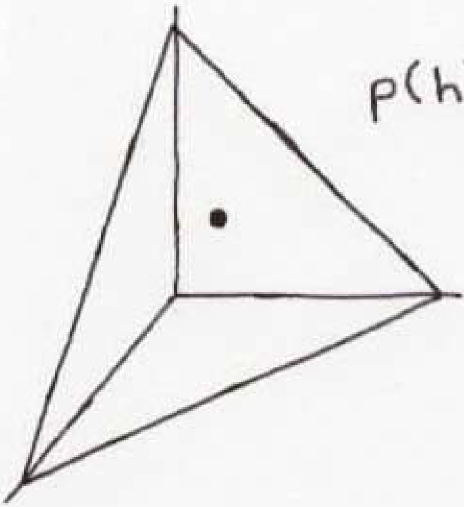
~~Variables~~

Bureau of Standards

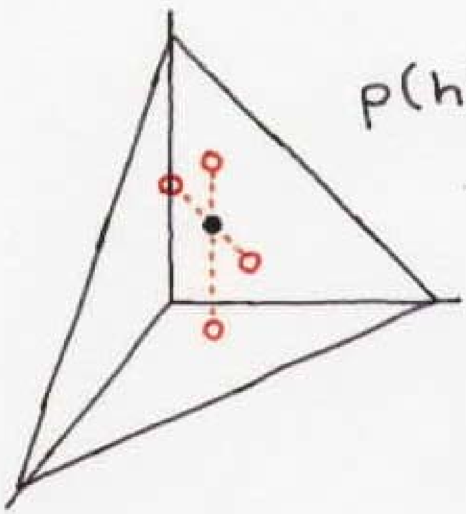




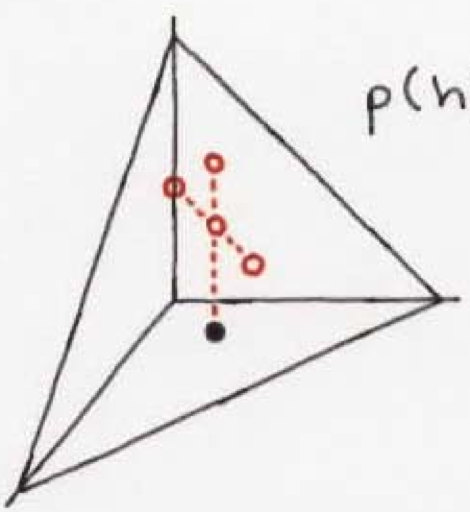
Accepting QM



$p(h)$

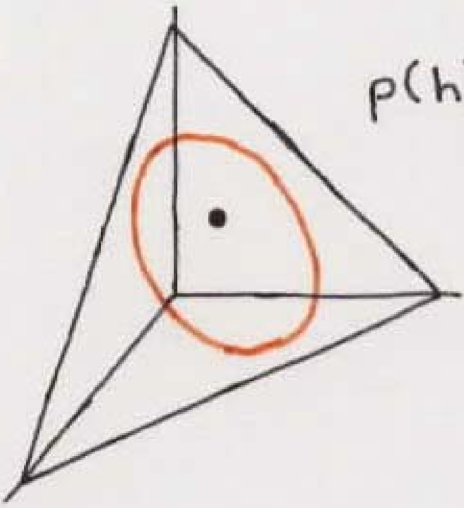


$p(h) = \sum_d p(d)\,p(h|d)$

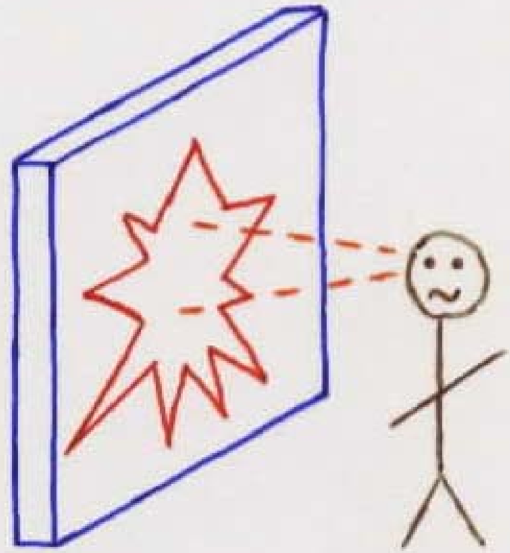


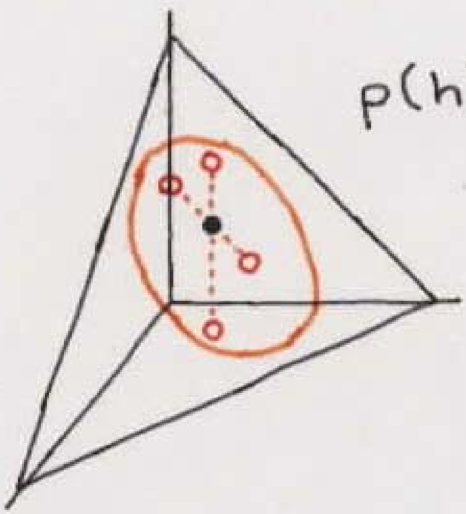
$p(h|d)$

Accepting QM

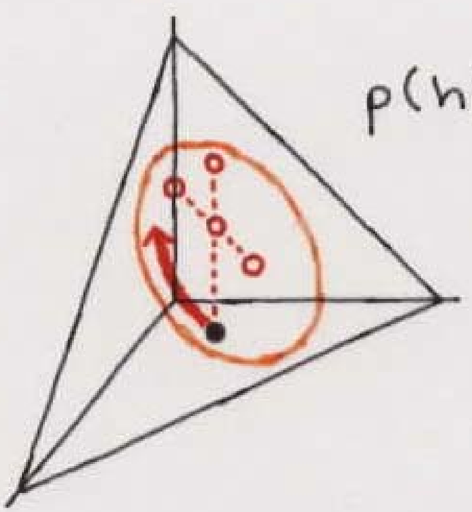


$p(h)$





$$p(h) = \sum_d p(d)\,p(h|d)$$



$p(h|d)$

1) too much certainty not prudent

2) measurement *is* refinement

3) extra conditionalization rule

von Neumann Measurements

"measurement"

$\Longleftrightarrow$ Hermitian operator

$$\mathcal{O} = \sum_i \alpha_i \Pi_i$$

eigenvalues

eigenprojectors

When state is ρ,

$$p(i) = \operatorname{tr} \rho \Pi_i \, .$$

Could say,

"measurement" $\Longleftrightarrow$ $\{\Pi_i\}$

POVMs

Positive Operator Valued Measures

— an immensely useful tool

Let $\mathcal{P} = \{ E : 0 \leq \langle \psi | E | \psi \rangle \leq 1 \ \forall |\psi\rangle \}$.

Any <u>set</u> of operators

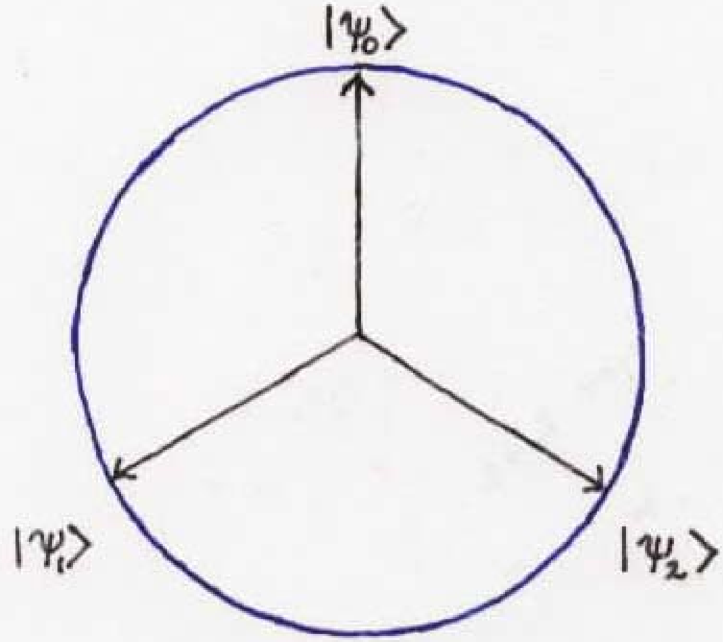
$$\{ E_b : E_b \in \mathcal{P}, \sum_b E_b = I \}$$

corresponds to a potential mmt.

Probability of outcome b,

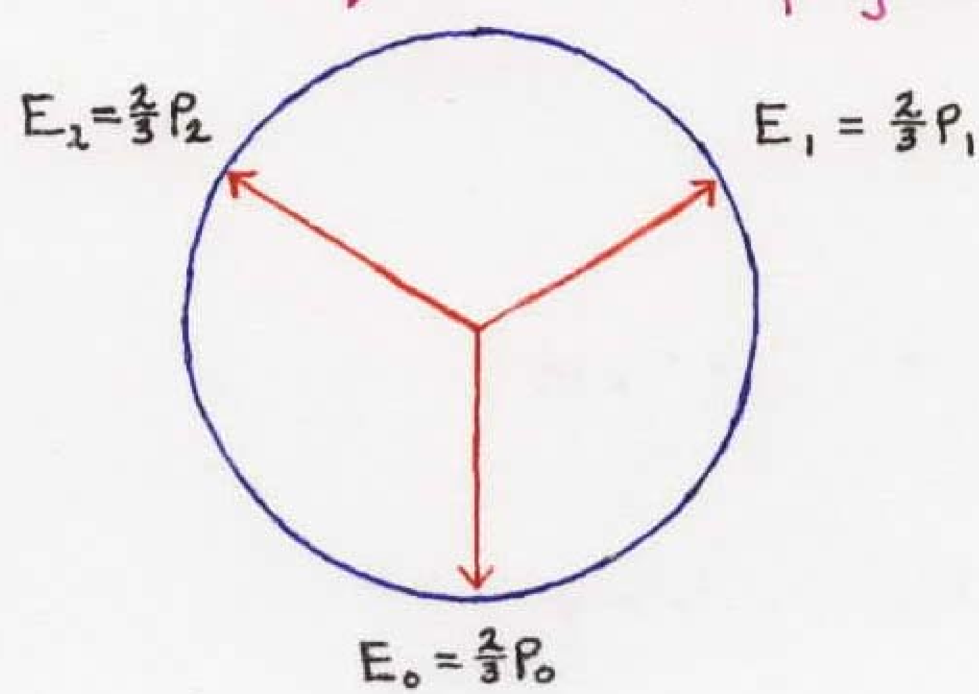
$$p_b = \operatorname{tr} \rho E_b .$$

But why POVMs ?



Needed!

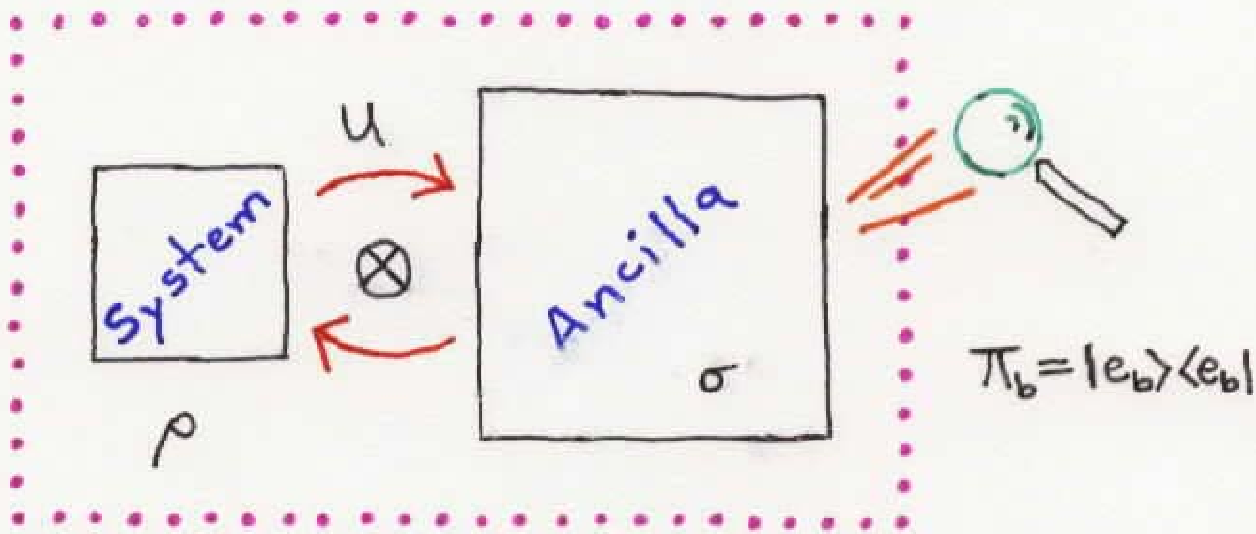
POVM proportional
to projectors



Why not take POVMs
as basic notion of measurement

?

Usual Justification



Interact:

$$\rho \otimes \sigma \longrightarrow U(\rho \otimes \sigma)U^\dagger$$

Measure Ancilla:

$$p(b) = \mathrm{tr}\left[U(\rho \otimes \sigma)U^\dagger (I \otimes \Pi_b)\right]$$

Rewrite:

$$p(b) = \mathrm{tr}(\rho E_b)$$

where

$$E_b = \mathrm{tr}_A\left[(I \otimes \sigma)U^\dagger (I \otimes \Pi_b) U\right]$$

Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_b\}$
$\langle\psi\vert\Pi_i\vert\psi\rangle \geq 0,\ \forall\vert\psi\rangle$	$\langle\psi\vert E_b\vert\psi\rangle \geq 0,\ \forall\vert\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_b E_b = I$
$p(i) = \mathrm{tr}\,\rho\Pi_i$	$p(b) = \mathrm{tr}\,\rho E_b$
$\Pi_i\Pi_j = \delta_{ij}\Pi_i$	—

Does this extra assumption really make the process any less mysterious

?

Informational Completeness

quantum states

$\rho \in \mathcal{L}(\mathcal{H}_D)$ — D^2-dimensional vector space

Choose POVM $\{E_h\}$, $h = 1, \ldots, D^2$, with E_h all linearly independent.
(Can be done.)

D^2 numbers $p(h) = \operatorname{tr} \rho E_h$ determine ρ.

projection of ρ onto E_h

Any $\{E_h\}$ can be the standard quantum measurement.

State Change

When measure POVM $\{E_b\}$ **efficiently**

$$\rho \xrightarrow{\;b\;} \tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^\dagger$$

where

$$p_b = \operatorname{tr} \rho E_b \quad \text{and} \quad A_b^\dagger A_b = E_b \,.$$

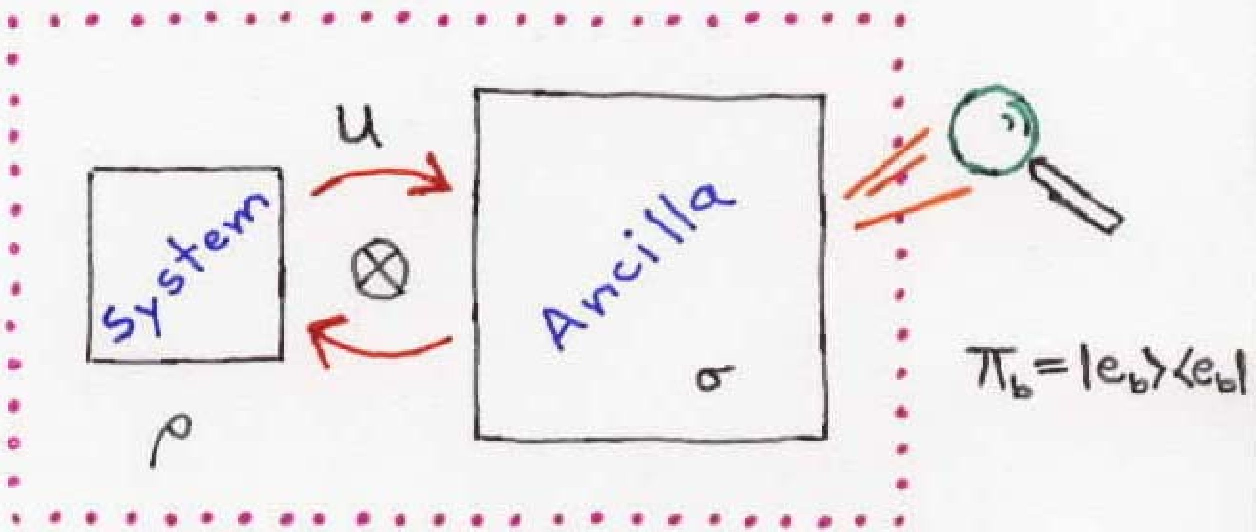
By polar decomposition theorem

$$\tilde{\rho}_b = \frac{1}{p_b} U_b \underbrace{E_b^{1/2} \rho E_b^{1/2}}_{\text{"collapse"}} U_b^\dagger$$

"feedback"

U_b depends upon detailed form of measurement interaction

Usual Justification



1) $\rho \otimes \sigma \xrightarrow{\text{couple}} U(\rho \otimes \sigma)U^{+}$

2) $\xrightarrow{\text{collapse}} (I \otimes \Pi_b)U(\rho \otimes \sigma)U^{+}(I \otimes \Pi_b)$

3) $\xrightarrow{\text{discard}} \mathrm{tr}_A[\cdots] \xrightarrow{\text{renormalize}}$

Upshot

$$\rho \xrightarrow{b} \tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^{+}$$

Collapse

$\Updownarrow$?

Bayesian Conditionalization

Bayes' Rule

$$p(H) = \sum_D p(H,D)$$

$$= \sum_D p(D)\, p(H|D)$$

$$p(H) \xrightarrow{D} p(H|D) = \frac{p(H)\,p(D|H)}{p(D)}$$

Ignorance decreases on average:

$$S(H) \geq S(H|D)$$

by concavity of Shannon entropy.

Note

Unlike with Bayes rule

$$\underset{\text{pre-measurement state}}{\rho} \neq \sum_b p_b \underset{\text{post-mmt. states}}{\tilde{\rho}_b}$$

Moreover, not even if we delete the "feedback"

$$\tilde{\rho}_b = \frac{1}{p_b} \cancel{U_b} E_b^{1/2} \rho E_b^{1/2} \cancel{U_b^\dagger}$$

<u>Too</u> <u>Bad</u> we can't build state change rule like this:

When given POVM $\{E_b\}$, $E_b = A_b^\dagger A_b$,

define ρ_b by

$$\rho = \rho^{1/2} I \rho^{1/2} = \sum_b \rho^{1/2} E_b \rho^{1/2}$$

$$= \sum_b p_b \rho_b$$

canonical refinement

where $\rho_b = \frac{1}{p_b} \rho^{1/2} E_b \rho^{1/2}$.

Unfortunately,

$$\tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^\dagger \neq \frac{1}{p_b} \rho^{1/2} A_b^\dagger A_b \rho^{1/2} = \rho_b$$

real post-mmt state

too-bad state

But!

ρ_b and $\tilde{\rho}_b$ have identical spectra

(Since for any operator B, $B^\dagger B$ and $BB^\dagger$ have identical spectra.)

Bohring Bayesians?

Given POVM $\{E_b\}$, write

$$\rho = \sum_b p_b \rho_b \quad \text{where} \quad \rho_b = \frac{1}{p_b}\rho^{1/2} E_b \rho^{1/2}.$$

Might as well say state changes according to a kind of conditionalizing

$$\rho \xrightarrow{\;b\;} \tilde{\rho}_b = V_b \rho_b V_b^\dagger$$

"learning"

"mental readjust"

V_b depends upon measurement interaction and initial state of knowledge ρ.

Emphasis

Classical

$$p(H) = \sum_D p(D)\, p(H|D)$$

$$p(H) \xrightarrow{D} p(H|D)$$

Quantum

$$\rho = \sum_b p_b\, \rho_b$$

$$\rho \xrightarrow{b} \rho_b$$

modulo a further
unitary
readjustment

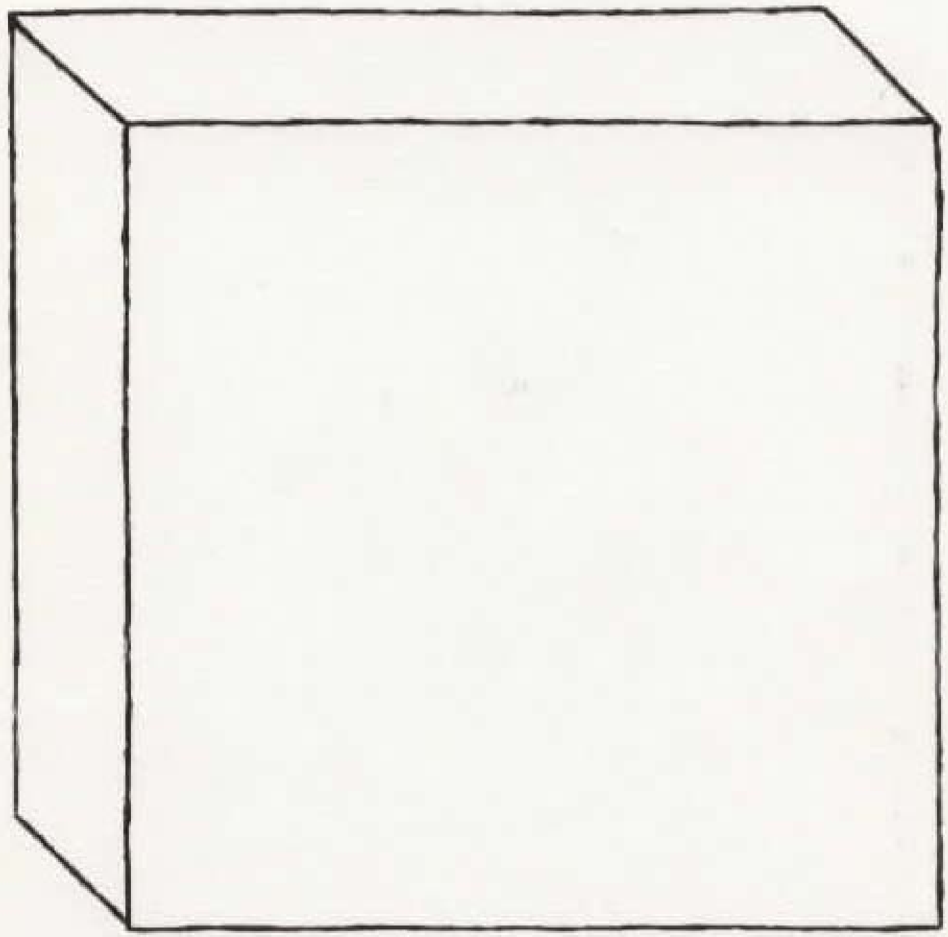
Limiting Cases

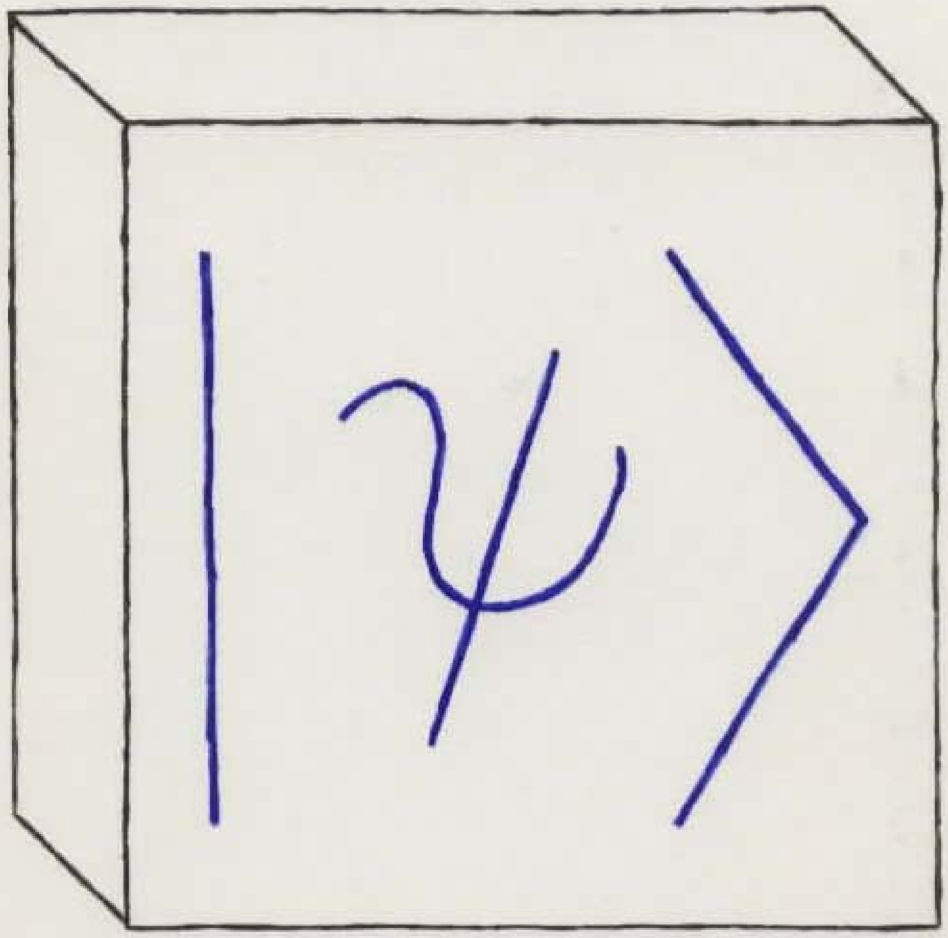
1) When no "back action" possible, measurement should lead to pure Bayesian conditionalization.

$$\rho \longrightarrow \rho^{1/2} E_b^T \rho^{1/2}$$

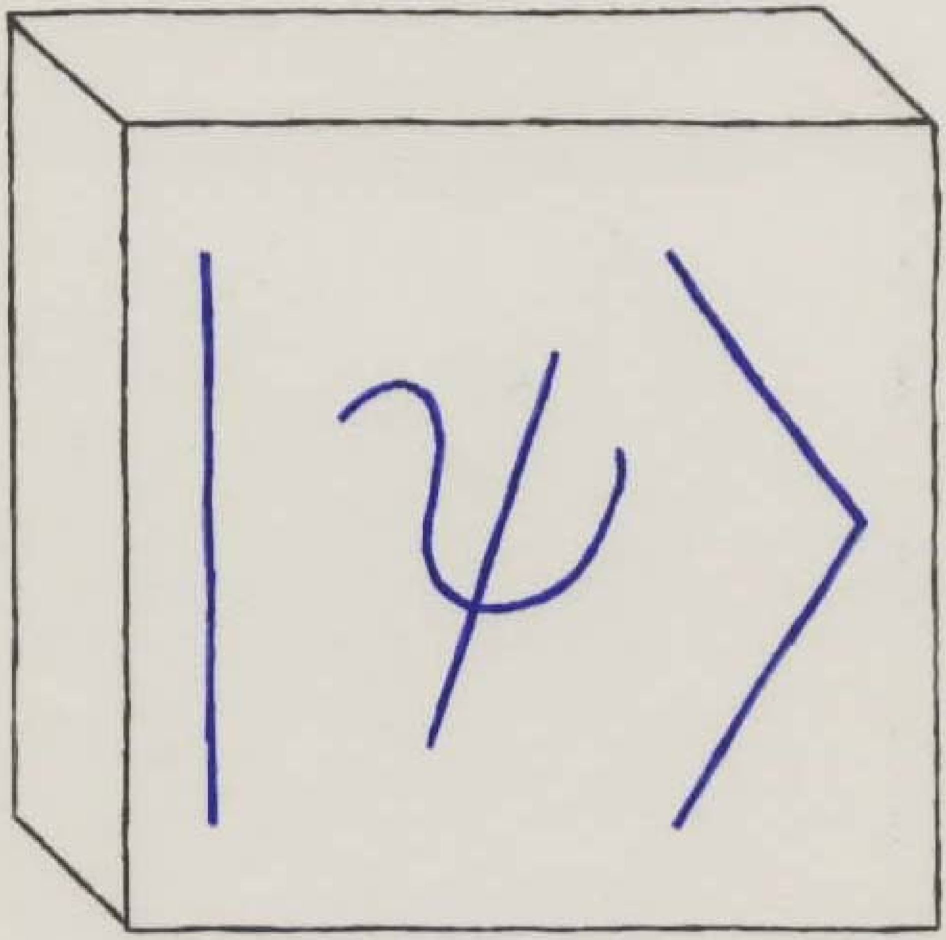
2) When $\rho = |\psi\rangle\langle\psi|$, nothing left to learn.

$$\rho \longrightarrow \rho^{1/2} E_b \rho^{1/2} \propto \rho \longrightarrow V_b \rho V_b^\dagger$$



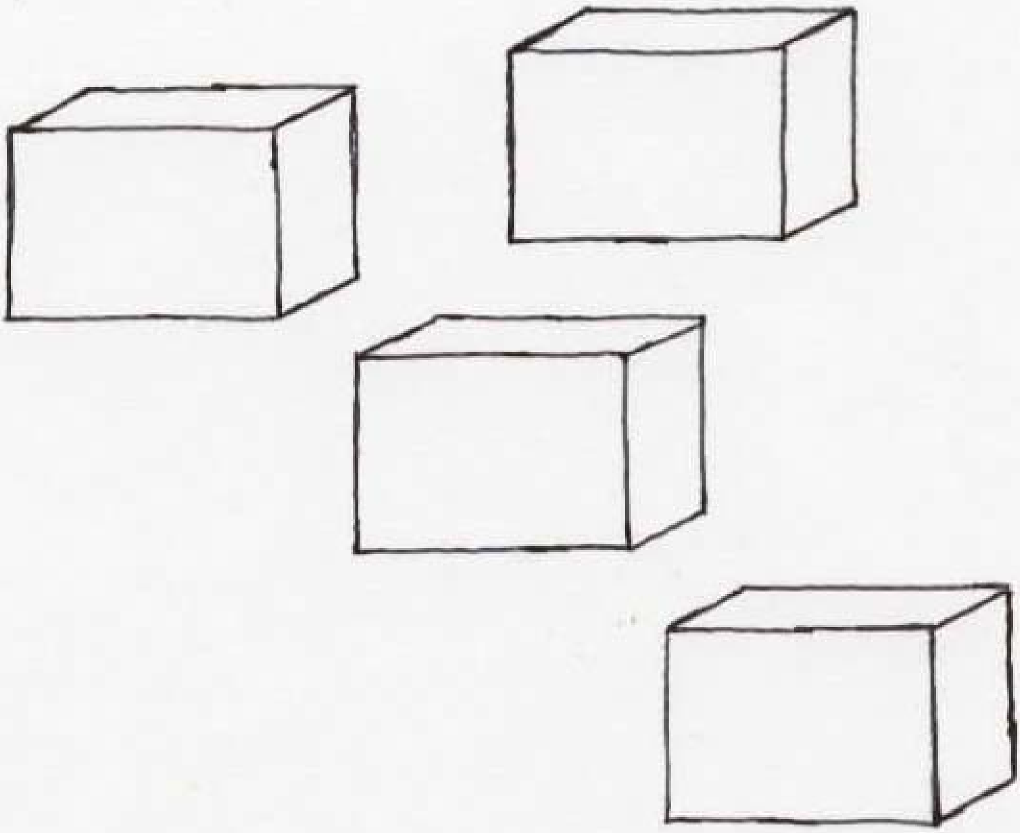

$|\psi\rangle$

Information / knowledge about what?



... the consequences of our experimental interventions into the course of Nature.

What is real about a system?



Irritable
Bricks