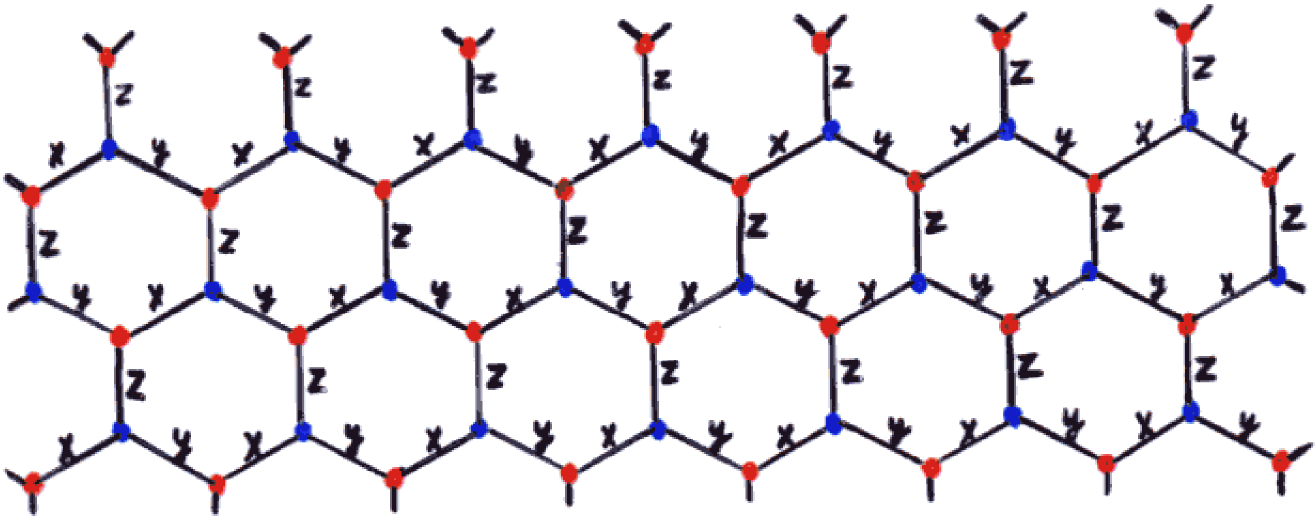
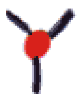
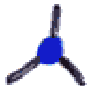
Anyons in a hexagonal lattice model

1. The model



Spin $\frac{1}{2}$ on each site.

Two sublattices:  and 

Three types of links: 

Spin-spin interactions: $K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x & \text{for } x\text{-links} \\ \sigma_j^y \sigma_k^y & \text{for } y\text{-links} \\ \sigma_j^z \sigma_k^z & \text{for } z\text{-links} \end{cases}$

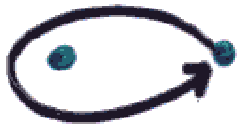
$$\boxed{H_0 = -J_x \sum_{x\text{-links}} K_{jk} - J_y \sum_{y\text{-links}} K_{jk} - J_z \sum_{z\text{-links}} K_{jk}}$$

Anyons

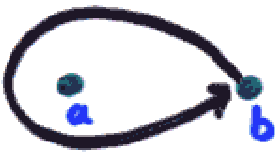
Bosons: 

$|\psi\rangle \mapsto |\psi\rangle$

Fermions: $|\psi\rangle \mapsto -|\psi\rangle$

$\Bigg\}$  $|\psi\rangle \mapsto |\psi\rangle$

Abelian anyons (in FQHE systems)

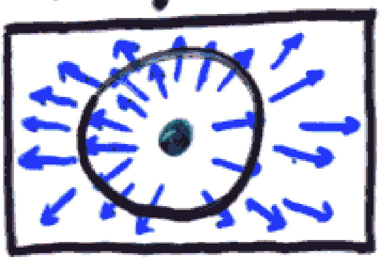


$$|\psi\rangle \mapsto e^{i\varphi_{ab}}|\psi\rangle$$

Laughlin quasiparticles: $\varphi_{++} = \frac{2\pi}{3}$

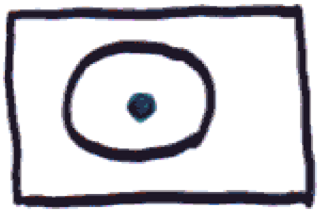
$\varphi_{+-} = -\frac{2\pi}{3}$

Anyons $\approx$ quantum vortices (but no order parameter)



Classical vortex

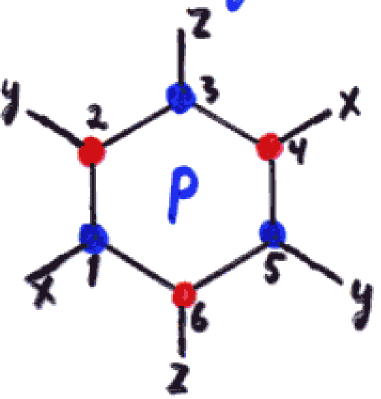
Local measurements + integration



Anyon

The presence can be detected by joint measurement

2. Integrals of motion



$$\boxed{B = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z}$$

A separate operator B_p for each face p

$[B_p, K_{jk}] = 0$, hence $\boxed{[B_p, H_0] = 0}$

individual term in the Hamiltonian

the Hamiltonian

$[B_p, B_s] = 0$

$B_p^2 = 1$, $B_p^\dagger = B_p$ $\Rightarrow$ the eigenvalues $= \pm 1$

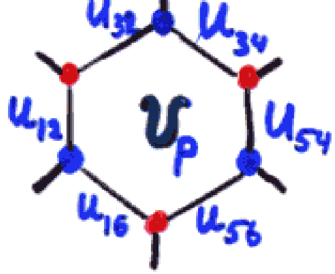
Hilbert space decomposition:

$$\boxed{\mathcal{H} = \bigoplus_{v_1,\ldots,v_M} \mathcal{H}_{v_1,\ldots,v_M} \quad (v_p = \pm 1)}$$

"gauge sectors"

$$\boxed{|\psi\rangle \in \mathcal{H}_{v_1,\ldots,v_M} \iff B_p|\psi\rangle = v_p|\psi\rangle}$$

$v_p = +1$ – non-vortex $\qquad$ $v_p = -1$ – vortex



$$v_p = \prod_{p} u_{jk} \qquad u_{jk} = \pm 1$$

($\mathbb{Z}_2$ gauge field)

Counting degrees of freedom.

N spins $\longleftrightarrow$ $N/2$ hexagons

$$\frac{2^N}{2^{N/2}} = 2^{N/2} \text{ states}$$

$\uparrow$ fixing v_p

($\sqrt{2}$ states per vertex)

$\uparrow$ real

(Majorana) fermions.

In fact, each sector can be described by free (real) fermions

$c_j^+ = c_j$	$c_j c_k + c_k c_j = 2\,\delta_{jk}$

Ordinary fermions:

$$a_m = c_{2m-1} + i\,c_{2m}$$
$$a_m^+ = c_{2m-1} - i\,c_{2m}$$

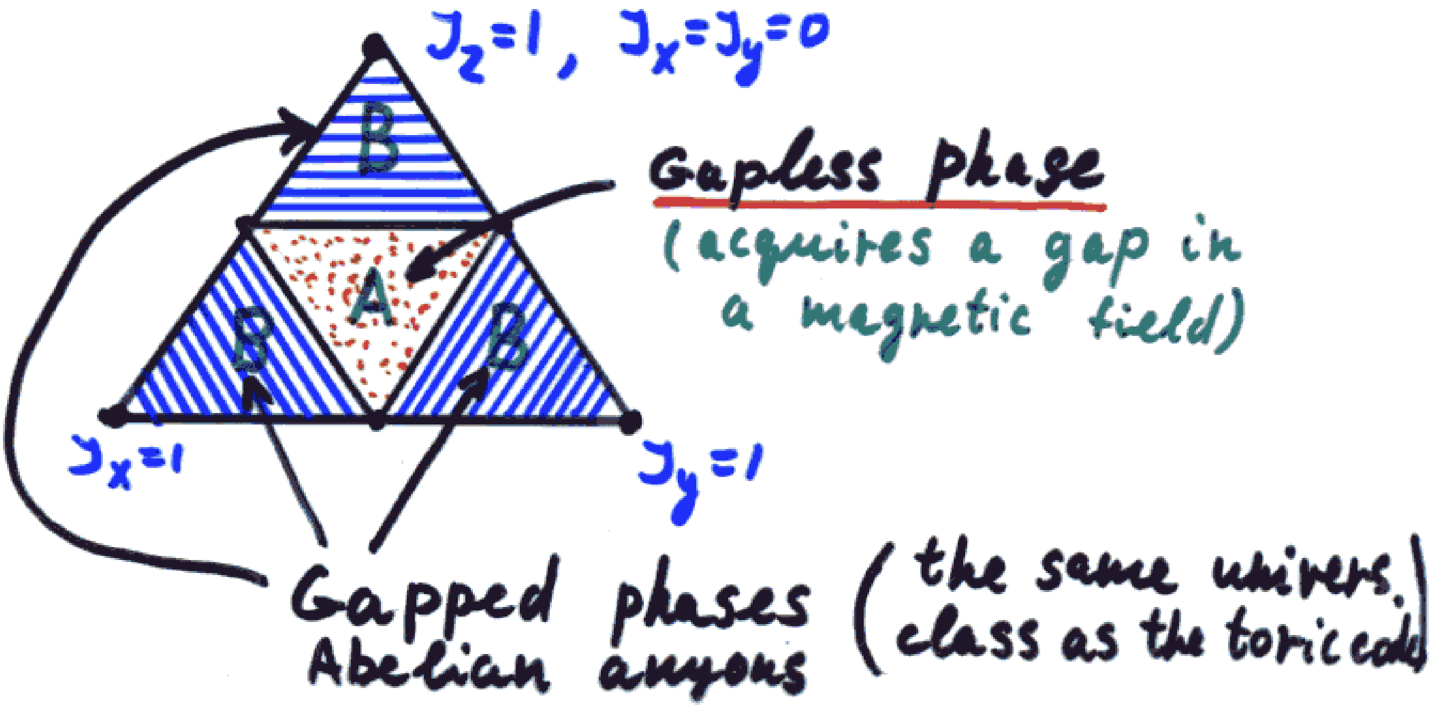
$$a_m a_n^+ + a_n^+ a_m = \delta_{mn}$$
$$a_m a_n + a_n a_m = 0$$

For each "vortex configuration" $(v_1, \ldots, v_{N/2})$ one can compute the energy of the fermionic vacuum

$$E(v_1, \ldots, v_{N/2})$$

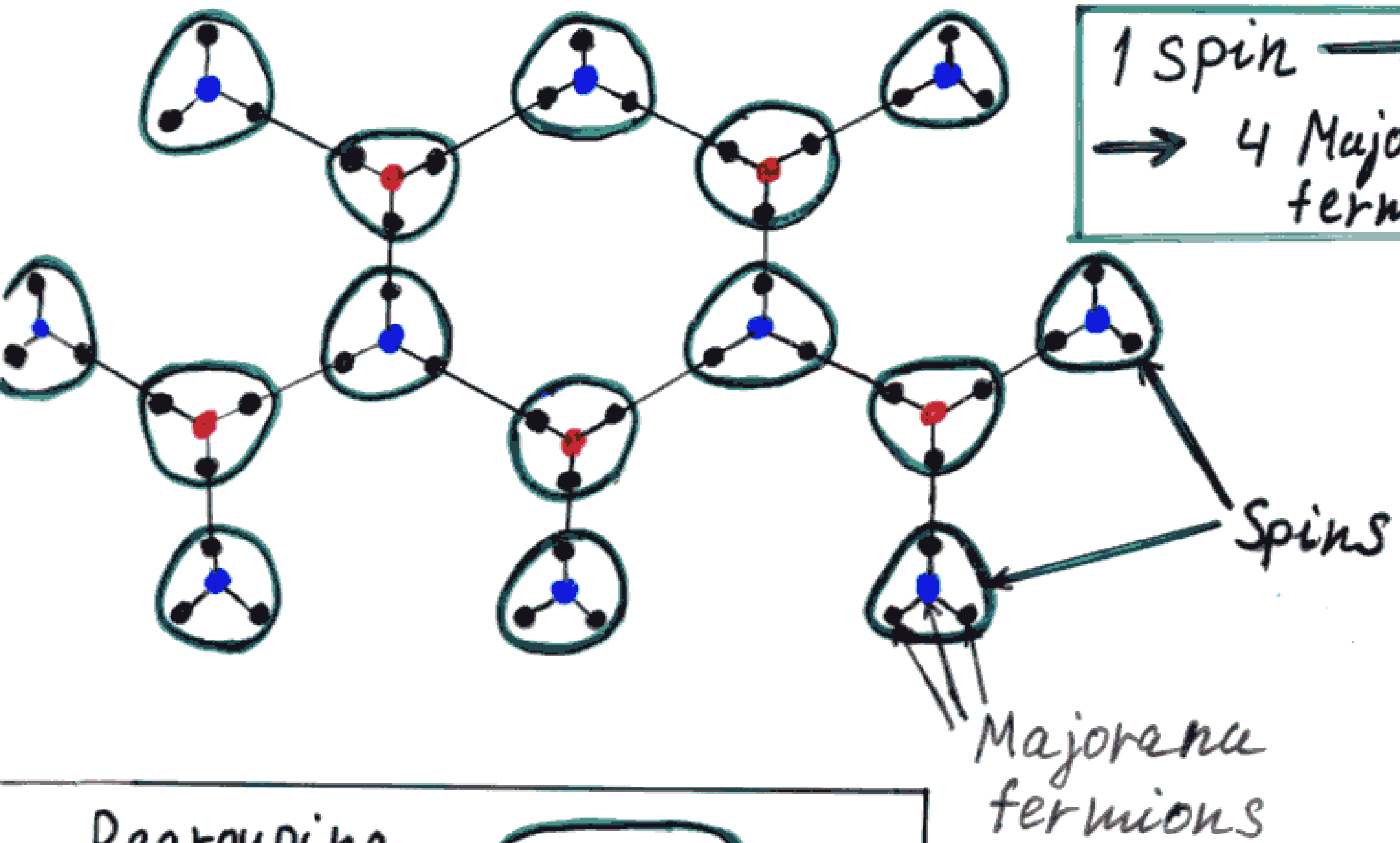
Numerical fact : $E(v_1, \ldots, v_{N/2})$ is minimal if $v_j = +1$ (no vortices) for all $J_x, J_y, J_z > 0$.

Phase diagram

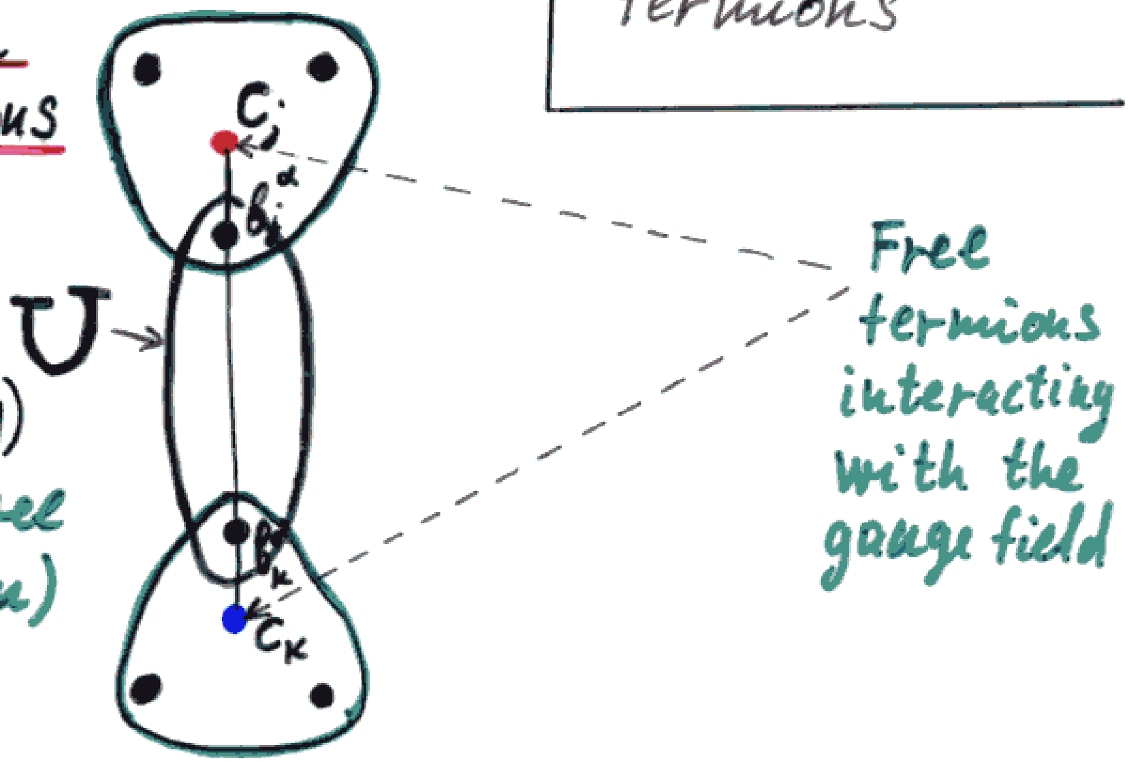


Fermionization

1 spin → 4 Majorana fermions



Regrouping the fermions



$U = b_j^\alpha b_k^\alpha \ (= \pm 1)$

(Gauge degree of freedom)

Each sector can be described by free real fermions in the gauge field

3. Representing spins by fermions
(a general procedure)

Two fermionic modes: $\square \quad \boxed{\bullet}$

Fock space: $\mathcal{F}(2) \cong \mathbb{C}^4$: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

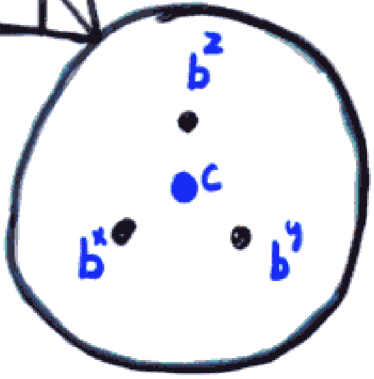
$a_1,\ a_1^\dagger,\ a_2,\ a_2^\dagger$

(annihilation and creation operators) or

$$\boxed{\begin{aligned} c_0 &= a_1 + a_1^\dagger = c & c_2 &= a_2 + a_2^\dagger = b^y \\ c_1 &= \frac{a_1 - a_1^\dagger}{i} = b^x & c_3 &= \frac{a_2 - a_2^\dagger}{i} = b^z \end{aligned}}$$

(Majorana operators)

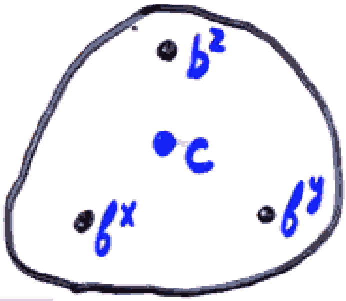
$$\boxed{c_j^\dagger = c_j \qquad c_j c_k + c_k c_j = 2\delta_{jk}}$$



The even subspace

$\mathcal{L} \subseteq \mathcal{F}(2)$: $|00\rangle$, $|11\rangle$

$$|\psi\rangle \in \mathcal{L} \iff \boxed{b^x b^y b^z c\, |\psi\rangle = |\psi\rangle}$$

1 spin = 

(subspace $\mathcal{L}$)

$$D = b^x b^y b^z c \quad \text{(a stabilizer operator)}$$

$$\sigma^\alpha \equiv i b^\alpha c \quad (\alpha = x, y, z)$$

"equal on $\mathcal{L}$"

$[\sigma^\alpha, D] = 0 \quad \Longrightarrow \quad \sigma^\alpha$ preserves the subspace $\mathcal{L}$

$$\sigma^x \sigma^y \sigma^z = i^3 b^x c \, b^y c \, b^z c = i \underbrace{b^x b^y b^z c}_{D} \equiv i$$

$b^x_j, \; b^y_j, \; b^z_j, \; c_j, \quad D_j = b^x_j b^y_j b^z_j c_j$

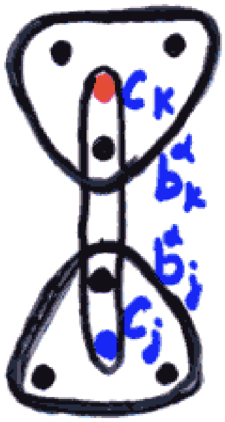
for each site j

Any spin Hamiltonian $= H\{i b^\alpha_j c_j\}$

$$[H, D_j] = 0 \qquad [D_j, D_k] = 0$$

$$D_j |\Psi\rangle = |\Psi\rangle \quad \text{(gauge invariance)}$$

4. Special properties of the Hamiltonian H_0.



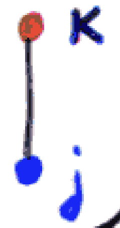
$$\hat{H}_0 = -\sum_{(j,k)} J_{\alpha(j,k)} \hat{K}_{jk}$$

$\alpha(j,k) = x, y, z$

$$\underbrace{\hat{K}_{jk}} = (i\hat{b}_j^\alpha \hat{c}_j)(i\hat{b}_k^\alpha \hat{c}_k) = \underbrace{-i\hat{U}_{jk}\hat{c}_j\hat{c}_k}$$

where $\boxed{\hat{U}_{jk} = i\hat{b}_j^\alpha \hat{b}_k^\alpha}$

$\underbrace{\hat{U}_{kj} = -\hat{U}_{jk}}$. Sign convention: $\hat{U}_{\underline{\underline{j}}\underline{\underline{k}}}$



$$\boxed{\begin{aligned} \hat{H}_0 &= \frac{i}{2}\sum_{(\underline{\underline{j}},\underline{\underline{k}})} \hat{A}_{jk}\hat{c}_j\hat{c}_k \\ \hat{A}_{jk} &= 2J_{\alpha(j,k)}\hat{U}_{jk} \end{aligned}} = \frac{i}{4}\sum_{j,k} \hat{A}_{jk}\hat{c}_j\hat{c}_k$$

(each pair (j,k) appears twice)

New integrals of motion: $\hat{U}_{jk}$ $\quad \underbrace{[\hat{U}_{jk}, \hat{H}_0] = 0}$

Did not exist in the spin model because $[\hat{D}_j, \hat{U}_{jk}] \neq 0$

Old integrals of motion:

$$\boxed{\hat{B}_p = \prod_{\text{(p)}} \hat{U}_{jk}}$$

$$[\hat{D}_j, \hat{B}_p] = 0$$

5. A special "fermionization" procedure for H_0

0) Choose a vortex configuration $\{v_p\}$. $(v_p = \pm 1)$

1) Fix a gauge: $\underbrace{\hat{U}_{jk} |\psi\rangle = u_{jk} |\psi\rangle}$

$$\boxed{\prod_{\text{(p)}} u_{jk} = v_p} \qquad u_{jk} = \pm 1$$

2) Solve for the ground state of

$$\boxed{\begin{aligned} \hat{H}_{\{u\}} &= \frac{i}{4} \sum_{j,k} A_{jk}\, \hat{c}_j \hat{c}_k \\ A_{jk} &= 2 J_{\alpha(j,k)}\, u_{jk} \end{aligned}}$$

$$E = E(v_1, \ldots, v_{N/2})$$

vortex configuration

A is a real skew-symmetric matrix

3) Symmetrize over "gauge transformations" $\hat{D}_j$ while keeping $\underbrace{v_p = \prod_{\text{(p)}} u_{jk}}$ fixed.

(projecting onto the physical subspace)

6. The one-particle spectrum

$$H_{\{u\}} = \frac{i}{4} \sum_{j,k} A_{jk}\, c_j c_k$$

Canonical form: $\displaystyle \frac{i}{2} \sum_m \varepsilon_m\, b'_m b''_m = \sum_m \varepsilon_m a^\dagger_m a_m$ + const

normal modes

$$\begin{pmatrix} b'_1 \\ b''_1 \\ \vdots \\ b'_N \\ b''_N \end{pmatrix} = W \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2N-1} \\ c_{2N} \end{pmatrix} \qquad WAW^T = \begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & 0 \\ & & \ddots & & \\ & 0 & & 0 & \varepsilon_N \\ & & & -\varepsilon_N & 0 \end{pmatrix}$$

$$\underline{W \in O(2N)}$$

$\underline{\varepsilon_m \geq 0}$

Ground state: $\underline{-i\, b'_m b''_m |\psi\rangle = |\psi\rangle}$

$$E = -\frac{1}{2} \sum_m \varepsilon_m$$

(Translational invariance allows further simpification: $\varepsilon(q)$ $\quad (\varepsilon(-q) = -\varepsilon(q))$)

The spectrum

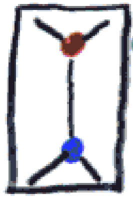
$$\boxed{H=\sum_{j\alpha,k\beta}\frac{i}{4}A_{j\alpha,k\beta}\,C_{j\alpha}\,C_{k\beta}}$$

$$\tilde{A}_{\alpha,\beta}(q)=\sum_{k}A_{0\alpha,k\beta}\,e^{i(\vec{q},\vec{r}_k)}$$

$C_{j\alpha}$

cell number

index of a Majorana operator in the cell



Momentum representation $\longrightarrow$ $H=\sum_{q,\alpha,\beta}\frac{i}{4}\tilde{A}_{\alpha\beta}(q)\,\tilde{C}_\alpha(q)\,\tilde{C}_\beta(-q)$

$$\left(\tilde{C}_\beta(-q)=\tilde{C}_\beta(q)^+\right)$$

$$\tilde{A}(q)=\begin{pmatrix}0 & f(q)\\ -f(q)^* & 0\end{pmatrix}\implies \varepsilon(q)=\pm|f(q)|$$

Eigenvalues $=-i\varepsilon(q)$

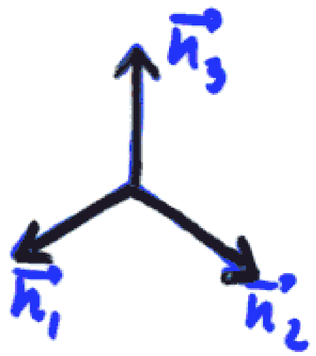
"double spectrum"

$$\boxed{f(q)=J_x\,e^{i(\vec{n}_1,\vec{q})}+J_y\,e^{i(\vec{n}_2,\vec{q})}+J_z\,e^{i(\vec{n}_3,\vec{q})}}$$

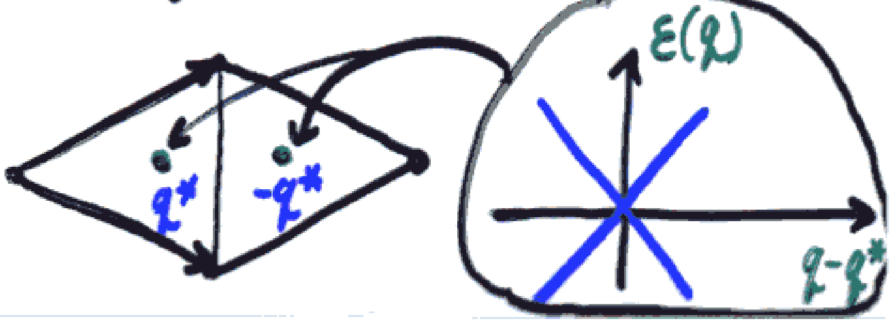
$J_x<J_y+J_z$, $J_y<J_x+J_z$,

$J_z<J_x+J_y$ $\implies$

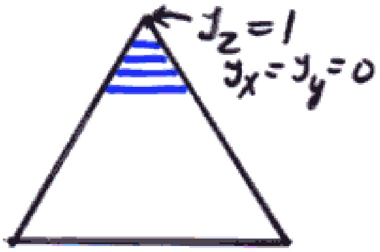
gapless spectrum



Momentum space

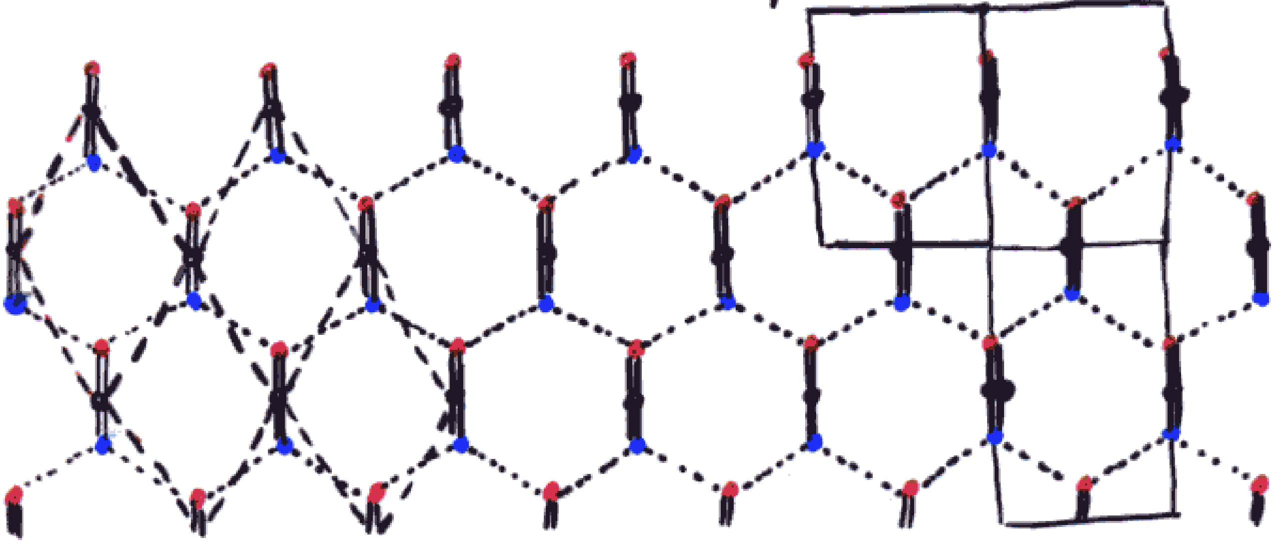


9. Abelian anyons

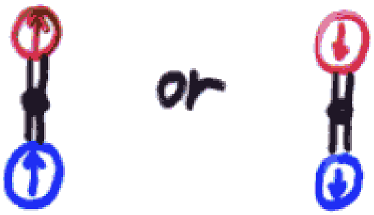


Let $J_z \gg J_x, J_y$

1-st order of the perturbation theory:



Low-energy subspace



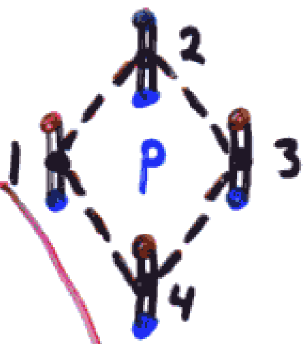
(an effective spin: ↑ or ↓)

4-th order of the perturbation theory

$$\boxed{H_{eff} = -\frac{J_x^2 J_y^2}{16 J_z^3} \sum_p B_p}$$

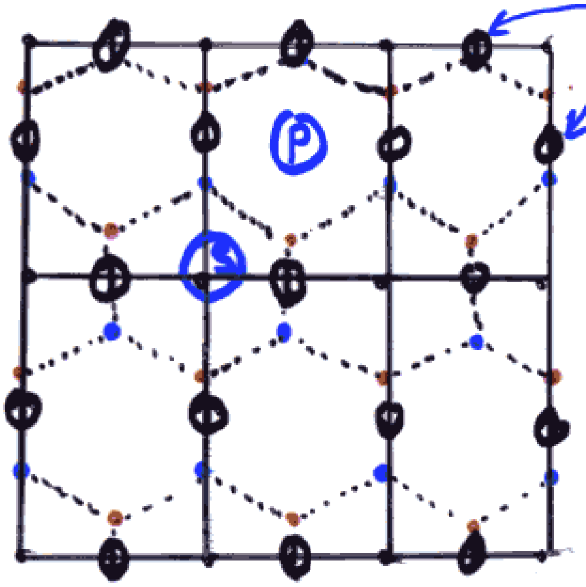
$$\boxed{B_p \equiv \sigma_1^y \sigma_2^z \sigma_3^y \sigma_4^z}$$

effective spins



An equivalent model

effective spins



$$H_{eff} = -\sum_{\text{plaquettes}} V_p - \sum_{\text{vertices}} V_s$$

Renaming $\sigma_j^y \leftrightarrow \sigma_j^x$ on some links $\Rightarrow$

$$\Rightarrow \quad V_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z$$

$$V_s = \prod_{j \in \text{star}(s)} \sigma_j^x$$

quant-ph/9707021

Excitations:

"Magnetic vortices" on plaquettes

"Electric charges" on vertices

10 Superselection sectors for Abelian anyons

Without vortices:

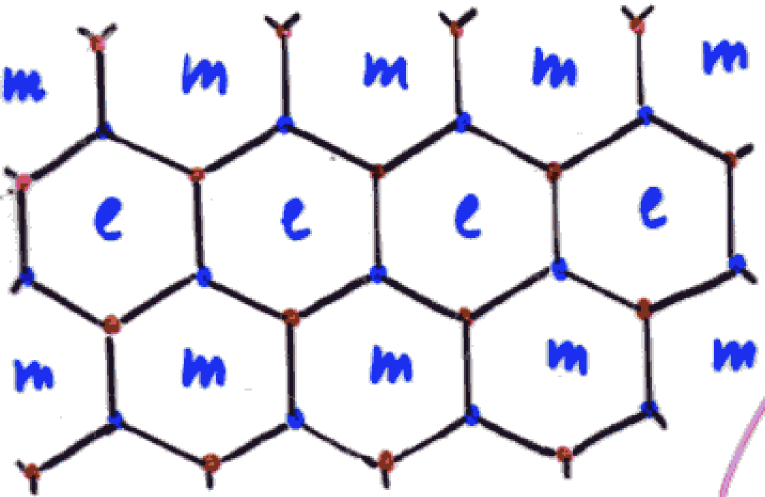
1 (the vacuum)

ε (the fermion)

Vortices (assuming that the fermions form the lowest energy state)

m ("magnetic vortex")

e ("electric charge")



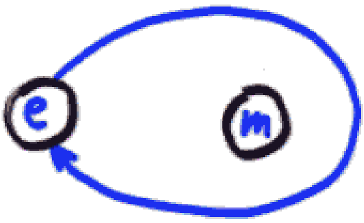
$e \times e = 1$

$m \times m = 1$

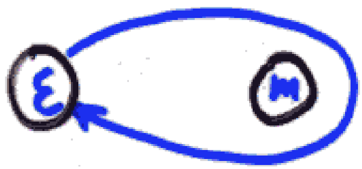
$e \times m = \varepsilon$

$\varepsilon \times \varepsilon = 1 \qquad e \times \varepsilon = m \qquad m \times \varepsilon = e$

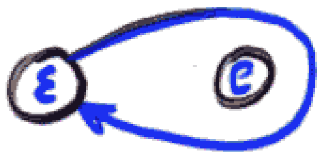
fusion rules



$= -1$



$= -1$



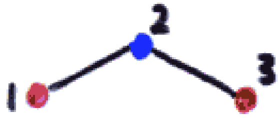
$= -1$

II The gapless phase acquires a gap in a magnetic field

$$H' = H + \sum_j (\vec{h}, \vec{\sigma}_j)$$

$$|\vec{h}| \ll J_x, J_y, J_z$$

3-rd order of the perturbation theory

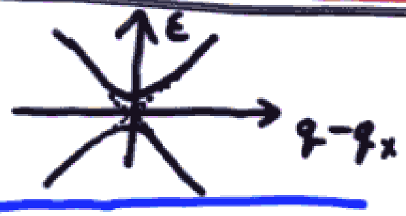


$$V^{(3)} \backsim h_x h_y h_z \underbrace{\sigma_1^x \sigma_2^y \sigma_3^z}_{i\, c_1 c_3} + \text{similar terms}$$

The Hamiltonian is still quadratic in c_j!

$$\tilde{A}(q) = \begin{pmatrix} i\, g(q) & f(q) \\ -f(q)^* & -i\, g(q) \end{pmatrix} \qquad g(q) \backsim h_x h_y h_z$$

$f(q) = 0$ for $q = \pm q_*$



$$\varepsilon(q) = \pm\sqrt{|f(q)|^2 + g(q)^2} \sim \underbrace{\sqrt{|\vec{q} - \vec{q}_*|^2 + g_*^2}}_{\text{conical point at } q = q_*,\ g = 0}$$

$$f(q) \backsim (q_x - q_{*x}) + i\,(q_y - q_{*y})$$

Nonabelian anyons (in phase A with magnetic field)

Three superselection sectors:

The vacuum : 1

the fermion : ε

only one type of vortex : σ

$$\varepsilon \times \varepsilon = 1$$

$$\sigma \times \varepsilon = \sigma$$

$$\sigma \times \sigma = 1 + \varepsilon$$

Each vortex p carries a Majorana operator $\hat{W}_p$

$$= 1 + \varepsilon$$

$$(-i\,\hat{W}_1\hat{W}_2)|\psi\rangle = |\psi\rangle$$

$$(-i\hat{W}_1\hat{W}_2)|\psi\rangle = -|\psi\rangle$$

$$= e^{i\varphi\nu}\exp\left(\frac{\pi}{4}\nu\,\hat{W}_1\hat{W}_2\right)$$

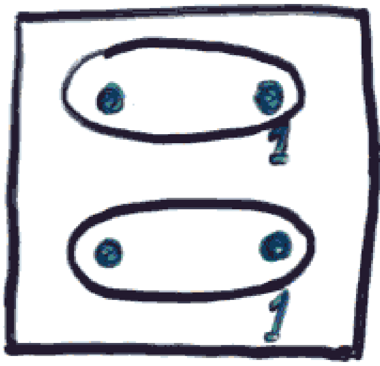
$\varphi = \frac{3\pi}{8}$?

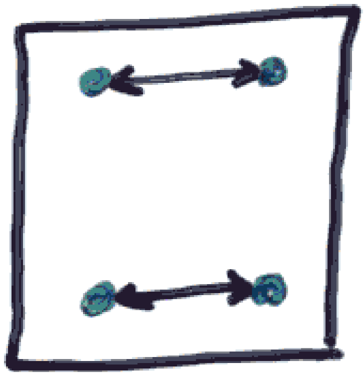
unknown phase

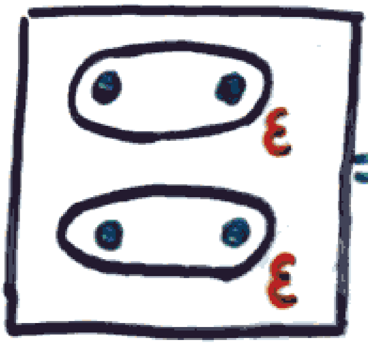
$\nu = \mathrm{sgn}(h_x h_y h_z)$

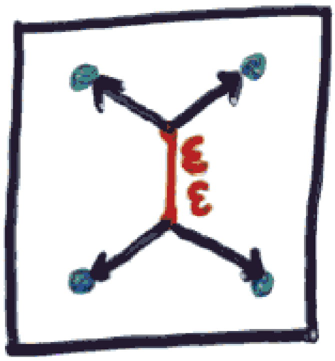
magnetic field components

Encoding a qubit into 4 vortices with total "charge" 1.

$|0\rangle_L \longmapsto$ 

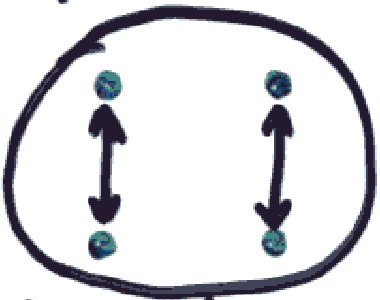
$= |1\,1\rangle$ 

$|1\rangle_L \longmapsto$ 

$= |\varepsilon\,\varepsilon\rangle$ 

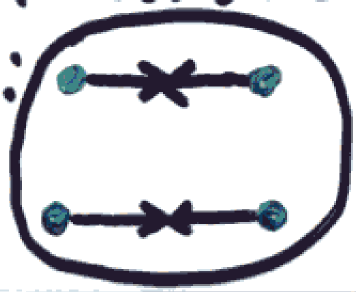
Thought experiment

1) Create 2 pairs



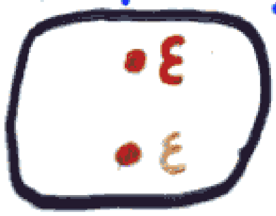
$$\psi = \frac{1}{\sqrt{2}}|0\rangle_L + \frac{1}{\sqrt{2}}|1\rangle_L$$

2) Annihilate them in a different way:

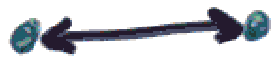
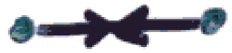


with prob. $\frac{1}{2}$

with prob. $\frac{1}{2}$



Basic computational operations

1) Initializing a qubit	
2) Measuring a qubit	 (ε or 1 ?)
3) Unitary gates	 $\boxed{\begin{array}{l} \lvert 1\rangle \mapsto e^{i\frac{\pi}{8}} \lvert 1\rangle \\ \lvert \varepsilon\rangle \mapsto e^{-i\frac{3\pi}{8}} \lvert \varepsilon\rangle \end{array}}$

- Unfortunately, the braiding gate is not universal (for this type of anyons)

- The anyons are still good for the realization of quantum memory and may relax the threshold requirement