The Oyster and the Quantum

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I want you to frame a question, as sharp and clear as possible—one to which you do not yet know the answer, but desperately want to know, and expect someday to know.

Pretend to be David Hilbert. The Millennium is approaching. Issue a challenge to the quantum theorists of the 21st century. List the key questions they should seek to answer. Hard questions, but not hopelessly hard, questions whose answers could transform our understanding of how the physical world works.

— John Preskill
12 August 1998

Axioms

0) Systems exist.

1) Associated with each is a complex vector space $\mathcal{H}$.

2) Measurements correspond to orthonormal bases $|e_i\rangle$ on $\mathcal{H}$.

3) States correspond to density operators ρ on $\mathcal{H}$.

4) Systems combine by tensor producting their vector spaces, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

5) When no measurement is performed, states evolve by unitary maps $\mathcal{U}$.

Special Relativity

c is constant.

Physics is constant.

Tegmark Poll

Interpretation	Votes
Copenhagen	13
Many Worlds	8
Bohm	4
Consistent Histories	4
Modified Dynamics (GRW)	1
None of the above/ undecided	18

Axioms

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   Give an information theoretic reason.

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QM is about Information.

$$H(X)$$

Plain old ordinary Shannon information:

ignorance

lack of predictability

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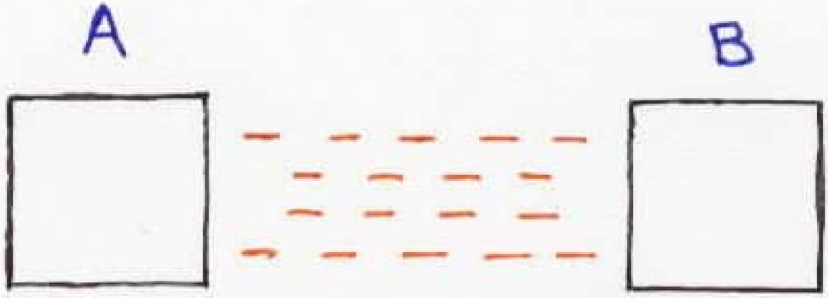
$H(X)$

Mostly!

Plain old ordinary Shannon information:

ignorance

lack of predictability



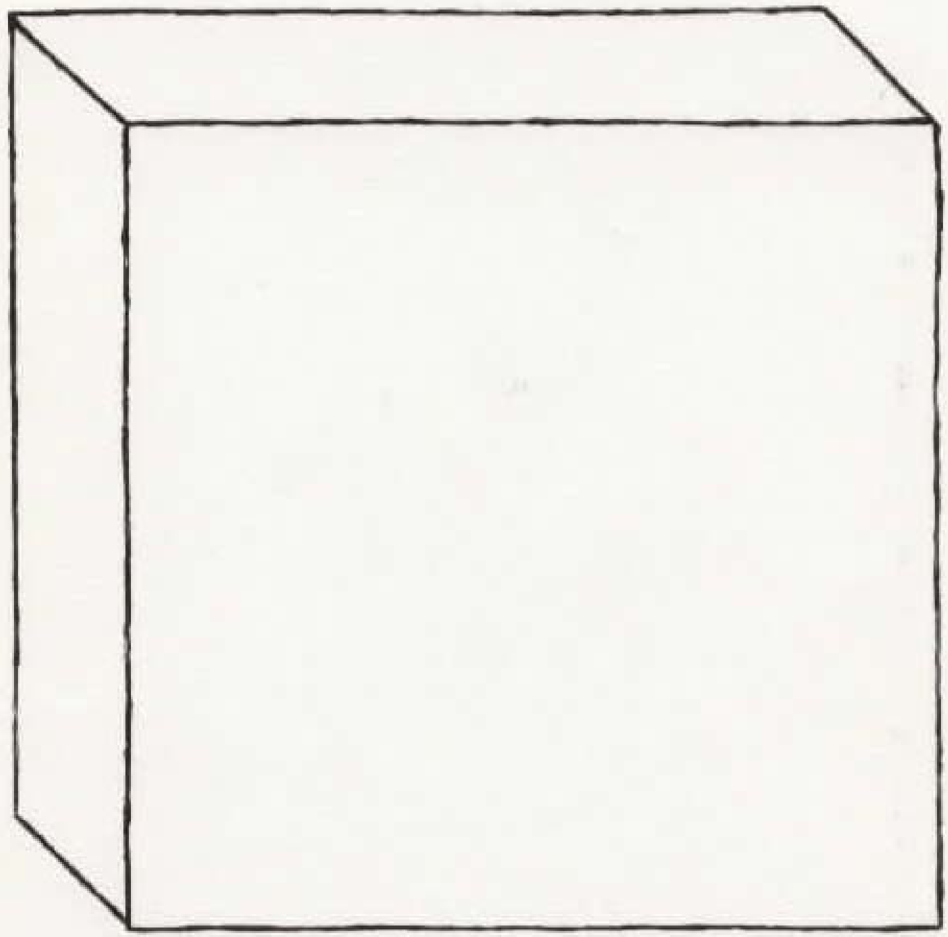
$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

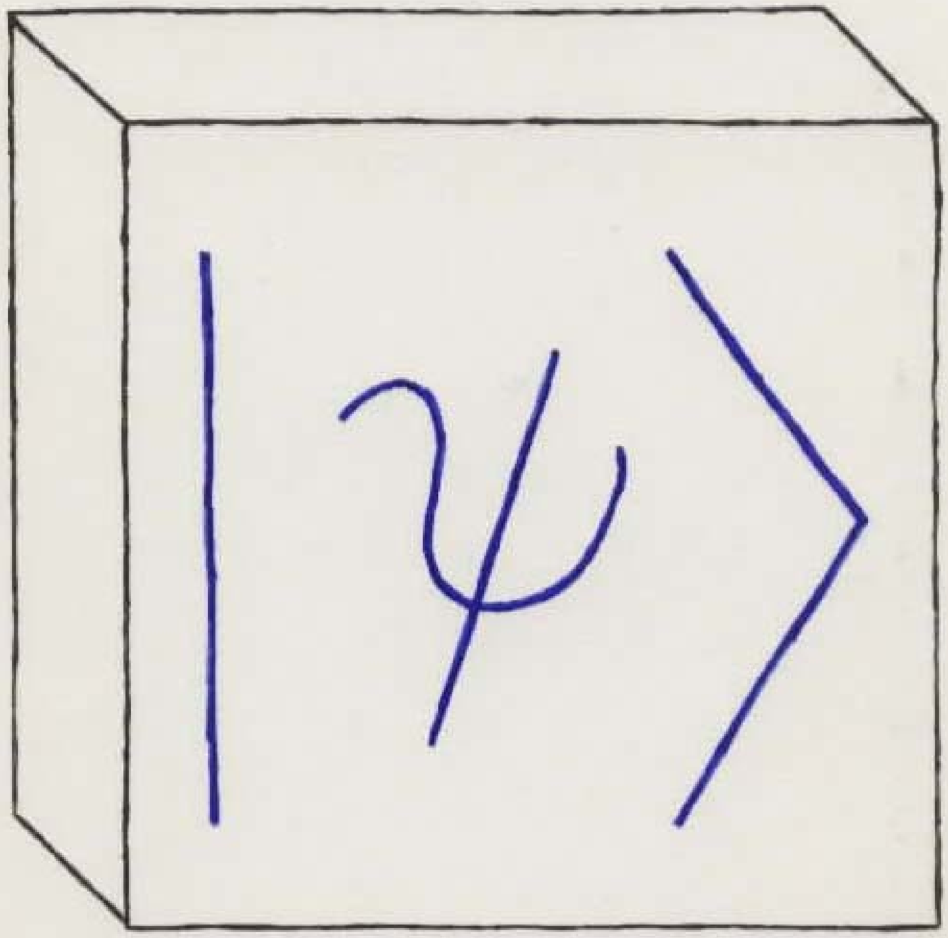
Let Alice measure $|\uparrow\rangle, |\downarrow\rangle$ basis.
Bob's system will be in state
$|\uparrow\rangle$ or $|\downarrow\rangle$ afterward.

Let Alice measure $|\rightarrow\rangle, |\leftarrow\rangle$ basis.
Bob's system will be in state
$|\leftarrow\rangle$ or $|\rightarrow\rangle$ afterward.

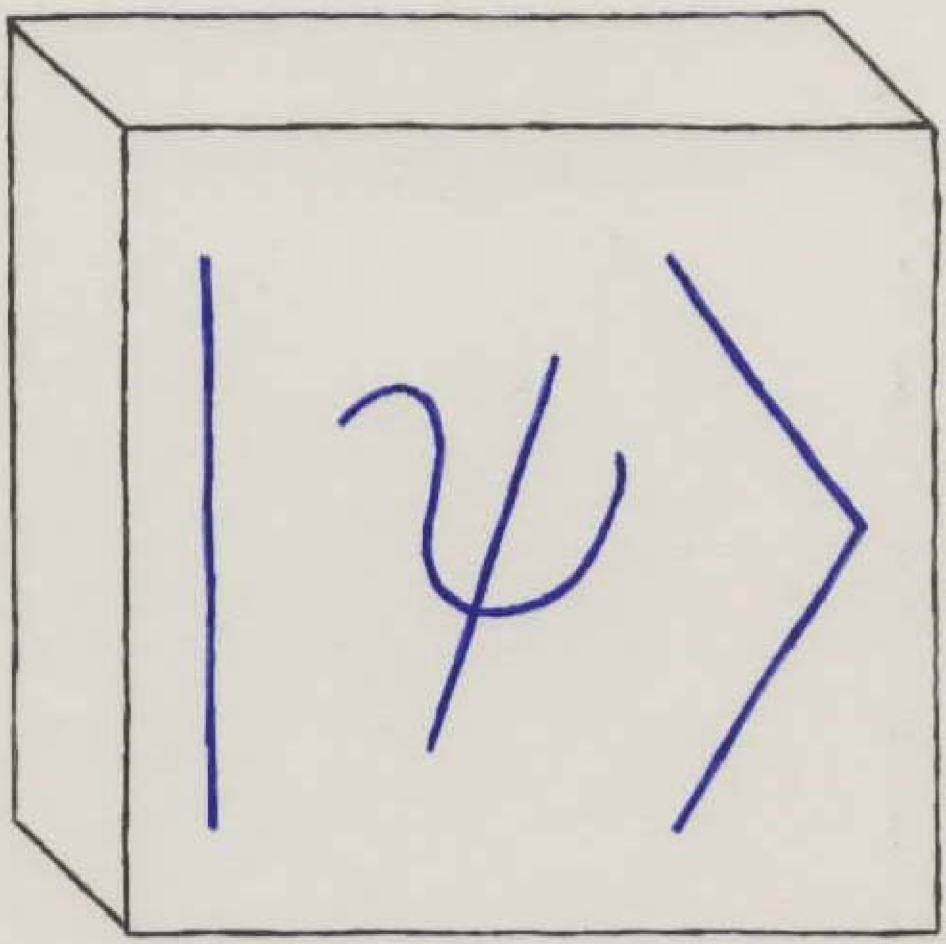
Conclusion

$|\psi\rangle$ is information.




$|\psi\rangle$

Information / knowledge about what?



... the consequences of our experimental interventions into the course of Nature.

What is **real** about a system?

The Probability Rule

The Collapse Rule

Why Entanglement

Irritable Bricks

(not a theorem)

Quantum Probability

Given a state ρ,

and an observable

$$H = \sum_i \alpha_i \Pi_i \quad \text{(eigendecomp)},$$

<u>the</u> probability of outcomes

is

$$p_i = \operatorname{tr} \rho \Pi_i .$$

Why this rule and not some other?

Gleason's Theorem

Let $\mathbb{P}(\mathcal{H}_d)$ be the set of 1-D projectors onto a (real or complex) vector space $\mathcal{H}_d$ of dimension $d \geq 3$.

Suppose there exists a function $f : \mathbb{P}(\mathcal{H}_d) \longrightarrow [0,1]$ such that

$$\sum_i f(\Pi_i) = 1$$

whenever $\{\Pi_i\}$ forms a complete orthogonal set.

Theorem: Then there exists a density operator ρ, such that

$$f(\Pi) = \mathrm{tr}\, \rho \Pi .$$

Main Assumptions (my take):

1. Measurements are incompatible. No good notion of measuring $\{\Pi_i\}$ AND $\{\tilde{\Pi}_i\}$.
2. Noncontextuality of probabilities. $\mathrm{Prob}(i \mid \{\Pi_k\}) = f(\Pi_i)$.

Comments

1) Proof is long, hard, ugly.

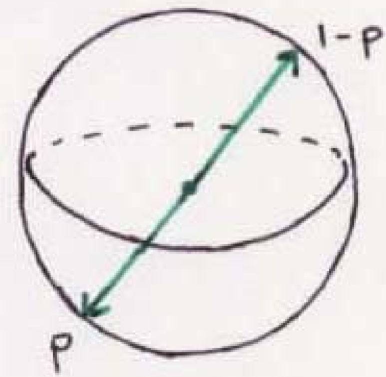
   a) Reduce problem to checking $\mathbb{R}^3$.

   b) Prove continuity of f.

   c) Expand f in spherical harmonics and massage.

2) Doesn't work for $d=2$.

Prompts people to say "there's nothing quantum about a single qubit."



3) Gave Dave Meyer another PRL!

Theorem fails for fields other than $\mathbb{C}$ and $\mathbb{R}$, like

$\mathbb{Q}$ — the rationals.

POVMs

Positive Operator Valued Measures

— an immensely useful tool

Let $\mathcal{P} = \{ E : 0 \le \langle \psi | E | \psi \rangle \le 1 \ \forall |\psi\rangle \}$.

Any <u>set</u> of operators

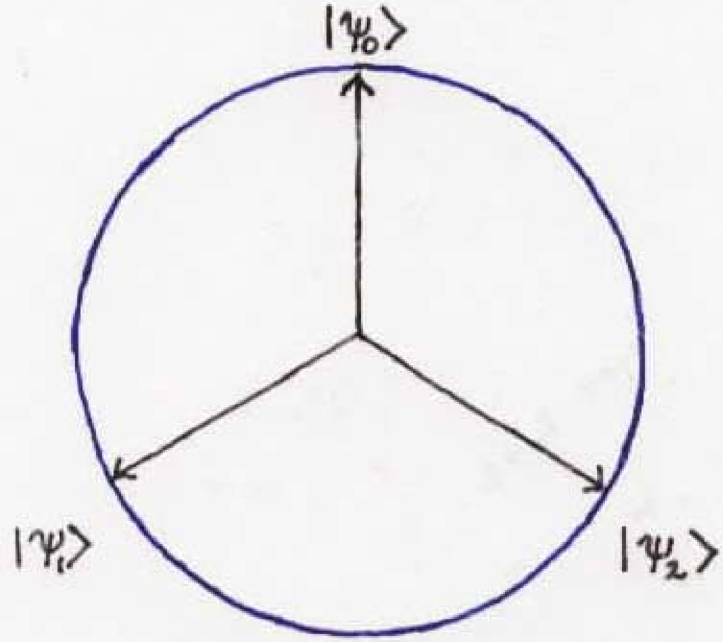
$$\{ E_b : E_b \in \mathcal{P}, \sum_b E_b = I \}$$

corresponds to a potential mmt.

Probability of outcome b,

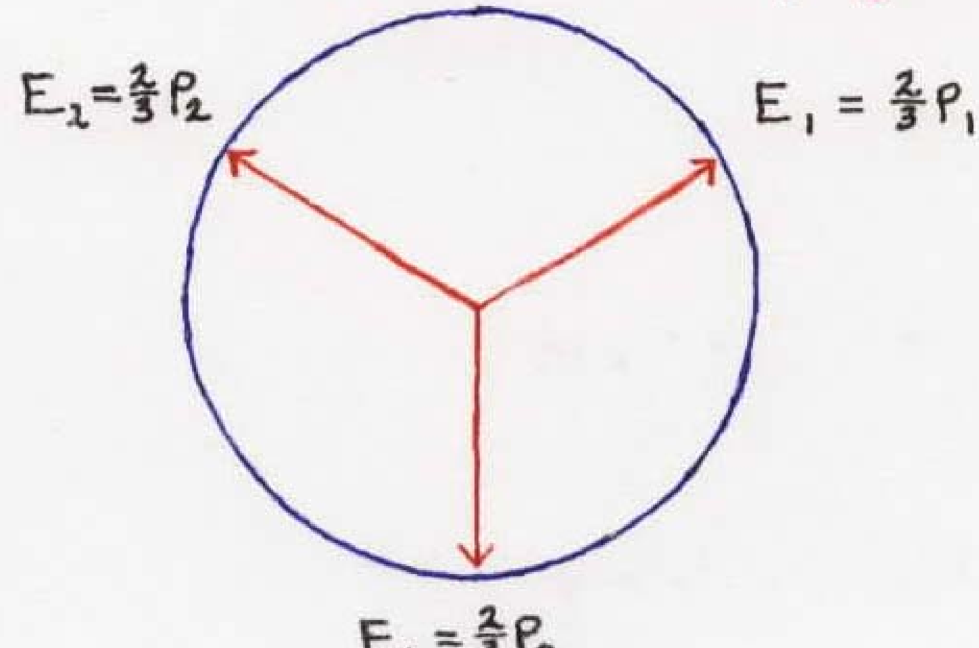
$$p_b = \mathrm{tr}\, \rho E_b .$$

But why POVMs ?



Needed!

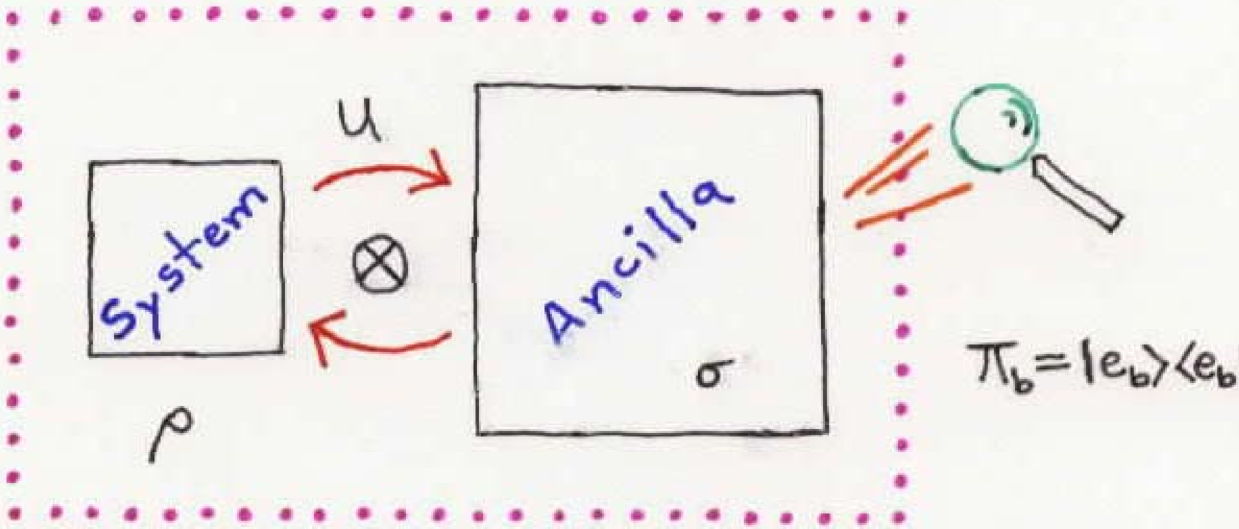
POVM proportional to projectors



Why not take POVMs
as basic notion of measurement

?

Usual Justification



Interact:

$$\rho \otimes \sigma \longrightarrow U(\rho\otimes\sigma)U^\dagger$$

Measure Ancilla:

$$p(b) = \mathrm{tr}\left[U(\rho\otimes\sigma)U^\dagger(I\otimes\Pi_b)\right]$$

Rewrite:

$$p(b) = \mathrm{tr}(\rho E_b)$$

where

$$E_b = \mathrm{tr}_A\left[(I\otimes\sigma)U^\dagger(I\otimes\Pi_b)U\right]$$

Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_b\}$
$\langle\psi\vert\Pi_i\vert\psi\rangle \geq 0,\ \forall\vert\psi\rangle$	$\langle\psi\vert E_b\vert\psi\rangle \geq 0,\ \forall\vert\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_b E_b = I$
$p(i) = \mathrm{tr}\,\rho\Pi_i$	$p(b) = \mathrm{tr}\,\rho E_b$
$\Pi_i\Pi_j = \delta_{ij}\Pi_i$	—

Does this extra assumption really make the process any less mysterious

?

Gleason-like Theorems

Renes, Manne and co.

Busch

Assumptions

1) Measurements = POVMs
2) Noncontextuality

$$\text{Prob}(E_i \mid \{E_k\}) = \text{Prob}(E_i \mid \{\tilde{E}_k\})$$

when $\{E_k\}$ and $\{\tilde{E}_k\}$ share E_i.

I.e. Let $f: \mathcal{P} \to [0,1]$ be such that $\sum_b f(E_b) = 1$ whenever $\sum_b E_b = I$.

Thm: $\exists\, \rho$, s.t. $f(E) = \text{tr}\,\rho E$.

Payoff:

1) Works for $d=2$.
2) Works for rational $\mathcal{H}$.
3) Proof easy.

Proof (for rational $\mathcal{H}$)

Consider $E_1, E_2 \in \mathcal{P}$ s.t. $E_1 + E_2 \in \mathcal{P}$.
Embed in 3-outcome POVM.

$$f(E_1) + f(E_2) + f(E_3) = 1$$

$$f(E_1 + E_2) + f(E_3) = 1$$

$$\Rightarrow \quad f(E_1 + E_2) = f(E_1) + f(E_2)$$

Similarly for integers p, q,

$$f(E) = p\, f(\tfrac{1}{p}E) = q\, f(\tfrac{1}{q}E) \;\Rightarrow\; f(\tfrac{p}{q}E) = \tfrac{p}{q} f(E).$$

etc.

Note $\mathcal{P}$ spans the space of operators.
Choose a complete basis $E_i \in \mathcal{P}$, $i = 1, \ldots, d^2$.

$$f(E) = f\Big(\sum_i \alpha_i E_i\Big) = \sum_i \alpha_i f(E_i)$$

Define ρ to satisfy d^2 equations

$$\operatorname{tr} \rho E_i = f(E_i) \leftarrow \text{numbers}$$

and use linearity of trace.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

When two systems, of which we know the states ... enter into temporary physical interaction due to known forces ... then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that "one" but rather "the" characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives have become entangled.

— E. Schrödinger
1935

<u>Why</u> <u>Entanglement</u> ?

$$\widetilde{\mathcal{H}} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Why tensor products ?

Why <u>not</u> direct sum?

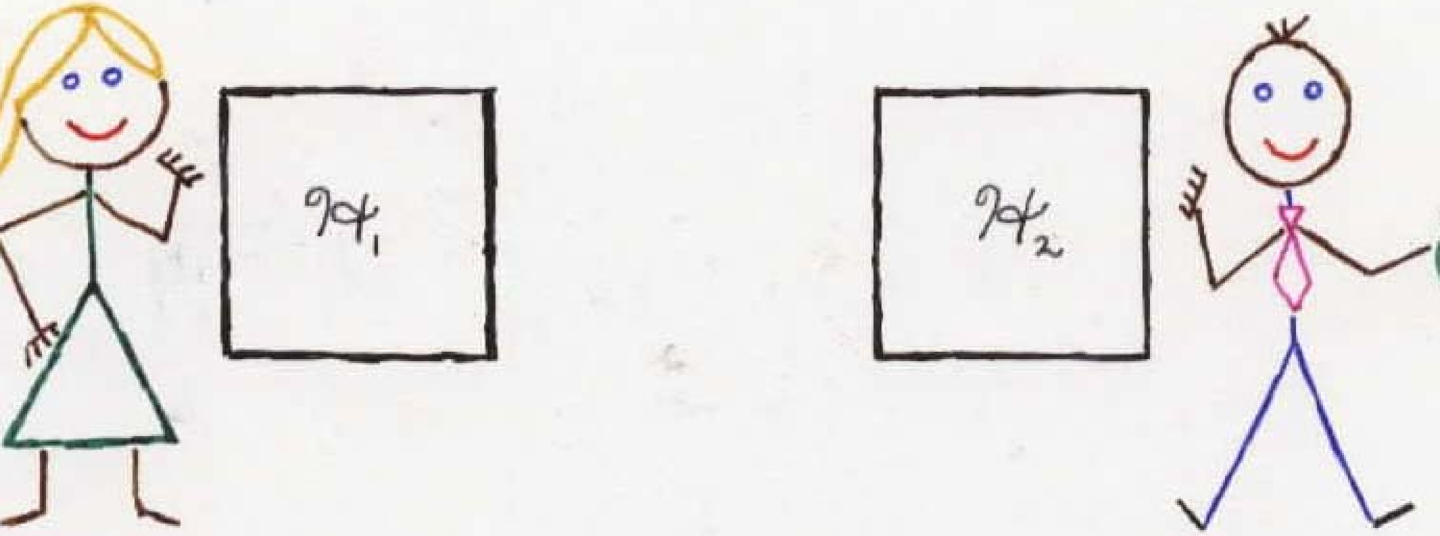
$$\widetilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

Why <u>not</u> Grasmann product ?

$$\widetilde{\mathcal{H}} = \mathcal{H}_1 \wedge \mathcal{H}_2$$

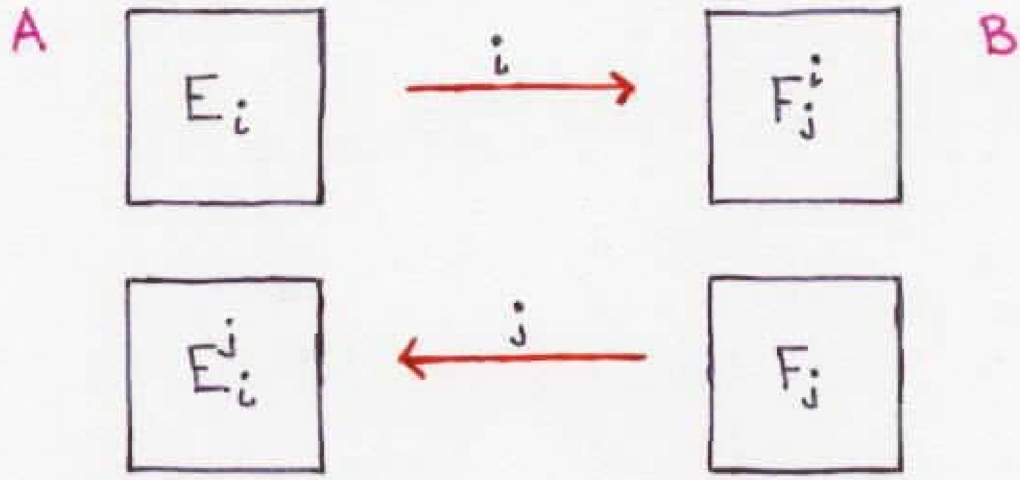
Etc.

Build it from a Gleason-like theorem for local measurements!



Why Tensor Products?

A sequential Gleason theorem



Suppose there exists a function

$f : \mathcal{E} \times \mathcal{E} \longrightarrow [0,1]$ such that

$\sum_{ij} f(S_{ij}) = 1$ whenever <u>either</u>

$S_{ij} = (E_i, F^i_j)$ with $\sum_i E_i = I$, $\sum_j F^i_j = I \;\; \forall i$

<u>or</u>

$S_{ij} = (E^j_i, F_j)$ with $\sum_j F_j = I$, $\sum_i E^j_i = I \;\; \forall j$

$\Rightarrow \; \exists \; \tilde{\rho}$ such that $f(E,F) = \operatorname{tr} \tilde{\rho} \, E \otimes F$,

Proof Idea

Consider all measurements of the form

$$\{(I-E, G_j), (E, H_k)\}$$

for fixed POVMs $\{I-E, E\}$, $\{G_j\}$.

By definition,

$$\sum_j f(I-E, G_j) + \sum_k f(E, H_k) = 1$$

$$\Rightarrow \quad \sum_k f(E, H_k) = \text{const.}$$

$\Rightarrow$ $g_E(F) = f(E,F)$ extendible to a linear function in F

$\Rightarrow$ $f(E,F)$ bilinear

Open Question

$\exists$ linear operator $\tilde{\rho} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ such that

$$f(E,F) = \mathrm{tr}[\tilde{\rho}(E \otimes F)].$$

But enforcing $f(E,F) \geq 0$ does <u>not</u> enforce that $\tilde{\rho}$ is positive semi-definite.

For instance define

$$\tilde{\rho}(E \otimes F) = \frac{1}{d^2} F \otimes E.$$

<u>Give</u> natural condition to enforce $\tilde{\rho} \geq 0$.

State Change

When measure POVM $\{E_b\}$ efficiently

$$\rho \xrightarrow{\;b\;} \tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^\dagger$$

where

$$p_b = \operatorname{tr} \rho E_b \quad \text{and} \quad A_b^\dagger A_b = E_b \,.$$

By polar decomposition theorem

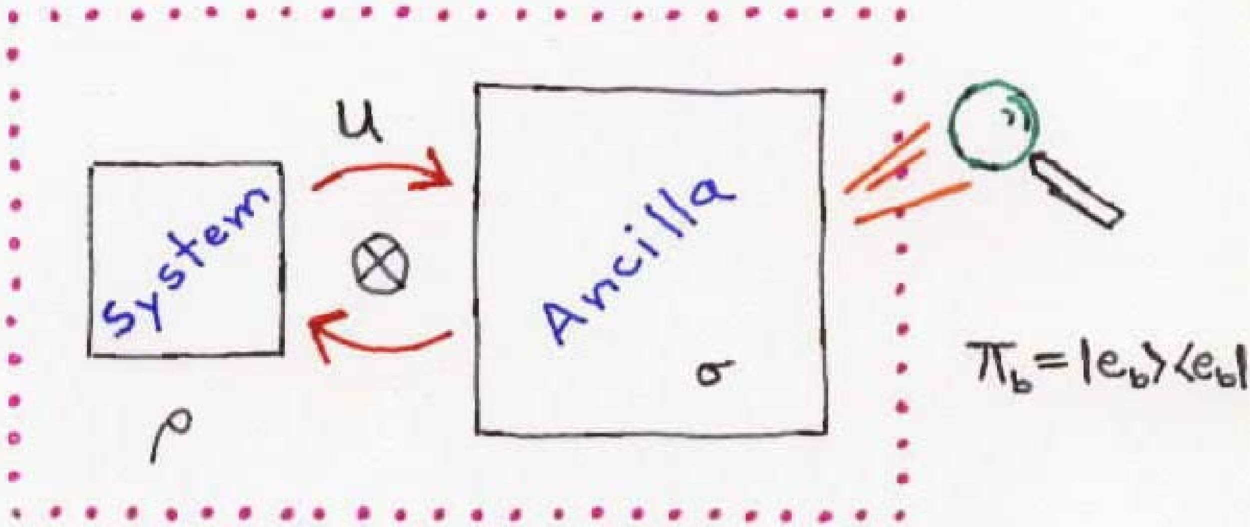
$$\tilde{\rho}_b = \frac{1}{p_b} U_b E_b^{1/2} \rho E_b^{1/2} U_b^\dagger$$

"feedback"

"collapse"

U_b depends upon detailed form of measurement interaction

Usual Justification



1) $\rho \otimes \sigma \xrightarrow{\text{couple}} U(\rho \otimes \sigma)U^{+}$

2) $\xrightarrow{\text{collapse}} (I \otimes \Pi_b)\,U(\rho \otimes \sigma)U^{+}(I \otimes \Pi_b)$

3) $\xrightarrow{\text{discard}} \mathrm{tr}_A[\cdots] \xrightarrow{\text{renormalize}}$

Upshot

$$\rho \xrightarrow{b} \tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^{+}$$

Collapse

$\Updownarrow$?

Bayesian Conditionalization

Bayes' Rule

$$p(H) = \sum_D p(H, D)$$

$$= \sum_D p(D)\, p(H|D)$$

$$p(H) \xrightarrow{D} p(H|D) = \frac{p(H)\,p(D|H)}{p(D)}$$

Ignorance decreases on average:

$$S(H) \geq S(H|D)$$

by concavity of Shannon entropy.

Note

Unlike with Bayes rule

$$\rho \neq \sum_b p_b \tilde{\rho}_b$$

post-mmt. states

pre-measurement state

Moreover, not even if we delete the "feedback"

$$\tilde{\rho}_b = \frac{1}{p_b} \cancel{U_b} E_b^{1/2} \rho E_b^{1/2} \cancel{U_b^\dagger}$$

Back to Quantum Collapse

In what sense do we gain information in a quantum measurement?

Information about what?

Consider any concave function $f(\rho)$.
unitarily invarient

Examples:

1) vN entropy $S(\rho) = -\mathrm{tr}\, \rho \log \rho$
(best mmt)

$$= -\sum_i \lambda_i \log \lambda_i$$

2) Wootters measure
(random mmt)

$$Q(\rho) = \left(\tfrac{1}{2} + \cdots + \tfrac{1}{d}\right) - \sum_i \prod_{k \neq i} \left(\frac{\lambda_i}{\lambda_i - \lambda_k}\right) \lambda_i \log \lambda_i$$

Assertion:

$$f(\rho) \geq \sum_b p_b f(\tilde{\rho}_b)$$

Proof:

With any POVM $\{E_b\}$, associate a canonical decomposition of ρ:

$$I = \sum_b E_b \implies \rho = \sum_b \rho^{1/2} E_b \rho^{1/2}$$
$$= \sum_b p_b \rho_b$$

where

$$\rho_b = \frac{1}{p_b} \rho^{1/2} E_b \rho^{1/2} .$$

Concavity implies $f(\rho) \geq \sum_b p_b f(\rho_b)$.

However

$$\tilde{\rho}_b = \frac{1}{p_b} A_b \rho A_b^\dagger \neq \frac{1}{p_b} \rho^{1/2} A_b^\dagger A_b \rho^{1/2} = \rho_b .$$

But!

$$\vec{\lambda}(\tilde{\rho}_b) = \vec{\lambda}(\rho_b)$$

identical spectra

$B^\dagger B$ and $BB^\dagger$ have same eigenvalues

So $f(\rho_b) = f(\tilde{\rho}_b)$.

Bohring Bayesians?

Given POVM $\{E_b\}$, write

$$\rho = \sum_b p_b \rho_b \quad \text{where} \quad \rho_b = \frac{1}{p_b}\rho^{1/2} E_b \rho^{1/2}.$$

Might as well say state changes according to a kind of conditionalizing

$$\rho \xrightarrow{\;b\;} \tilde{\rho}_b = V_b \rho_b V_b^{\dagger}$$

"learning"

"mental readjust"

V_b depends upon measurement interaction and initial state of knowledge ρ.

Emphasis

Classical

$$
\begin{tikzcd}
p(H) = \sum_D p(D)\, p(H|D) \arrow[d] \\
p(H) \xrightarrow{D} p(H|D)
\end{tikzcd}
$$

$$p(H) = \sum_D p(D)\, p(H|D)$$

$$p(H) \xrightarrow{\;D\;} p(H|D)$$

Quantum

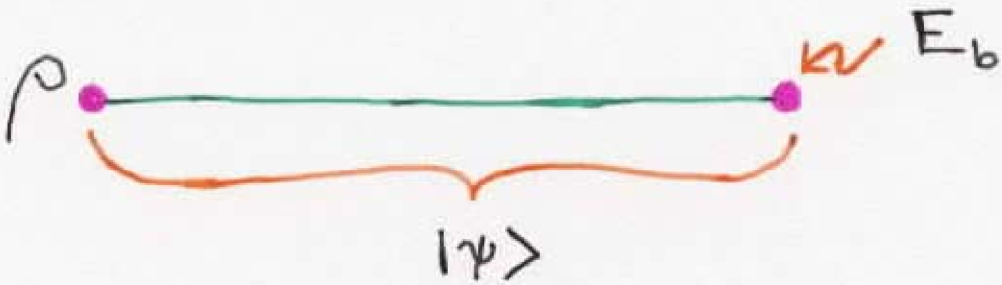
$$\rho = \sum_b p_b\, \rho_b$$

$$\rho \xrightarrow{\;b\;} \rho_b$$

modulo a further unitary readjustment

Limiting Cases

1) When no "back action" possible, measurement should lead to pure Bayesian conditionalization.



$$\rho \longrightarrow \rho^{1/2} E_b^{T} \rho^{1/2} \quad ✓$$

2) When $\rho = |\psi\rangle\langle\psi|$, nothing left to learn.

$$\rho \longrightarrow \rho^{1/2} E_b \rho^{1/2} \propto \rho \longrightarrow V_b \rho V_b^{\dagger}$$

My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this:

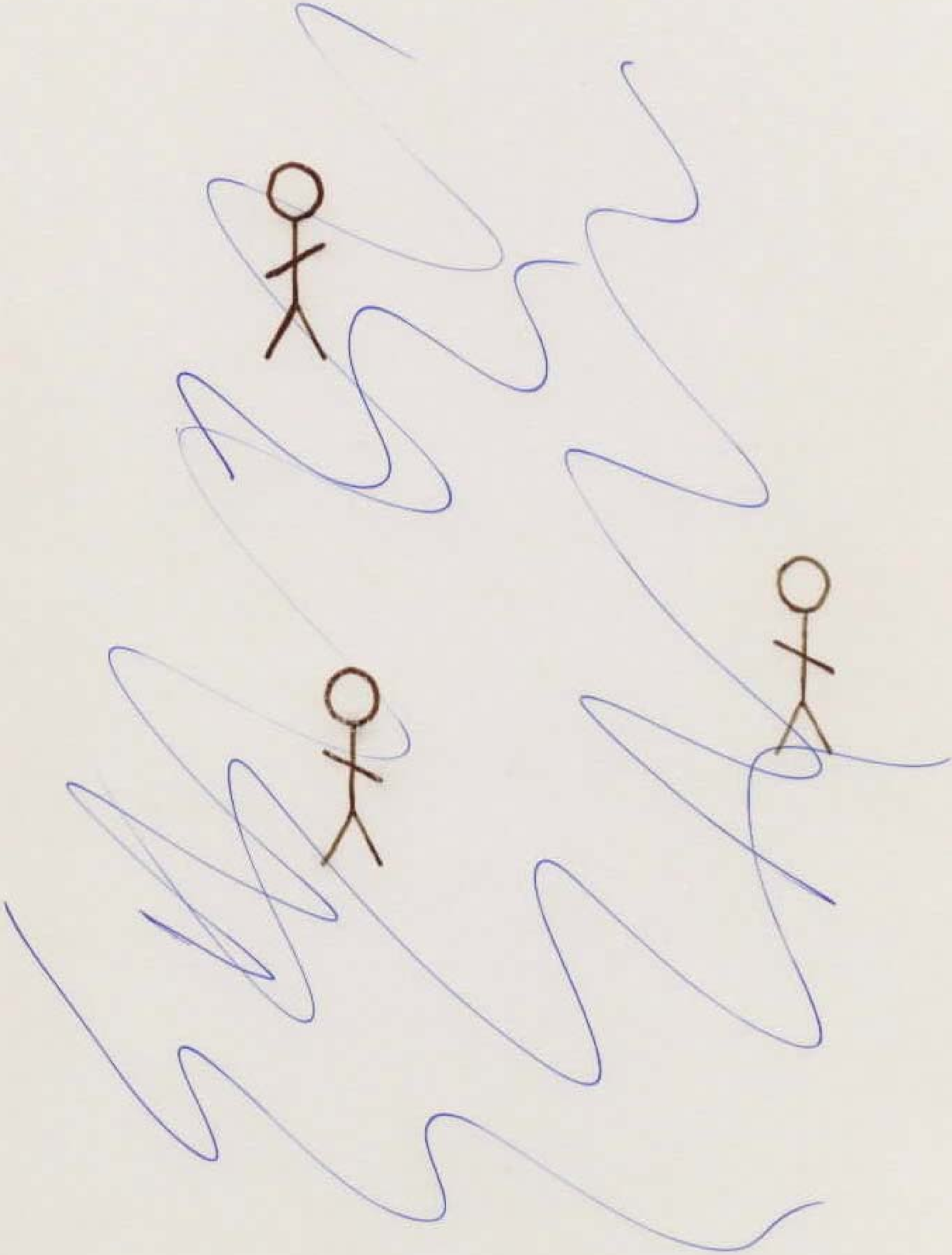
QUANTUM STATES DO NOT EXIST

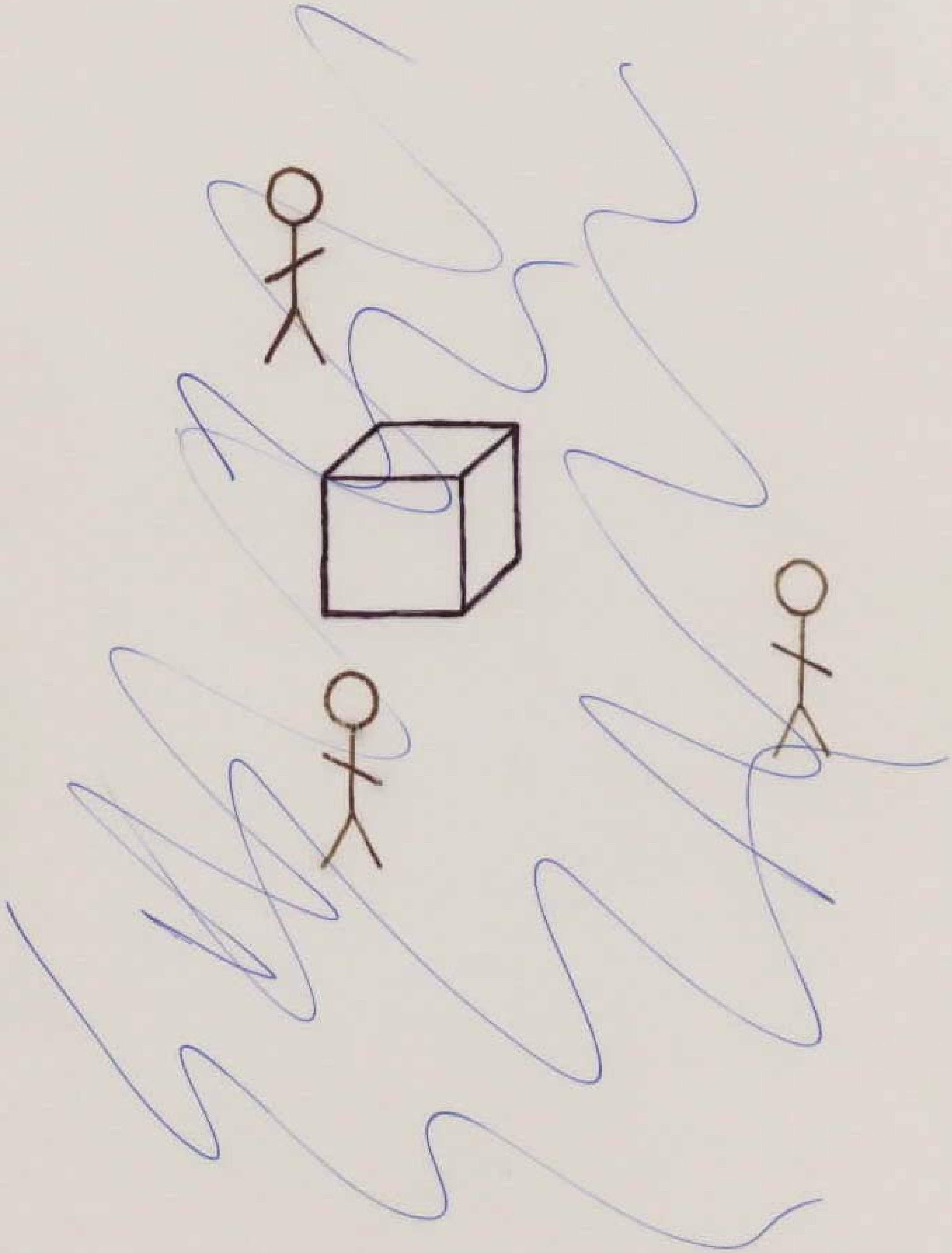
The abandonment of superstitious beliefs about the existence of phlogiston, the cosmic ether, absolute space and time, or fairies and witches, was an essential step along the road to scientific thinking. The quantum state, too, if regarded as something endowed with some kind of objective existence, is no less a misleading conception, an illusory attempt to exteriorize or materialize the information we possess.

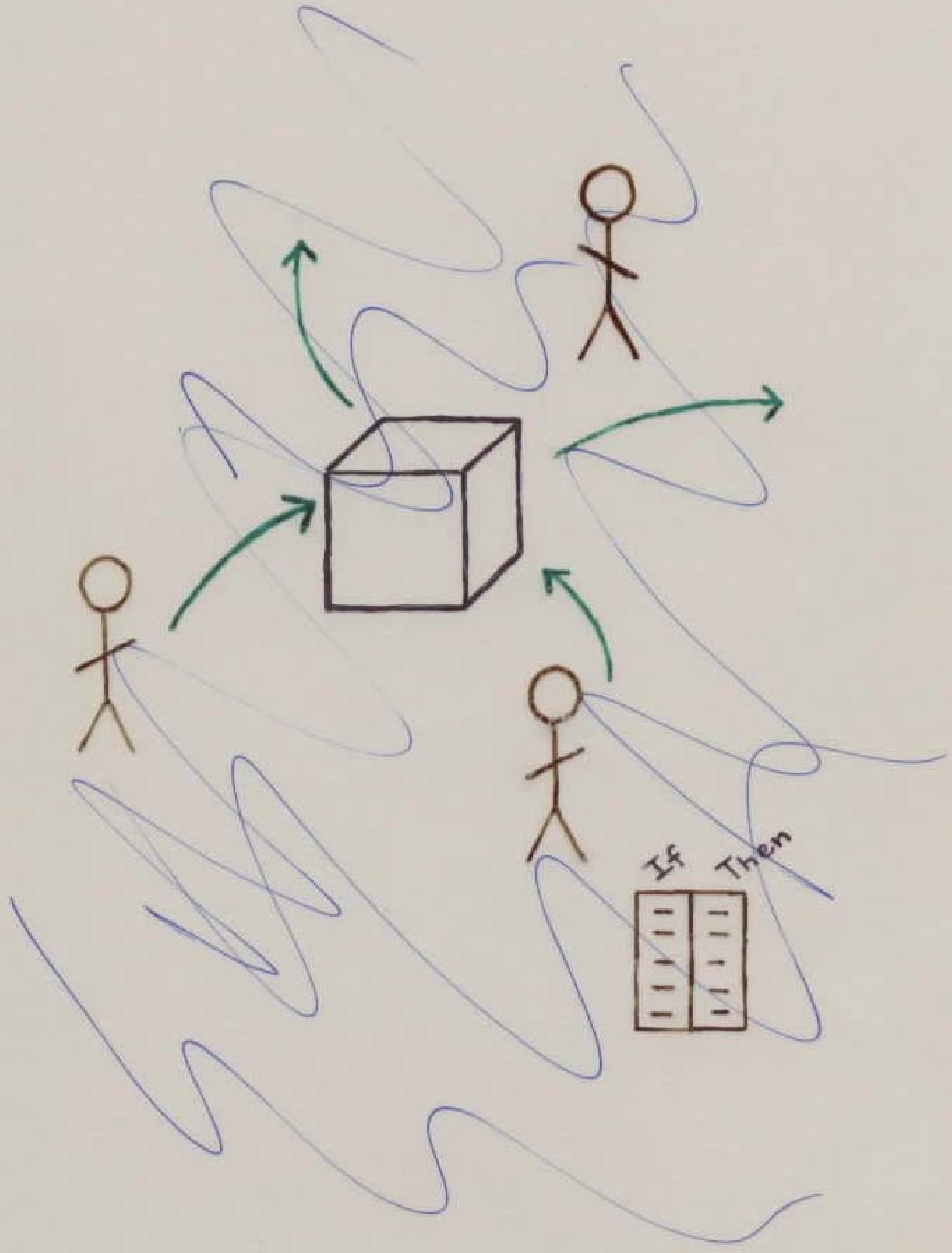
— the ghost of Bruno de Finetti

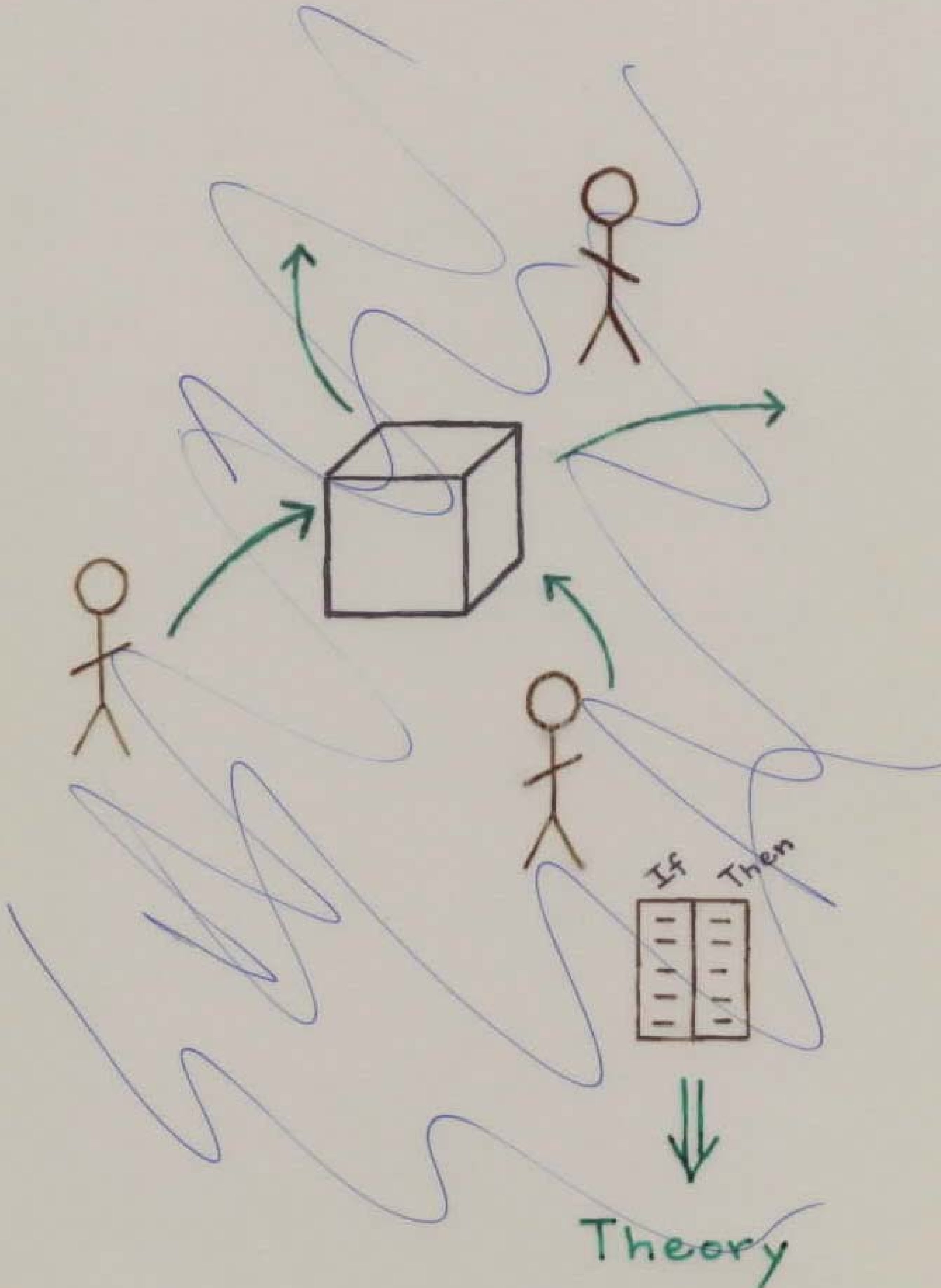
The History of the World

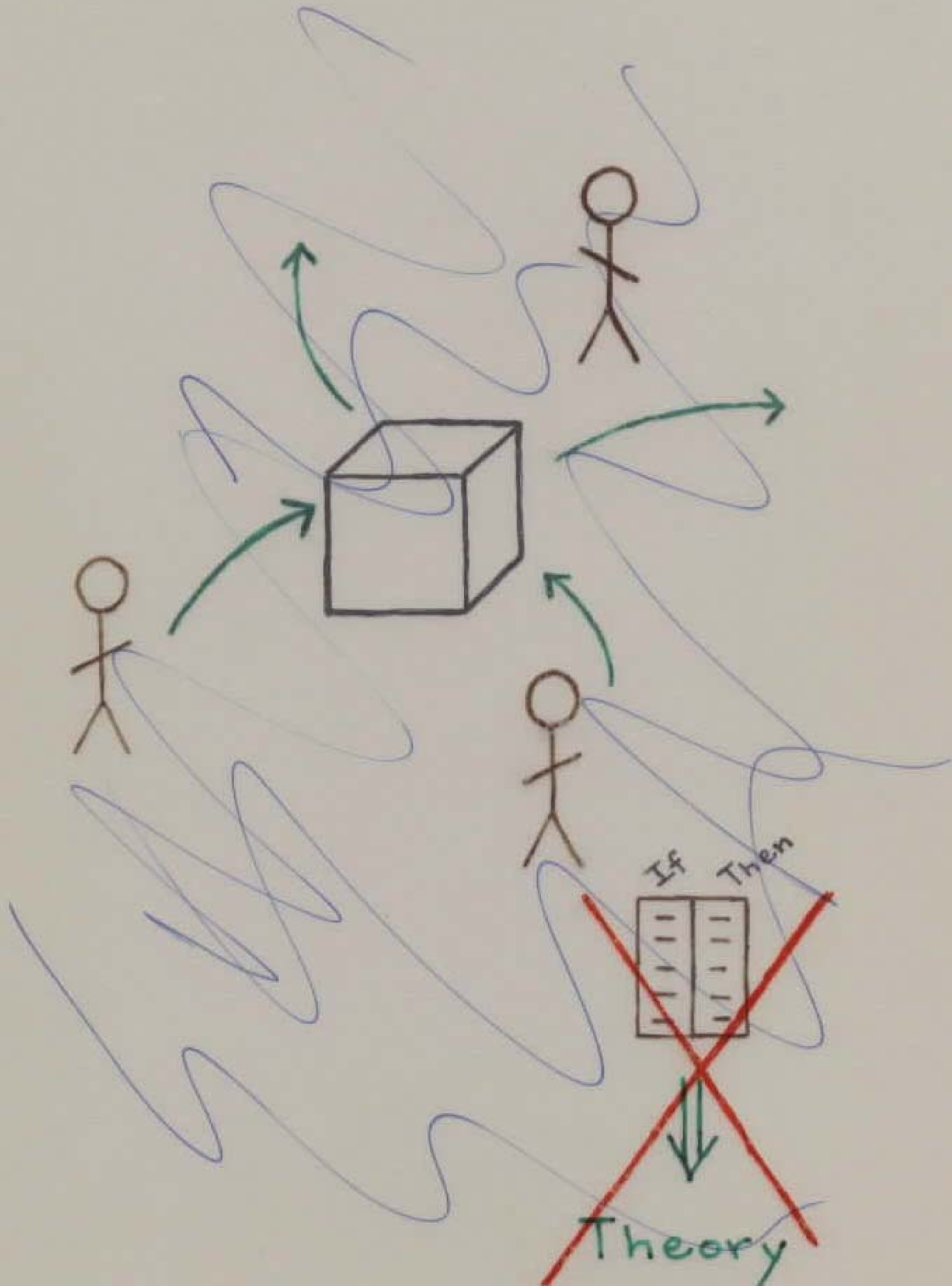
(five parts)






If
Then


If
Then
Theory


If
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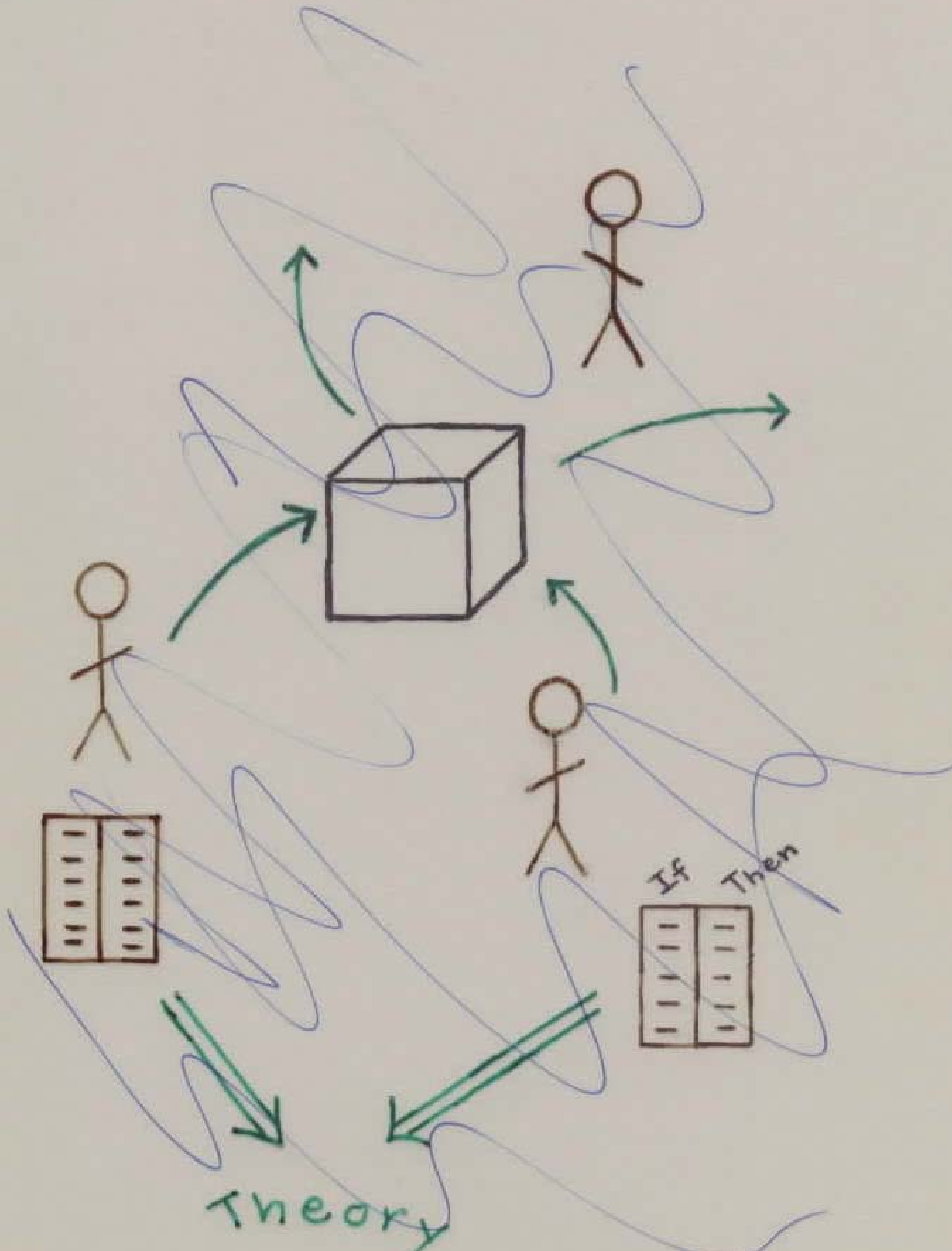
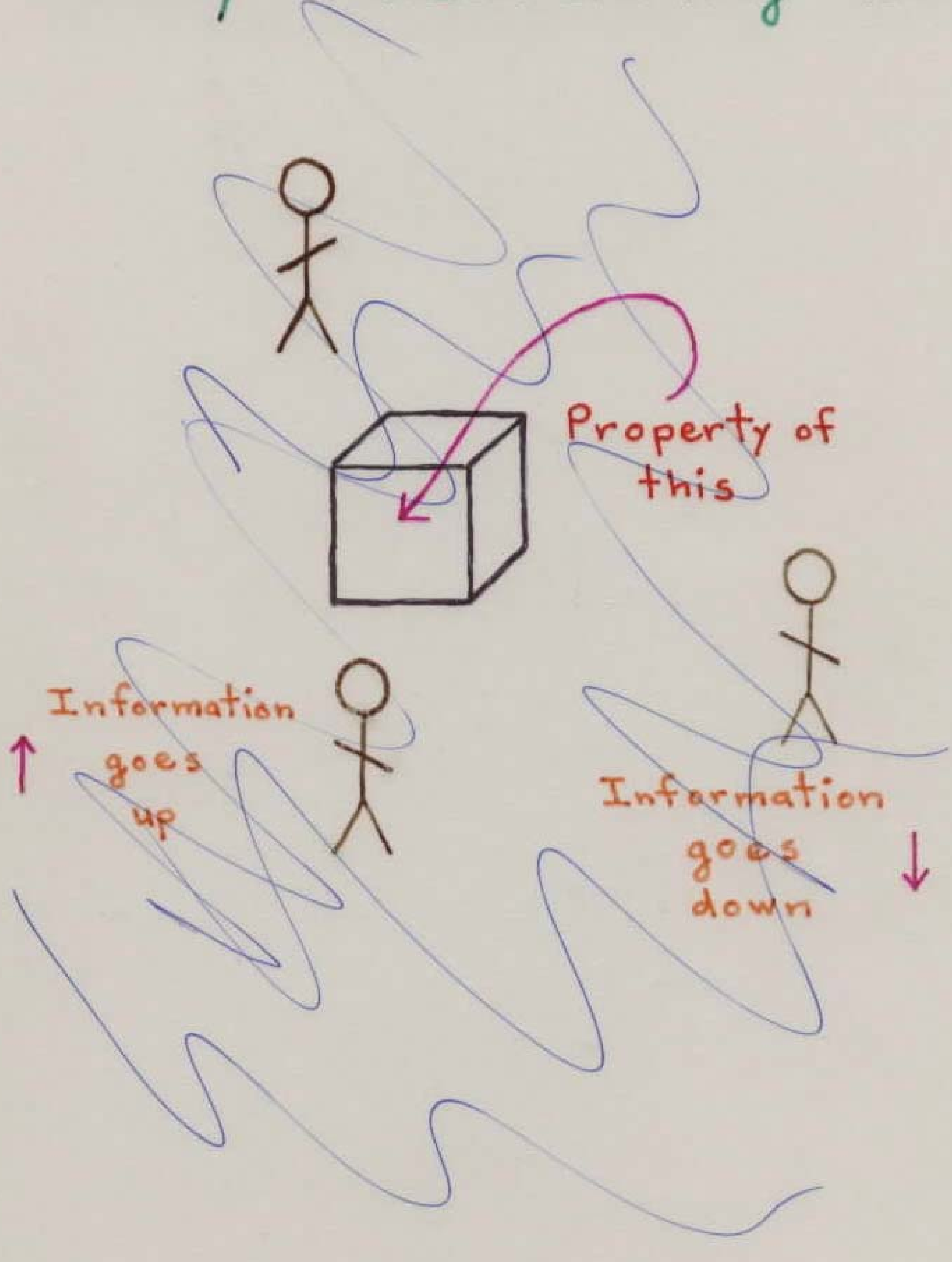


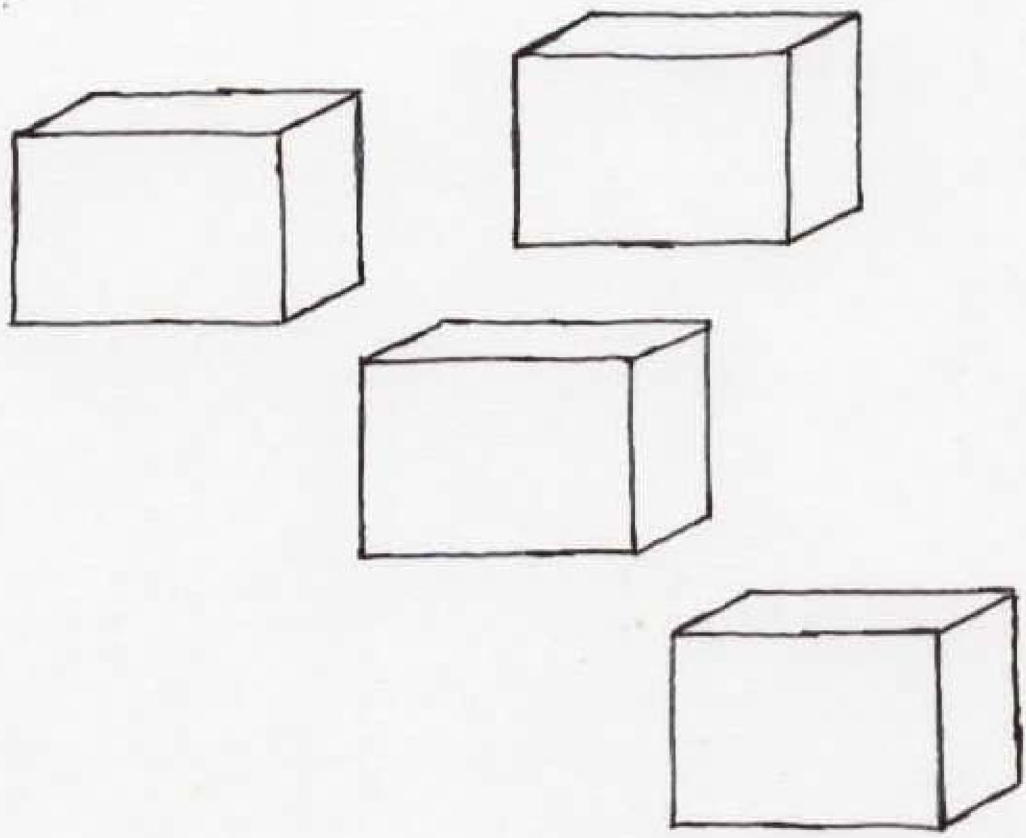
If
Then
Theory

Slowly realized starting ~1900



Research Direction

Quantum theory represents the best agreement we can come to in light of this fundamental situation.



Irritable

Bricks

What does energy look like?

Philosophy

Just as energy, Zing! has no good picture.

Understanding comes from applications!

- Quantum Computing
- Quantum Cryptography
- Teleportation
- Superdense Coding
- Error Correction for Q. Computing
- Entanglement-enhanced classical communication
- Atomic Frequency Standards Control

Our whole problem is to
make the mistakes as
fast as possible!

— John Archibald Wheeler