Analysis of the Superoperator Obtained by Process Tomography

of the Quantum Fourier Transform in a Liquid-State NMR Experiment

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Superoperators and Quantum Process Tomography

A general transformation is given by a completely positive linear map, or **"superoperator":**

Performing state tomography on the output states for a complete set of input states completely specifies the superoperator for the transformation:

$$\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}\begin{bmatrix} . & R^{in} & . \\ . & . & . \\ . & . & . \end{bmatrix} = \begin{bmatrix} . & R^{out} & . \\ . & . & \\ . & . & \end{bmatrix}$$

where each column of $\boldsymbol{R^{in}}$ and $\boldsymbol{R^{out}}$ is a vector of length N^2 obtained by stacking the columns of the associated N x N density matrix, such that,

N^2 X N^2 Supermatrix

$$\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}\begin{bmatrix} \\ \rho_{initial} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \rho_{final} \\ \\ \end{bmatrix}$$

Density matrices as N^2 x 1 vectors

Quantum Fourier Transform

$$U_{QFT}|y\rangle = \frac{1}{\sqrt{N}} \sum_x \exp(i2\pi xy/N)|x\rangle$$

The quantum Fourier transform is implemented via a sequence of one and two qubit quantum gates.

For 3 qubits the gate-sequence is:

$$QFT_8 = Swap_{1,3} H_3 B_{2,3} B_{1,3} H_2 B_{1,2} H_1$$

$\boldsymbol{H_j}$ is the one-qubit Hadamard gate on qubit j.

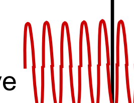
$\boldsymbol{B_{jk}}$ is the two-qubit conditional phase gate. It applies a z-phase to qubit j only if qubit k is one.

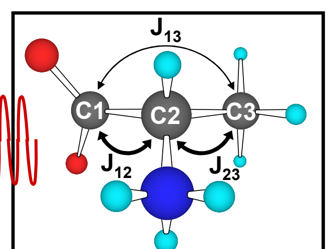
The QFT is a fundamental component of all practical algorithms that potentially offer exponential speed-ups, ie. Shor's algorithm and quantum simulation.

NMR Hamiltonian for Liquid Solution of Alanine

$$H_{total}(t) = H_{int} + H_{ext}(t)$$

RF Wave





Alanine

$$H_{int} = \overbrace{\omega_1 I^1_z + \omega_2 I^2_z + \omega_3 I^3_z}^{\text{static B field}}$$

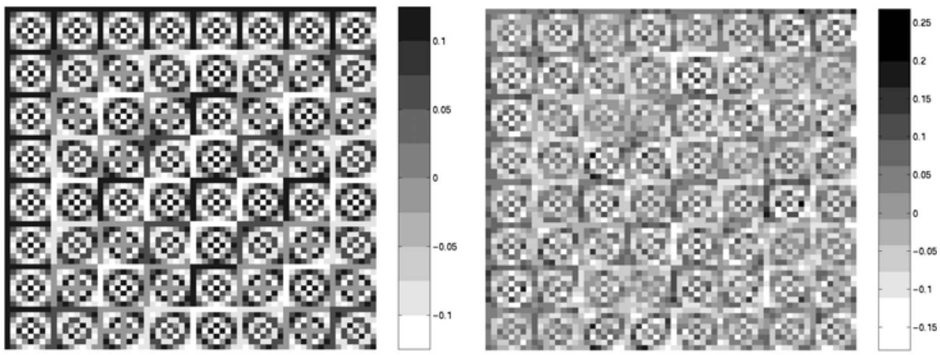
$$+ \underbrace{2\pi J_{12} I^1_z I^2_z + 2\pi J_{13} I^1_z I^3_z + 2\pi J_{23} I^2_z I^3_z}_{\text{spin-spin coupling in high-field approximation}}$$

$$H_{ext}(t) = \underbrace{\omega_{RFx}(t)\cdot(I^1_x + I^2_x + I^3_x) + \omega_{RFy}(t)\cdot(I^1_y + I^2_y + I^3_x)}_{\text{time-dependent control from applied RF field}}$$

Dominant decoherence source: residual inter-molecular dipolar interactions.

Measured QFT Supermatrix

$S_{QFT} = U^*_{QFT} \otimes U_{QFT}$ $S_{\exp}$

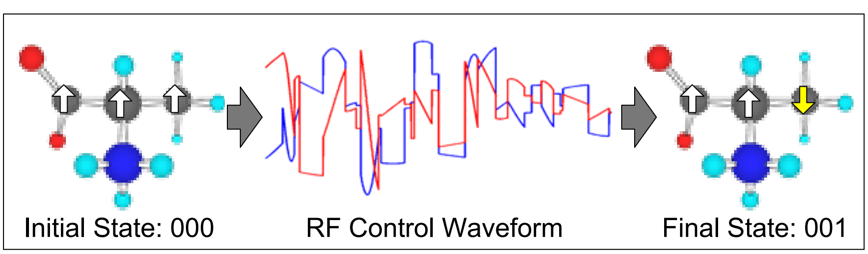


Plotted is the real part of supermatrices in the computational basis.

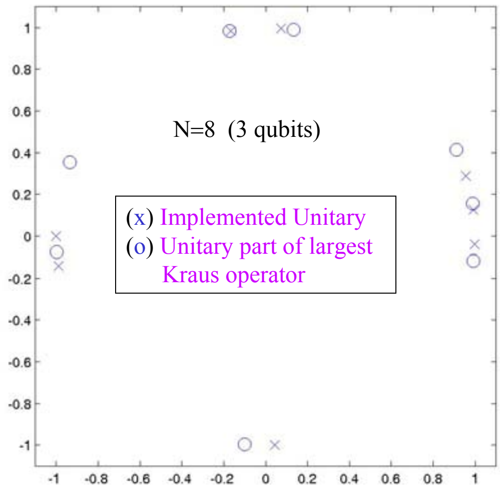
How can we determine the relative importance of different error and decoherence sources (and what can we do about them)?

Implemented Unitary is an Approximation to the Exact QFT

Time-domain resolution limitations in the RF control will produce a unitary which only approximates the exact (desired) gate:



Eigenvalues of Implemented Unitary



Note:
the exact QFT has four degenerate eigenvalues: (1,1,1,i,i,-1,-1,-i)

The implemented unitary is no longer degenerate due to the cumulative unitary errors in the gate sequence.

Kraus Decomposition

Given the supermatrix we can construct a canonical Kraus sum from the eigenvectors of the "Choi" matrix *(see T. Havel, J. Math Phys, 2002)* :

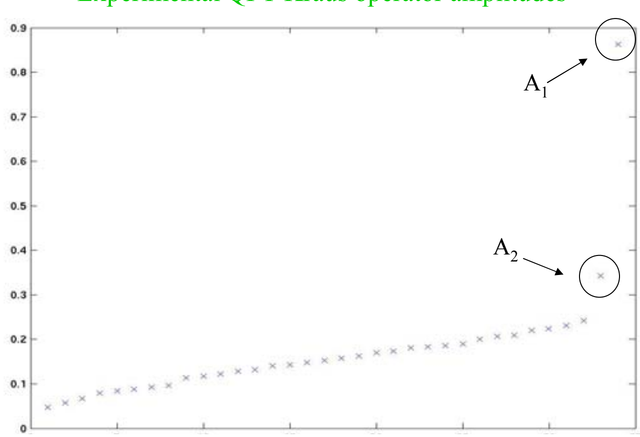
Kraus Decomposition:

$$\rho_{out} = \sum_k A_k \rho_{in} A_k^+$$

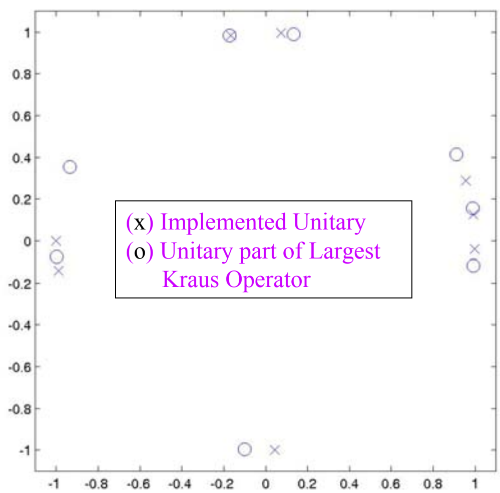
Trace-preserving condition:

$$1 = \sum_k A_k^+ A_k$$

Experimental QFT Kraus operator amplitudes

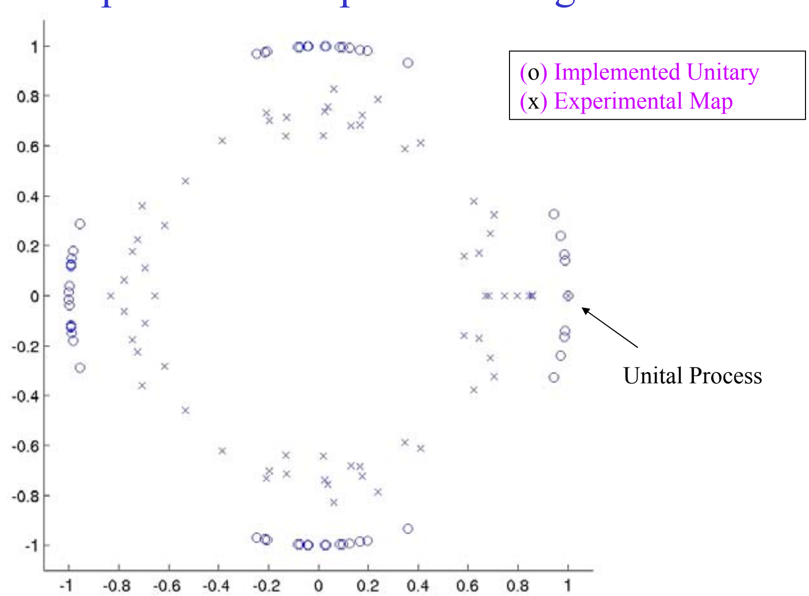


Information from the Largest Kraus Operator



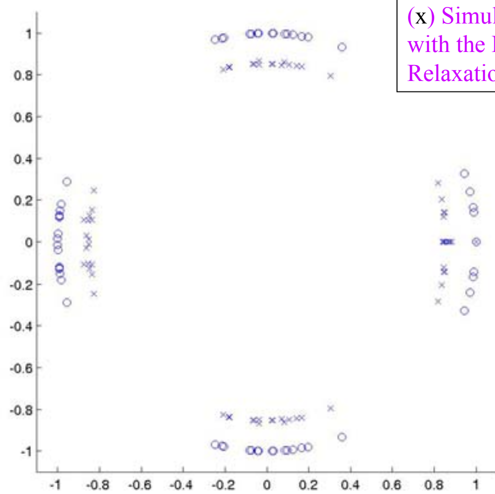
Obviously this cumulative unitary error, once identified, can be removed by additional pulses… though we do not learn much about our sources of error from this process.

Experimental Supermatrix Eigenvalues



Numerical Simulation of the Experiment with the Measured "Natural" Relaxation Superoperator

(o) Implemented Unitary
(x) Simulation of Experiment with the Measured "Natural" Relaxation Superoperator



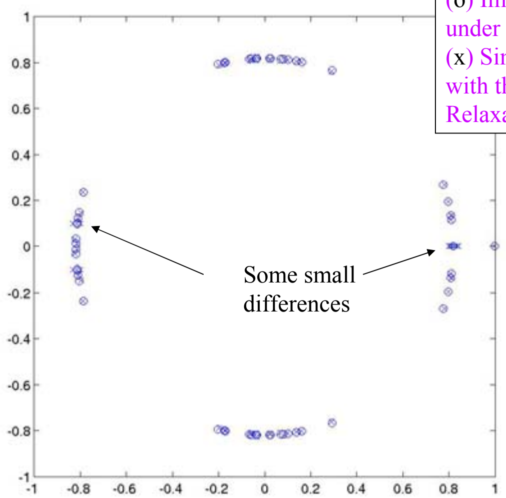
Uniform Eigenvalue Attenuation under the Depolarizing Channel

$$\mathcal{E}_{dep}(\rho) = (\mathbf{p}/\mathbf{N})\ \mathrm{I} + (1-\mathbf{p})\ \rho$$

The superoperator S_{dep} for this process has the **N** eigenvalues **(1,a,a,…,a),** where eigenvalue **1** is for the identity eigenvector, and **a = 1-p** is an attenuation constant.

S_{dep} is thus diagonal is the eigenbasis of any trace-preserving, unital transformation, and uniformly attenuates its N-1 non-identity eigenvalues by the factor **1-p**.

The QFT Sequence “Evens-Out” the Non-Uniform Natural System Decoherence



(o) Implemented Unitary under Depolarizing Channel
(x) Simulation of Experiment with the Measured “Natural” Relaxation Superoperator

The relaxation super-operator under the QFT may/should be very different than for the internal Hamiltonian.

More importantly, the in-homogeneity of the RF over the sample introduces “incoherence” effects…

Signatures of Incoherence

$$S_{\text{inc}} = \int d\omega\, p(\omega)\, U^{*}(\omega) \otimes U(\omega)$$

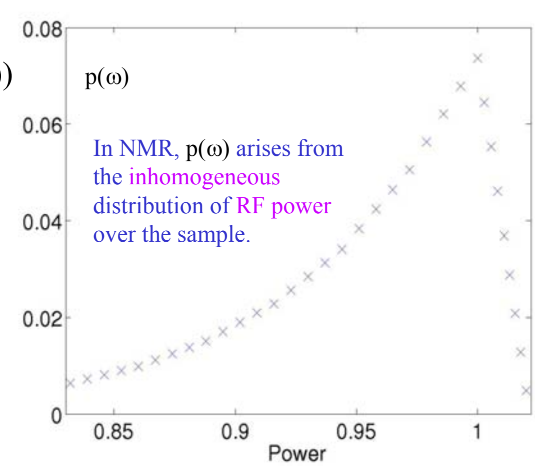
Not environment-induced decoherence, but “ignorance-induced” decoherence.

where,

$$U(\omega) = U_{QFT} \exp(iK(\omega))$$

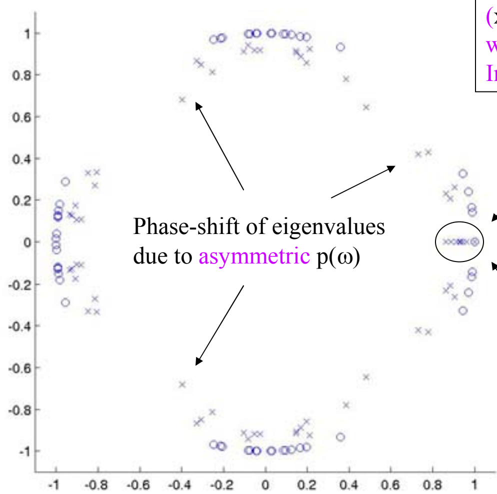
is unitary.

Some basic features of the eigenvalues of S_{inc} are determined from generic properties of $p(\omega)$.



Numerical Simulation of Experiment with the Measured RF Inhomogeneity Distribution

(o) Implemented Unitary
(x) Simulation of Sequence with the Measured RF Inhomogeneity Distribution



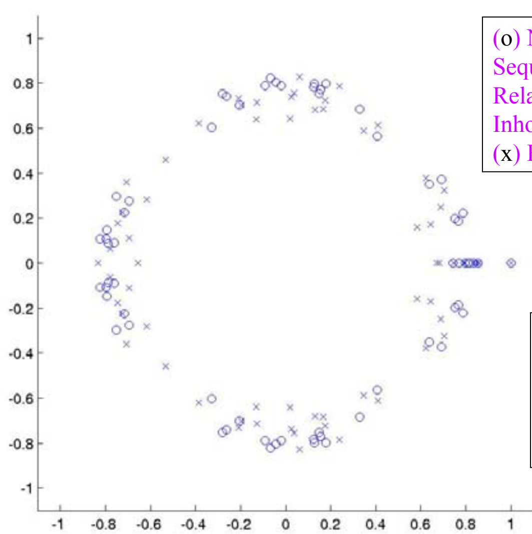
Eigenvalues along real axis are from the degenerate unperturbed eigenvalues.

Spreading determined by properties of a doubly-stochastic matrix .

Under strong but physical assumptions the spectral gap can be related to the width of the inhomogeneity

Numerical Simulation with RF Inhomogeneity and the Natural Relaxation Superoperator

(o) Numerical Simulation of the Sequence with the Natural Relaxation Superoperator and RF Inhomogeneity
(x) Experimental Map



Differences give some indication of the dependence of the relaxation super-operator on the applied transformation.

Conclusions and Future Work

- From the largest operator in the Kraus decomposition we can identify a unitary “close” to the target unitary. Is this the closest unitary? Is there information in the smaller operators?

- The supermatrix eigenvalues exhibit distinctive signatures for different types of decoherence: models and perturbation theory provide estimates of the “strength” of different noise sources.

- Do other maps mix the noise generators as uniformly as the QFT? Explore relation between cumulative error and underlying error model? Try regular vs chaotic/random unitary maps…

- As the system increases in size we need to develop algorithms and statistical methods to efficiently estimate the few scalar quantities of most interest. Can the universal statistics of random maps help?